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On the Time Value of Ruin

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Abstract

This paper studies the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. The classical model is generalized by discounting with respect to the time of ruin. We show how to calculate an expected discounted penalty, which is due at ruin, and may depend on the deficit at ruin and the surplus immediately before ruin. The expected discounted penalty, considered as a function of the initial surplus, satisfies a certain renewal equation, which has a probabilistic interpretation. Explicit answers are obtained for zero initial surplus, very large initial surplus, and arbitrary initial surplus if the claim amount distribution is exponential or a mixture of exponentials. We generalize D.C.M. Dickson's formula, which expresses the joint distribution of the surplus immediately prior to and at ruin in terms of the probability of ultimate ruin. Explicit results are obtained when dividends are paid out to the stockholders according to a constant barrier strategy.

1. Introduction

Collective risk theory has started in 1903 with the doctoral thesis of Filip Lundberg, a Swedish actuary, and it has been developed throughout this century. It is now an area rich in useful ideas and sophisticated techniques. Many of its tools can be applied to solve problems in other areas. A recent example is the method of *Esscher transforms*, which was used by Gerber and Shiu [23] to price financial derivatives.

Two particular questions of ruin theory are (a) the *severity of ruin*, and (b) the *time of ruin*, both of which have been treated separately in the literature. In this paper certain answers to both questions are given at the same time. We study the joint distribution of the deficit and the time of ruin. From a mathematical point of view a crucial role is played by the surplus immediately before ruin occurs. We incorporate the time of ruin in the model by discounting. We show how to calculate an *expected discounted penalty*, which is due at ruin and may depend on the deficit at ruin and the surplus immediately prior to ruin. The expected discounted penalty, considered as a function of the initial surplus, satisfies a certain renewal equation. The renewal equation and its convolution series solution have natural probabilistic interpretations. For the former one considers the first time when the surplus falls below the initial level, and distinguishes according to whether or not ruin takes place at that time. For the latter the distinction is made according to which of the record lows of the surplus process causes ruin. Explicit answers are obtained for zero initial surplus, very large initial surplus, and arbitrary initial surplus if the claim amount distribution is exponential or a mixture of exponentials. Additional insight is obtained from a pair of exponential martingales. As an application we generalize D.C.M. Dickson's [7] formula, which expresses the joint distribution of the surplus immediately prior to and at ruin in terms of the probability of ultimate ruin. Similarly, explicit results are obtained when dividends are paid out to the stockholders according to a constant barrier strategy.

The paper generalizes and adds to a better understanding of classical ruin theory, which can be retrieved by setting the interest rate equal to zero. For example, in the classical model, the *adjustment coefficient* is the solution of an implicit equation, which has 0 as another solution. If the interest rate is positive, the situation is suddenly symmetric: the corresponding equation, called *Lundberg's fundamental equation,* has a positive solution and a negative solution. Both solutions are important and are used to construct exponential martingales. We also present some results concerning the finite-time ruin function and its Laplace transforms.

This paper was motivated by the problem of pricing *American options.* The classical model uses the geometric Brownian motion to model the stock price process. Such a process has continuous sample paths, which facilitate the analysis of an American option: the option is exercised as soon as the stock price arrives on the optimal exercise boundary, and the price of the option is the expected discounted payoff. On the other hand, we would like to price an American option in a perhaps more realistic model where the stock price may have jumps. The resulting mathematical problem is more intricate, because now, at the time of the exercise, the stock price is not *on* but *beyond* the optimal exercise boundary. If the logarithm of the stock price is modeled by a shifted compound Poisson process, this leads to the type of problems that are discussed in this paper. Evidently "penalty at ruin" has to be replaced by "payoff at exercise." Thus the paper lays the mathematical bases for a financial application. Since it is substantial and of an independent interest, we decided to present the application to the pricing of American options in a subsequent paper.

2. When and How Does Ruin Occur

We follow the notation in Chapter 12 of *Actuarial Mathematics* [4]. Thus $u \geq 0$ is the insurer's initial surplus. The premiums are received continuously at a constant rate c per unit time. The aggregate claims constitute a compound Poisson process, $\{S(t)\}$, given by the Poisson parameter λ and individual claim amount distribution function $P(x)$ with $P(0) = 0$. That is,

$$S(t) = \sum_{j=1}^{N(t)} X_j, \tag{2.1}$$

where $\{N(t)\}$ is a Poisson process with mean per unit time λ and $\{X_j\}$ are independent random variables with common distribution $P(x)$. Then

$$U(t) = u + ct - S(t) \tag{2.2}$$

is the surplus at time t, $t \geq 0$. For simplicity we assume that $P(x)$ is differentiable, with

$$P'(x) = p(x)$$

being the individual claim amount probability density function.

Let T denote the *time of ruin*,

$$T = \inf\{t \mid U(t) < 0\} \tag{2.3}$$

($T = \infty$ if ruin does not occur). We consider the probability of ultimate ruin as a function of the initial surplus $U(0) = u \geq 0$,

$$\psi(u) = \Pr[T < \infty \mid U(0) = u]. \tag{2.4}$$

Let p_1 denote the mean of the individual claim amount distribution,

$$p_1 = \int_0^\infty x\, p(x)\, dx = E(X_j).$$

We assume

$$c > \lambda p_1 \tag{2.5}$$

to ensure that $\{U(t)\}$ has a positive drift; hence

$$\lim_{t \to \infty} U(t) = \infty \tag{2.6}$$

with certainty, and

$$\psi(u) < 1. \tag{2.7}$$

We also consider the random variables $U(T-)$, the surplus immediately before ruin, and $U(T)$, the surplus at ruin. See Figure 1. For given $U(0) = u \geq 0$, let $f(x, y, t \mid u)$ denote the joint probability density function of $U(T-)$, $|U(T)|$ and T. Then

$$\int_0^\infty\int_0^\infty\int_0^\infty f(x, y, t \mid u)\, dxdydt \;=\; \Pr[T < \infty \mid U(0) = u] \;=\; \psi(u). \tag{2.8}$$

Because of (2.7), $f(x, y, t \mid u)$ is a *defective* probability density function. We note that

$$f(x, y, t \mid u) \;=\; 0$$

for $x > u + ct$.

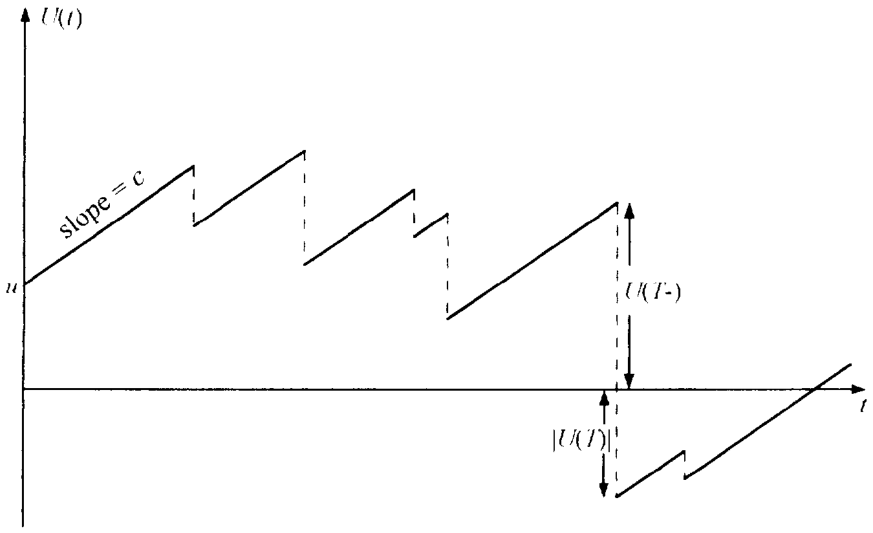

Figure 1. The Surplus Immediately before and at Ruin

It is easier to analyze the following function, the study of which is a central theme in this paper. For $\delta \geq 0$, define

$$f(x, y \mid u) \;=\; \int_0^\infty e^{-\delta t} f(x, y, t \mid u)\, dt. \tag{2.9}$$

Here δ can be interpreted as a force of interest, or, in the context of Laplace transforms, as a dummy variable. Note that the symbol $f(x, y \mid u)$ does not exhibit the dependence on δ. If $\delta = 0$, (2.9) is the defective joint probability density function of $U(T\text{-})$ and $|U(T)|$, given $U(0) = u$. Also, if $\delta > 0$, then

$$e^{-\delta T} \;=\; e^{-\delta T}\, I(T < \infty),$$

where I denotes the indicator function, i.e., $I(A) = 1$ if A is true and $I(A) = 0$ if A is false.

Let w(x, y) be a nonnegative function of x > 0 and y > 0. We consider, for $u \geq 0$,

$$\phi(u) = E[w(U(T-), |U(T)|)\, e^{-\delta T}\, I(T < \infty) \mid U(0) = u] \tag{2.10}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty w(x, y)\, e^{-\delta t}\, f(x, y, t \mid u)\, dt\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty w(x, y)\, f(x, y \mid u)\, dx\, dy. \tag{2.11}$$

With $\delta = 0$ and $w(x, y) = w(-y)$, $\phi(u)$ is denoted as $\psi(u; w)$ in the Proof of Theorem 12.4 of *Actuarial Mathematics* [4]. If we interpret δ as a force of interest and w as some kind of penalty when ruin occurs, then $\phi(u)$ is the expectation of the discounted penalty. We should clarify that, while it can be very helpful to consider δ as a force of interest in this paper, we are dealing with the classical model in which the surplus does not earn any interest.

Our immediate goal is to derive a functional equation for $\phi(u)$ by applying the law of iterated expectations to the right-hand side of (2.10). For h > 0, consider the time interval (0, h), and condition on the time t and the amount x of the first claim in this time interval. Note that the probability that there is no claim up to time h is $e^{-\lambda h}$, the probability that the first claim occurs between time t and time t + dt is $e^{-\lambda t}\lambda dt$, and

$$x > u + ct$$

means that ruin has occurred with the first claim. Hence

$$\phi(u) = e^{-(\delta+\lambda)h}\phi(u + ch) + \int_0^h \left[\int_0^{u+ct} \phi(u + ct - x)p(x)dx\right] e^{-(\delta+\lambda)t}\lambda dt$$

$$+ \int_0^h \left[\int_{u+ct}^\infty w(u + ct, x - u - ct)p(x)dx\right] e^{-(\delta+\lambda)t}\lambda dt. \tag{2.12}$$

Differentiating (2.12) with respect to h and setting h = 0, we obtain

$$0 = -(\delta + \lambda)\phi(u) + c\phi'(u) + \lambda\int_0^u \phi(u - x)\, p(x)\, dx$$

$$+ \lambda\int_u^\infty w(u, x - u)\, p(x)\, dx$$

$$= -(\delta + \lambda)\phi(u) + c\phi'(u) + \lambda\int_0^u \phi(u - x)\, p(x)\, dx + \lambda\omega(u), \tag{2.13}$$

where

$$\omega(u) = \int_u^\infty w(u, x - u)\, p(x)\, dx. \tag{2.14}$$

With X denoting the individual claim amount random variable,

$$\omega(u) = E[w(u, X-u)\, I(X > u)]. \tag{2.15}$$

Also, by a change of variable,

$$\omega(u) = \int_0^{\infty} w(u, y)\, p(u+y)\, dy. \tag{2.16}$$

For further analysis, we use the technique of *integrating factors*. Let

$$\phi_\rho(u) = e^{-\rho u}\, \phi(u), \tag{2.17}$$

where ρ is a nonnegative number to be specified later. Multiplying (2.13) with $e^{-\rho u}$, applying the product rule for differentiation, and rearranging yields

$$c\phi_\rho'(u) = (\delta + \lambda - c\rho)\phi_\rho(u) - \lambda\int_0^u \phi_\rho(u-x)e^{-\rho x}p(x)dx - \lambda e^{-\rho u}\omega(u). \tag{2.18}$$

Define

$$\ell(\xi) = \delta + \lambda - c\xi; \tag{2.19}$$

hence the coefficient of $\phi_\rho(u)$ in (2.18) is $\ell(\rho)$. In this paper we let $\hat{f}$ denote the Laplace transform of a function f,

$$\hat{f}(\xi) = \int_0^{\infty} e^{-\xi x}\, f(x)\, dx. \tag{2.20}$$

The Laplace transform of p, $\hat{p}(\xi)$, is defined for all nonnegative numbers ξ, and is a decreasing convex function because

$$\hat{p}'(\xi) = -\int_0^{\infty} e^{-\xi x}\, x\, p(x)\, dx < 0$$

and

$$\hat{p}''(\xi) = \int_0^{\infty} e^{-\xi x}\, x^2\, p(x)\, dx > 0.$$

Consider the equation

$$\ell(\xi) = \lambda\hat{p}(\xi). \tag{2.21}$$

Since

$$\ell(0) = \delta + \lambda \geq \lambda = \lambda\hat{p}(0)$$

and by (2.5)

$$\ell'(0) = -c < -\lambda p_1 = \lambda\hat{p}'(0),$$

equation (2.21) has a unique nonnegative root, say ξ_1. It is obvious from Figure 2 (which corresponds to Figure 11.7.1 of Panjer and Willmot [27]) that ξ_1 is an increasing function of δ, with $\xi_1 = 0$ when $\delta = 0$. Furthermore, if the individual claim amount density function p is sufficiently regular, equation (2.21) has one more root, say ξ_2, which is negative. This negative root, which will be denoted as $-R$, plays an important role later. As we shall see in Section 8, both roots are related to the construction of exponential martingales. When $\delta = 0$, (2.21) is equivalent to (12.3.1) in *Actuarial Mathematics* [4] and R is the *adjustment coefficient*. Equation (2.21) is equivalent to Beekman [3, p. 41, top equation], Panjer and Willmot [27, (11.7.8)], and Seal [34, (4.24)]. Lundberg [26, p. 144] points out that the equation is "fundamental to the whole of collective risk theory," and Seal [34, p. 111] calls it "Lundberg's (1928) 'fundamental' equation." (It is incorrect for Seal [34, p. 112] to assert that the second root is also positive.)

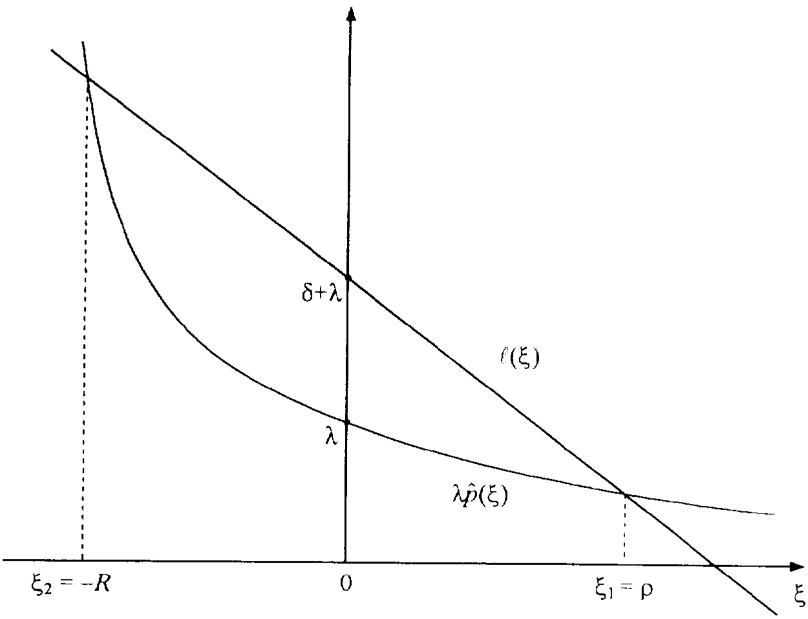

Figure 2. The Two Roots of Lundberg's Fundamental Equation

The trick for solving (2.18) is to choose

$$\rho = \xi_1, \tag{2.22}$$

so that (2.18) becomes

$$c\phi_\rho'(u) = \lambda\hat{p}(\rho)\phi_\rho(u) - \lambda\int_0^u \phi_\rho(u-x)e^{-\rho x}p(x)dx - \lambda e^{-\rho u}\omega(u)$$
$$= \lambda\Big[\hat{p}(\rho)\phi_\rho(u) - \int_0^u \phi_\rho(x)e^{-\rho(u-x)}p(u-x)dx - e^{-\rho u}\omega(u)\Big]. \qquad (2.23)$$

For $z > 0$, we integrate (2.23) from $u = 0$ to $u = z$. After a division by λ, the resulting equation is

$$\frac{c}{\lambda}[\phi_\rho(z) - \phi_\rho(0)]$$
$$= \hat{p}(\rho)\int_0^z \phi_\rho(u)du - \int_0^z\Big[\int_0^u \phi_\rho(x)e^{-\rho(u-x)}p(u-x)dx\Big]du - \int_0^z e^{-\rho u}\omega(u)du$$
$$= \hat{p}(\rho)\int_0^z \phi_\rho(u)du - \int_0^z\Big[\int_x^z e^{-\rho(u-x)}p(u-x)du\Big]\phi_\rho(x)dx - \int_0^z e^{-\rho u}\omega(u)du$$
$$= \int_0^z \phi_\rho(x)\Big[\int_{z-x}^\infty e^{-\rho y}p(y)dy\Big]dx - \int_0^z e^{-\rho u}\omega(u)du. \qquad (2.24)$$

For $z \to \infty$, the first terms on both sides of (2.24) vanish, which shows that

$$\phi_\rho(0) = \frac{\lambda}{c}\int_0^\infty e^{-\rho u}\omega(u)du = \frac{\lambda}{c}\hat{\omega}(\rho). \qquad (2.25)$$

Finally, substituting (2.25) in (2.24) and simplifying yields

$$\phi_\rho(z) = \frac{\lambda}{c}\Big\{\int_0^z \phi_\rho(x)\Big[\int_{z-x}^\infty e^{-\rho y}p(y)dy\Big]dx + \int_z^\infty e^{-\rho u}\omega(u)du\Big\}, \quad z \geq 0. \qquad (2.26)$$

For two integrable functions f_1 and f_2 defined on $[0, \infty)$, the *convolution* of f_1 and f_2 is the function

$$(f_1 * f_2)(x) = \int_0^x f_1(y)\, f_2(x-y)\, dy, \qquad x \geq 0. \qquad (2.27)$$

Note that

$$f_1 * f_2 = f_2 * f_1.$$

With the definitions

$$g_\rho(x) = \frac{\lambda}{c}\int_x^\infty e^{-\rho y}p(y)dy, \quad x \geq 0, \qquad (2.28)$$

and

$$h_\rho(x) = \frac{\lambda}{c}\int_x^\infty e^{-\rho u}\omega(u)du, \quad x \geq 0, \qquad (2.29)$$

equation (2.26) can be written more concisely as

$$\phi_\rho = \phi_\rho * g_\rho + h_\rho. \qquad (2.30)$$

In the literature of integral equations, (2.30) is classified as a *Volterra equation of the second kind.* The function g_ρ is a nonnegative function on $[0, \infty)$ and hence may be interpreted as a (not necessarily proper) probability density function; in probability theory, (2.30) is known as a *renewal equation* for the function ϕ_ρ.

Unlike ϕ, the function ϕ_ρ does not have a probabilistic interpretation. Hence it is preferable to work with the function ϕ. It follows from (2.27) that for each constant k

$$e^{kx}(f_1*f_2) = (e^{kx}f_1)*(e^{kx}f_2). \tag{2.31}$$

This enables us to convert (2.30) into a renewal equation for ϕ using (2.17), which is

$$\phi(x) = e^{\rho x}\phi_\rho(x), \qquad x \geq 0.$$

With the definition

$$\begin{aligned} g(x) &= e^{\rho x} g_\rho(x) \\ &= \frac{\lambda}{c}\int_x^\infty e^{-\rho(y-x)}\, p(y)\, dy && (2.32) \\ &= \frac{\lambda}{c}\int_0^\infty e^{-\rho z} p(x+z)\, dz, \qquad x \geq 0, && (2.33) \end{aligned}$$

and

$$\begin{aligned} h(x) &= e^{\rho x} h_\rho(x) \\ &= \frac{\lambda}{c}\int_x^\infty e^{-\rho(u-x)}\, \omega(u)\, du && (2.34) \\ &= \frac{\lambda}{c}\int_0^\infty e^{-\rho z}\, \omega(x+z)\, dz, \qquad x \geq 0, && (2.35) \end{aligned}$$

we have

$$\phi = \phi*g + h. \tag{2.36}$$

The solution of (2.36) can be expressed as an infinite series of functions, sometimes called a *Neumann series,*

$$\phi = h + g*h + g*g*h + g*g*g*h + g*g*g*g*h + \cdots . \tag{2.37}$$

This generalizes what is called *Beekman's convolution series* in the actuarial literature. One may obtain (2.37) from (2.36) by the method of successive substitution.

Remarks (i) Two useful expressions for h are

$$h(u) = \frac{\lambda}{c}\int_u^\infty\int_0^\infty e^{-\rho(x-u)}\, w(x, y)\, p(x+y)\, dy\, dx. \tag{2.38}$$

and

$$h(u) = \frac{\lambda}{c}\int_0^\infty\int_0^\infty e^{-\rho z}\, w(u+z, y)\, p(u+z+y)\, dy\, dz. \tag{2.39}$$

(ii) With $\delta = 0$ and hence $\rho = 0$, it is well known [4, Theorem 12.4] that the differential

$$g(y)\, dy = \frac{\lambda}{c}[1 - P(y)]dy, \tag{2.40}$$

can be interpreted as the probability that the surplus will ever fall below its initial level u, and will be between u – y and u – y – dy when it happens for the first time. Furthermore, with $\delta = 0$ and $w \equiv 1$, we have

$$h(x) = \int_x^\infty g(y)dy,$$

which is the probability that the surplus will ever fall below its initial level u, and will be below u – x when it happens for the first time. The renewal equation (2.36) generalizes Exercise 12.11 of *Actuarial Mathematics* [4]; see also (9.44) below. In Section 5 we shall see how the functions g and h can be interpreted for $\delta > 0$, and hence probabilistic explanations of (2.36) and (2.37) can be given.

(iii) It follows from the conditional probability formula,

$$\Pr(A \cap B) = \Pr(A) \Pr(B \mid A),$$

that the joint probability density function of U(T-), |U(T)| and T at the point (x, y, t) is the joint probability density function of U(T-) and T at the point (x, t) multiplied by the conditional probability density function of |U(T)| at y, given that U(T-) = x and T = t. The latter does not depend on t and is

$$\frac{p(x+y)}{\int_0^\infty p(x+y)\, dy} = \frac{p(x+y)}{1 - P(x)}, \qquad y \geq 0.$$

Hence

$$f(x, y, t \mid u) = \left[\int_0^\infty f(x, z, t \mid u)\, dz\right]\frac{p(x+y)}{1 - P(x)}. \tag{2.41}$$

With the definition

$$\begin{aligned} f(x \mid u) &= \int_0^\infty f(x, y \mid u)\, dy \\ &= \int_0^\infty\int_0^\infty e^{-\delta t} f(x, y, t \mid u)\, dt\, dy, \end{aligned} \tag{2.42}$$

multiplying (2.41) with $e^{-\delta t}$ and then integrating with respect to t yields

$$f(x, y \mid u) = f(x \mid u) \frac{p(x+y)}{1-P(x)}. \tag{2.43}$$

With $\delta = 0$, (2.43) was first pointed out by Dufresne and Gerber [13, (3)]; another proof can be found in Dickson and Egídio dos Reis [9]. Also, it follows from (2.10), (2.41), (2.42) and (2.16) that

$$\begin{aligned}
\phi(u) &= \int_0^\infty\int_0^\infty\int_0^\infty w(x, y)\, e^{-\delta t} f(x, y, t \mid u)\, dt\, dx\, dy \\
&= \int_0^\infty\int_0^\infty\int_0^\infty w(x, y)\, e^{-\delta t} \Big[\int_0^\infty f(x, z, t \mid u)\, dz\Big]\frac{p(x+y)}{1-P(x)}\, dt\, dx\, dy \\
&= \int_0^\infty\int_0^\infty w(x, y)\, f(x \mid u) \frac{p(x+y)}{1-P(x)}\, dx\, dy \\
&= \int_0^\infty \omega(x) \frac{f(x \mid u)}{1-P(x)}\, dx.
\end{aligned} \tag{2.44}$$

(iv) It follows from an integration by parts that

$$\hat{p}(\xi) = 1 - \xi\int_0^\infty e^{-\xi x}[1 - P(x)]\, dx, \tag{2.45}$$

with which we can rewrite Lundberg's fundamental equation (2.21) as

$$\delta = c\xi - \lambda[1 - \hat{p}(\xi)] \tag{2.46}$$

$$= \xi\{c - \lambda\int_0^\infty e^{-\xi x}[1 - P(x)]\, dx\}. \tag{2.47}$$

Hence

$$\lim_{\delta \to 0} \frac{\delta}{\rho} = c - \lambda\int_0^\infty [1 - P(x)]\, dx$$

$$= c - \lambda p_1, \tag{2.48}$$

which is the drift of $\{U(t)\}$. With $\delta = 0$, the negative root $\xi = \xi_2$ of (2.47) or (2.21) is determined by the equation

$$\int_0^\infty e^{-\xi x}[1 - P(x)]\, dx = \frac{c}{\lambda}; \tag{2.49}$$

this condition to identical to the one in Exercise 12.7 of *Actuarial Mathematics* [4].

(v) Equation (2.36) may be solved by the method of Laplace transforms (Spiegel [37]). Taking Laplace transforms, we have

$$\hat{\phi}(\xi) = \hat{\phi}(\xi)\hat{g}(\xi) + \hat{h}(\xi), \tag{2.50}$$

or

$$\hat{\phi}(\xi) = \frac{\hat{h}(\xi)}{1 - \hat{g}(\xi)}. \tag{2.51}$$

Hence ϕ can obtained by inverting or identifying the right-hand side of (2.51). In terms of complex integration,

$$\phi(u) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\hat{h}(\xi)}{1 - \hat{g}(\xi)} e^{\xi u}\, d\xi, \tag{2.52}$$

where $i = \sqrt{-1}$, and the path of integration is parallel to the imaginary axis in the complex plane, with the real number b being chosen so that all the singularities of the integrand lie to the left of the line of integration (Spiegel [37, p. 201]). Furthermore, if we expand the right-hand side of (2.51) as a geometric series, we obtain

$$\hat{\phi}(\xi) = \sum_{n=0}^{\infty} \hat{g}(\xi)^n\, \hat{h}(\xi), \tag{2.53}$$

which is the Laplace transform of (2.37).

(vi) From (2.32) and by changing of the order of integration we see that

$$\begin{aligned} \hat{g}(\xi) &= \frac{\lambda}{c}\int_0^\infty e^{-\xi x}\Big[\int_x^\infty e^{-\rho(y-x)}\, p(y)\, dy\Big]dx \\ &= \frac{\lambda}{c(\rho-\xi)}\int_0^\infty [e^{(\rho-\xi)y} - 1]e^{-\rho y}\, p(y)\, dy \\ &= \frac{\lambda}{c(\rho-\xi)}[\hat{p}(\xi) - \hat{p}(\rho)]. \end{aligned} \tag{2.54}$$

Because ρ satisfies Lundberg's fundamental equation (2.21), it follows that

$$\hat{g}(\xi) = \frac{\lambda\hat{p}(\xi) + c\rho - \delta - \lambda}{c(\rho-\xi)}, \tag{2.55}$$

or

$$1 - \hat{g}(\xi) = \frac{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi}{c(\rho-\xi)}. \tag{2.56}$$

We note that (2.21) is the condition that the numerator on the right-hand side of (2.56) vanishes. Hence the negative root ξ_2 of (2.21) is determined by the condition that

$$\hat{g}(\xi) = 1. \tag{2.57}$$

(vii) Writing (2.54) as

$$(\rho - \xi)\hat{g}(\xi) = \frac{\lambda}{c}[\hat{p}(\xi) - \hat{p}(\rho)],$$

and differentiating with respect to ξ, we obtain

$$\hat{g}'(\xi) = \frac{1}{\rho - \xi}\left[\hat{g}(\xi) + \frac{\lambda}{c}\hat{p}'(\xi)\right]. \tag{2.58}$$

Since the negative root ξ_2 satisfies (2.57), a particular case of (2.58) is

$$\hat{g}'(\xi_2) = \frac{1}{\rho - \xi_2}\left[1 + \frac{\lambda}{c}\hat{p}'(\xi_2)\right]. \tag{2.59}$$

(viii) From (2.34) and by changing the order of integration, we get

$$\begin{aligned}
\hat{h}(\xi) &= \frac{\lambda}{c}\int_0^\infty e^{-\xi x}\left[\int_x^\infty e^{-\rho(u-x)}\,\omega(u)\,du\right]dx \\
&= \frac{\lambda}{c(\rho - \xi)}\int_0^\infty [e^{(\rho - \xi)u} - 1]e^{-\rho u}\,\omega(u)\,du \\
&= \frac{\lambda}{c(\rho - \xi)}\int_0^\infty\int_0^\infty (e^{-\xi u} - e^{-\rho u})\,w(u, y)\,p(u + y)\,du\,dy.
\end{aligned} \tag{2.60}$$

(ix) Consider the special case with the penalty function $w(x, y) \equiv 1$. Then (2.60) becomes

$$\begin{aligned}
\hat{h}(\xi) &= \frac{\lambda}{c(\rho - \xi)}\int_0^\infty\int_0^\infty (e^{-\xi u} - e^{-\rho u})\,p(u + y)\,du\,dy \\
&= \frac{\lambda}{c(\rho - \xi)}\int_0^\infty (e^{-\xi u} - e^{-\rho u})\,[1 - P(u)]\,du \\
&= \frac{1}{c(\rho - \xi)}\left\{\lambda\int_0^\infty e^{-\xi u}\,[1 - P(u)]\,du + \frac{\delta}{\rho} - c\right\},
\end{aligned} \tag{2.61}$$

because ρ satisfies (2.47). Applying (2.45) yields

$$\hat{h}(\xi) = \frac{1}{c(\rho - \xi)}\left\{\frac{\lambda}{\xi}[1 - \hat{p}(\xi)] + \frac{\delta}{\rho} - c\right\}. \tag{2.62}$$

Hence, with $w(x, y) \equiv 1$,

$$\begin{aligned}
\hat{\phi}(\xi) &= \frac{\hat{h}(\xi)}{1 - \hat{g}(\xi)} \\
&= \frac{\lambda\rho[1 - \hat{p}(\xi)] + \xi(\delta - c\rho)}{\xi\rho\{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi\}}
\end{aligned} \tag{2.63}$$

by (2.62) and (2.56). In deriving (2.63) it is assumed that δ, and hence ρ, are positive. The case where $\delta = \rho = 0$, and hence $\phi = \psi$, is best treated as a limiting case: From (2.63) and (2.48) we obtain

$$\hat{\psi}(\xi) = \frac{\lambda[1 - p_1\xi - \hat{p}(\xi)]}{\xi\{\lambda[1 - \hat{p}(\xi)] - c\xi\}}, \tag{2.64}$$

which can be reconciled with (12.6.9) in *Actuarial Mathematics* [4] by the formula

$$\int_0^\infty e^{-\xi u}\,\psi'(u)\,du = -\psi(0) + \xi\,\hat{\psi}(\xi). \tag{2.65}$$

3. Finite-Time Ruin Probability

Adopting the notation in *Actuarial Mathematics* [4, (12.1.4)], we let

$$\psi(u, t) = \Pr[T \le t \mid U(0) = u] \tag{3.1}$$

be the probability of ruin by time t, $t \ge 0$. Then

$$\psi(u, t) = \int_0^t \Big[\int_0^\infty\int_0^\infty f(x, y, s \mid u)\, dxdy\Big] ds, \tag{3.2}$$

or

$$\frac{\partial}{\partial t}\psi(u, t) = \int_0^\infty\int_0^\infty f(x, y, t \mid u)\, dxdy. \tag{3.3}$$

Hence

$$\begin{aligned} E[e^{-\delta T}\, I(T < \infty) \mid U(0) = u] &= \int_0^\infty e^{-\delta t} \frac{\partial}{\partial t}\psi(u, t)\, dt \\ &= \delta\int_0^\infty e^{-\delta t}\, \psi(u, t)\, dt \end{aligned} \tag{3.4}$$

by an integration by parts. Formula (3.4) shows how the Laplace transform of the defective distribution of the time of ruin can be expressed in terms of the single Laplace transform of the finite-time ruin function. Because the left-hand side of (3.4) is $\phi(u)$ with $w(x, y) \equiv 1$, the double Laplace transform of the finite-time ruin function is

$$\begin{aligned} \hat{\psi}(\xi, \delta) &= \int_0^\infty\int_0^\infty e^{-\xi u - \delta t}\, \psi(u, t)\, dt\, du \\ &= \frac{1}{\delta}\int_0^\infty e^{-\xi u}\, \phi(u)\, du \\ &= \frac{1}{\delta}\hat{\phi}(\xi) \\ &= \frac{\lambda p[1 - \hat{p}(\xi)] + \xi(\delta - cp)}{\delta\xi p\{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi\}}, \qquad \xi > 0, \delta > 0, \end{aligned} \tag{3.5}$$

by (2.63).

Historically, some actuarial researchers have preferred to study the *survival* function

$$\sigma(u, t) = 1 - \psi(u, t). \tag{3.6}$$

Its double Laplace transform is

$$\begin{aligned} \hat{\sigma}(\xi, \delta) &= \int_0^\infty\int_0^\infty e^{-\xi u - \delta t}\, \sigma(u, t)\, dt\, du \\ &= \frac{1}{\xi\delta} - \hat{\psi}(\xi, \delta) \end{aligned}$$

$$= \frac{\rho - \xi}{\xi\rho\{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi\}}$$

$$= \frac{\frac{1}{\xi} - \frac{1}{\rho}}{\lambda[1 - \hat{p}(\xi)] + \delta - c\xi}, \qquad \xi > 0, \delta > 0, \tag{3.7}$$

which is the same as Seal [34, (4.26)] and Panjer and Willmot [27, Theorem 11.7.4]. Through an integration by parts, (3.7) is equivalent to Beekman [3, p. 40, Corollary 1]. Note that the denominator on the right-hand side of (3.7) is the difference of the two sides in Lundberg's fundamental equation (2.21).

A particularly elegant result is the Laplace transform of the survival function with zero initial surplus, $\sigma(0, t)$, which can be readily obtained with the *Initial Value Theorem* of Laplace Transforms (Spiegel [37, p. 5, Theorem 1-16]). The theorem states that, for a sufficiently regular function f,

$$\lim_{u \to 0} f(u) = \lim_{\xi \to \infty} \xi \hat{f}(\xi). \tag{3.8}$$

Applying it to (3.7) yields

$$\int_0^\infty e^{-\delta t} \sigma(0, t)\, dt = \lim_{\xi \to \infty} \xi \hat{\sigma}(\xi, \delta)$$

$$= \frac{1}{c\rho}, \qquad \delta > 0. \tag{3.9}$$

Formula (3.9), which is the same as Seal [34, (4.25)] and Panjer and Willmot [27, Theorem 11.7.2], may also be derived using (4.9) below. An explicit formula for $\sigma(0, t)$ is given by (8.23).

Most of the results in this section can be found in Arfwedson [1]. However, Seal [34, p. 111] points out that they are "implicit in Segerdahl's (1939) work on the moments of the time to ruin." To conclude this section, let us suggest an analogy with Life Contingencies which may be useful to some readers. Interpret δ as a force of interest. Identify T, the time of ruin, with T(x), the remaining time till death for a person now aged x. Then $\psi(u, t)$ and $\sigma(u, t)$ correspond to the probabilites ${}_tq_x$ and ${}_tp_x$, respectively. With $w \equiv 1$, $\phi(u)$ corresponds to $\bar{A}_x$, while $\int_0^\infty e^{-\delta t} \sigma(u, t)dt$ corresponds to $\bar{a}_x$.

4. Zero initial surplus

In this section we study functions such as $f(x \mid 0)$ [defined by (2.42)] and $f(x, y \mid 0)$ [defined by (2.9)]. With initial surplus $U(0) = u = 0$, some very explicit results can be obtained. Since ϕ satisfies the renewal equation (2.36), it follows that

$$\phi(0) = h(0). \tag{4.1}$$

Applying (2.11) and (2.38) to (4.1) yields

$$\int_0^\infty\int_0^\infty w(x, y)\, f(x, y \mid 0)\, dx\, dy = \frac{\lambda}{c}\int_0^\infty\int_0^\infty e^{-\rho x}\, w(x, y)\, p(x + y)\, dx\, dy. \tag{4.2}$$

Because the function w is arbitrary, it follows that

$$f(x, y \mid 0) = \frac{\lambda}{c}\, e^{-\rho x}\, p(x + y), \qquad x > 0, y > 0. \tag{4.3}$$

This formula plays a central role; an alternative proof and additional insight will be given in Section 8. Some immediate consequences can be obtained by integrating over x, y, and both:

$$\begin{aligned}\int_0^\infty f(x, y \mid 0)\, dx &= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}\, p(x + y)\, dx \\ &= g(y),\end{aligned} \tag{4.4}$$

as defined by (2.33);

$$\begin{aligned}f(x \mid 0) &= \int_0^\infty f(x, y \mid 0)\, dy \\ &= \frac{\lambda}{c} e^{-\rho x}\int_0^\infty p(x + y)\, dy \\ &= \frac{\lambda}{c} e^{-\rho x}[1 - P(x)];\end{aligned} \tag{4.5}$$

$$\begin{aligned}E[e^{-\delta T}\, I(T < \infty) \mid U(0) = 0] &= \int_0^\infty\int_0^\infty f(x, y \mid 0)\, dy\, dx \\ &= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}[1 - P(x)]\, dx.\end{aligned} \tag{4.6}$$

As a check, note that (4.3) and (4.5) satisfy (2.43) with $u = 0$.

With $\delta = 0$, and hence $\rho = 0$, (4.3) reduces to a result of Dufresne and Gerber [13, (10)]. Here

$$f(x, y \mid 0) = f(y, x \mid 0). \tag{4.7}$$

Dickson [7] has pointed out that this symmetry can be explained in terms of "duality." Figures 3.a and 3.b illustrate the duality. Further discussion can be found in Dickson and

Egídio dos Reis [9], and in Section 9 below. For $\delta > 0$, formula (4.7) does not hold any longer.

For $\delta = 0$, (4.6) reduces to the famous formula

$$\psi(0) = \frac{\lambda}{c}\int_0^\infty [1 - P(x)]\,dx = \frac{\lambda p_1}{c}, \tag{4.8}$$

which, of course, can also be derived from (2.25). For $\delta > 0$, we can use (4.6) and the fact that ρ is a solution of (2.47) to see that

$$E[e^{-\delta T} \mid U(0) = 0] = E[e^{-\delta T} I(T < \infty) \mid U(0) = 0] = 1 - \frac{\delta}{c\rho}. \tag{4.9}$$

This result is equivalent to (3.9). Formula (4.8) can be obtained as a limiting case of (4.9) because of (2.48).

Example Let us look at the case of an exponential individual claim amount distribution,

$$p(x) = \beta e^{-\beta x}, \qquad x \geq 0, \tag{4.10}$$

with $\beta > 0$ and $c > \lambda p_1 = \frac{\lambda}{\beta}$. The number ρ is ξ_1, the nonnegative solution of (2.21), which is

$$\delta + \lambda - c\xi = \frac{\lambda\beta}{\beta + \xi},$$

or

$$c\xi^2 + (c\beta - \delta - \lambda)\xi - \beta\delta = 0. \tag{4.11}$$

Hence

$$\rho = \xi_1 = \frac{\lambda + \delta - c\beta + \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}{2c}. \tag{4.12}$$

(Note that, if $\delta = 0$, then $\rho = \xi_1 = 0$.) Then

$$f(x, y \mid 0) = \frac{\lambda\beta}{c} e^{-(\rho + \beta)x - \beta y} = \frac{\lambda}{c} e^{-(\rho + \beta)x} p(y);$$

$$g(y) = \frac{\lambda\beta}{c(\beta + \rho)} e^{-\beta y} = \frac{\lambda}{c(\beta + \rho)} p(y); \tag{4.13}$$

$$f(x \mid 0) = \frac{\lambda}{c} e^{-(\rho + \beta)x};$$

$$E[e^{-\delta T} I(T < \infty) \mid U(0) = 0] = \int_0^\infty \int_0^\infty f(x, y \mid 0)\, dy\, dx$$

$$= \frac{\lambda}{c(\beta + \rho)} \tag{4.14}$$

$$= \frac{2\lambda}{c\beta + \delta + \lambda + \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}. \tag{4.15}$$

An alternative to (4.14) and (4.15) is formula (4.9), which is simple and general at the same time. In Section 7 we shall show that

$$E[e^{-\delta T} I(T < \infty) \mid U(0) = u] = E[e^{-\delta T} I(T < \infty) \mid U(0) = 0]\, e^{\xi_2 u}, \tag{4.16}$$

where ξ_2 is the negative root of (4.11); see (8.33) and (8.38). Hence it follows from (3.4), (4.16) and (4.14) that

$$\int_0^\infty e^{-\delta t}\, \psi(u, t)\, dt = \frac{\lambda}{\delta c(\beta + \rho)}\, e^{\xi_2 u}. \tag{4.17}$$

On the other hand, using (4.9) instead of (4.14) yields

$$\int_0^\infty e^{-\delta t}\, \psi(u, t)\, dt = \left(\frac{1}{\delta} - \frac{1}{c\rho}\right) e^{\xi_2 u}. \tag{4.18}$$

Finally, we note that (2.43) can be simplified to

$$f(x, y \mid u) = f(x \mid u)\, p(y), \qquad u \geq 0,\ x > 0,\ y > 0. \tag{4.19}$$

5. Positive Initial Surplus

Results concerning "ruin" for zero initial surplus can be translated into results that are related to the event that the surplus falls below the initial level in the more general situation where the initial surplus is positive. We can use (4.3) and (4.4) to derive the renewal equation (2.36) by probabilistic reasoning. We condition on the first time when the surplus falls below the initial level. For given initial surplus $U(0) = u \geq 0$, the probability that this event occurs between time t and time t + dt, with

$$u + x \leq U(t\text{-}) \leq u + x + dx$$

and

$$u - y - dy \leq U(t) \leq u - y,$$

is

$$f(x, y, t \mid 0)\, dx\, dy\, dt. \tag{5.1}$$

Furthermore, the occurrence

$$y > u$$

means that ruin also takes place with this claim. Thus

$$\begin{aligned}\phi(u) &= \int_0^u\int_0^\infty\int_0^\infty e^{-\delta t}\,\phi(u-y)\,f(x, y, t\,|\,0)\,dt\,dx\,dy \\ &+ \int_u^\infty\int_0^\infty\int_0^\infty e^{-\delta t}\,w(x+u, y-u)\,f(x, y, t\,|\,0)\,dt\,dx\,dy \\ &= \int_0^u\int_0^\infty \phi(u-y)\,f(x, y\,|\,0)\,dx\,dy + \int_u^\infty\int_0^\infty w(x+u, y-u)\,f(x, y\,|\,0)\,dx\,dy. \qquad (5.2)\end{aligned}$$

Applying (4.4) and (4.3) to the right-hand side of (5.2) yields

$$\begin{aligned}\phi(u) &= \int_0^u \phi(u-y)g(y)dy + \frac{\lambda}{c}\int_u^\infty\int_0^\infty w(x+u, y-u)e^{-\rho x}p(x+y)dxdy \\ &= (\phi * g)(u) + \frac{\lambda}{c}\int_0^\infty\int_0^\infty w(x+u, s)e^{-\rho x}p(x+u+s)dxds \\ &= (\phi * g)(u) + h(u)\end{aligned}$$

by (2.39). This is the probabilistic proof of (2.36).

Let δ be considered as a force of interest. Then $\phi(u)$ is the expectation of the discounted penalty. From the calculations above we see that h(u) is the expectation of the discounted penalty if ruin occurs at the first time when the surplus falls below the initial level u. Because

$$g(y) = \int_0^\infty\int_0^\infty e^{-\delta t}\,f(x, y, t\,|\,0)\,dx\,dt, \qquad (5.3)$$

the differential g(y)dy is the discounted probability that the surplus will ever fall below its initial level u and will be between u – y and u – y – dy when it happens for the first time. The two terms on the right-hand side of the renewal equation (2.36) correspond to whether or not ruin occurs at the first time when the surplus falls below the initial level u.

Formula (2.37), the representation of ϕ as a series of functions, has a natural probabilistic interpretation as follows. We observe that ruin occurs at a time when the surplus process $\{U(t)\}$ attains a *record low* (cf. Section 12.6 of *Actuarial Mathematics* [4]). For j = 1, 2, 3, ... , let τ_j denote the time of the j-th record low of the surplus process. Then

$$I(T < \infty) = \sum_{j=1}^{\infty} I(T = \tau_j), \qquad (5.4)$$

and hence

$$\phi(u) = E[e^{-\delta T} w(U(T\text{-}), |U(T)|) I(T < \infty) \mid U(0) = u]$$
$$= \sum_{j=1}^{\infty} E[e^{-\delta T} w(U(T\text{-}), |U(T)|) I(T = \tau_j) \mid U(0) = u], \quad u \geq 0. \tag{5.5}$$

It remains to show by mathematical induction that the j-th term in (5.5) is identical to the j-th term in (2.37). From the calculations in (5.2) and (5.3), we see that

$$h(u) = \int_u^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\delta t} w(x + u, y - u) f(x, y, t \mid 0)\, dt\, dx\, dy$$
$$= E[e^{-\delta T} w(U(T\text{-}), |U(T)|) I(T = \tau_1) \mid U(0) = u], \quad u \geq 0. \tag{5.6}$$

Thus (2.37) and (5.5) have the same first term. Let j be an integer with $j \geq 2$. Assume that it has been proved that (2.37) and (5.5) have the same (j − 1)-th term,

$$E[e^{-\delta T} w(U(T\text{-}), |U(T)|) I(T = \tau_{j-1}) \mid U(0) = u] = (\underbrace{g*g* \ldots *g*g}_{j-2} *h)(u). \tag{5.7}$$

We are to show that (2.37) and (5.5) have the same j-th term. Because $\tau_j - \tau_1$ has the same distribution as τ_{j-1}, we have

$$E[e^{-\delta T} w(U(T\text{-}), |U(T)|) I(T = \tau_j) \mid U(0) = u]$$
$$= E[e^{-\delta \tau_j} w(U(T\text{-}), |U(T)|) I(T = \tau_j) \mid U(0) = u]$$
$$= E[e^{-\delta(\tau_1 + \tau_j - \tau_1)} w(U(T\text{-}), |U(T)|) I(T = \tau_j) \mid U(0) = u]$$
$$= [g*(\underbrace{g*g* \ldots *g*g}_{j-2} *h)](u)$$
$$= (\underbrace{g*g* \ldots *g*g}_{j-1} *h)(u), \tag{5.8}$$

which is the j-th term in (2.37). Hence (2.37) has a probabilistic interpretation.

If we consider $f(x \mid u)$ and $f(x, y \mid u)$ as functions of u, they satisfy renewal equations similar to (2.36). By distinguishing whether or not ruin occurs at the first time when the surplus falls below the initial value u, we see that

$$f(x, y \mid u) = \int_0^u f(x, y \mid u - z) g(z) dz + f(x - u, y + u \mid 0), \quad 0 \leq u < x, \tag{5.9}$$

and

$$f(x, y \mid u) = \int_0^u f(x, y \mid u - z) g(z) dz, \quad 0 < x \leq u. \tag{5.10}$$

By (4.3),

$$f(x - u, y + u \mid 0) = \frac{\lambda}{c} e^{-\rho(x-u)} p(x + y). \tag{5.11}$$

Hence, for $u \geq 0$, $x > 0$, $y > 0$,

$$f(x, y \mid u) = \int_0^u f(x, y \mid u - z)g(z)dz + \frac{\lambda}{c} e^{-\rho(x-u)} p(x + y)I(x > u). \tag{5.12}$$

From this and (2.43) it follows that, for $u \geq 0$, $x > 0$,

$$f(x \mid u) = \int_0^u f(x \mid u - z)g(z)dz + \frac{\lambda}{c} e^{-\rho(x-u)}[1 - P(x)]I(x > u). \tag{5.13}$$

We observe that, as a function of x, $f(x \mid u)$ has a discontinuity of amount

$$\frac{\lambda}{c}[1 - P(u)] \tag{5.14}$$

at $x = u$. Remarkably, it does not depend on δ.

Equations (5.12) and (5.13) can be viewed as special cases of (2.36). Let x_0 be a positive number. Consider $w(x, y)$ as the "generalized" density function with mass 1 for $x = x_0$ and 0 for $x \neq x_0$ (and independent of y). Then

$$\phi(u) = f(x_0 \mid u), \tag{5.15}$$

and, by (2.38),

$$\begin{aligned} h(u) &= \frac{\lambda}{c} \int_0^\infty \int_0^\infty I(x > u)\, e^{-\rho(x-u)}\, w(x, y)\, p(x + y)\, dy\, dx \\ &= \frac{\lambda}{c} e^{-\rho(x_0-u)}[1 - P(x_0)]\, I(x_0 > u) \\ &= f(x_0 \mid 0)\, e^{\rho u} I(x_0 > u). \end{aligned} \tag{5.16}$$

Hence (5.13) is a special case of (2.36).

From (5.13) it follows that

$$f(x \mid u) = h(u) + (g*h)(u) + (g*g*h)(u) + (g*g*g*h)(u) + \cdots, \tag{5.17}$$

with

$$h(u) = f(x \mid 0)\, e^{\rho u} I(x > u). \tag{5.18}$$

Note that this series has a probabilistic interpretation similar to (2.37): The j-th term represents the contribution to $f(x \mid u)$ of the event that $\tau_j = T$. With the definition

$$\eta(u) = e^{\rho u} I(x > u), \tag{5.19}$$

(5.18) becomes

$$h(u) = f(x \mid 0)\, \eta(u), \tag{5.20}$$

and we can rewrite (5.17) as

$$f(x \mid u) = f(x \mid 0)[\eta(u) + (g*\eta)(u) + (g*g*\eta)(u) + (g*g*g*\eta)(u) + \cdots]. \quad (5.21)$$

6. Key Renewal Theorem

Let f(x) and z(x) be two nonnegative functions on $[0, \infty)$. Consider the integral equation

$$Z(x) = (f*Z)(x) + z(x), \qquad x \geq 0, \quad (6.1)$$

which is a renewal equation for Z(x). The so-called *key renewal theorem,* originally formulated by Walter L. Smith, gives the asymptotic behavior of the solution of a renewal equation. It states that, if f is a proper probability density function, i.e.,

$$\int_0^\infty f(x)dx = 1, \quad (6.2)$$

and the function z is sufficiently regular, then the solution of the renewal equation (6.1) satisfies

$$\lim_{x \to \infty} Z(x) = \frac{\int_0^\infty z(y)\,dy}{\int_0^\infty y\,f(y)\,dy}. \quad (6.3)$$

In earlier days there was some confusion about this result because it was given under a variety of hypotheses. Finally, Feller [17, p. 362] clarified the situation with what he called the *direct Riemann integrability* condition. The condition requires that the function z be Riemann integrable and not oscillate too much in a neighborhood of infinity. This notion can be found in Norbert Wiener's work on Tauberian theorems. In our applications, this condition is always satisfied. A recent book with a long chapter on renewal theory is Resnick [32]. Several actuarial books ([5], [19], [34]) contain discussions on applying renewal theory to risk theory.

If the function f is not a proper probability density function, i.e., if

$$\hat{f}(0) = \int_0^\infty f(x)dx \neq 1, \quad (6.4)$$

then we try to find a real number R such that

$$\hat{f}(-R) = \int_0^\infty e^{Rx} f(x)dx = 1. \tag{6.5}$$

Applying (2.31) with $k = R$ to (6.1) yields a proper renewal equation for the function $e^{Rx}Z(x)$,

$$e^{Rx}Z = (e^{Rx}f)*(e^{Rx}Z) + e^{Rx}z. \tag{6.6}$$

The key renewal theorem is applicable to (6.6), yielding

$$\lim_{x\to\infty} e^{Rx}Z(x) = \frac{\int_0^\infty e^{Ry}\, z(y)\, dy}{\int_0^\infty y\, e^{Ry} f(y)\, dy} = \frac{\hat{z}(-R)}{-\hat{f}'(-R)}. \tag{6.7}$$

The number R satisfying (6.5) is unique, because

$$\frac{d}{d\xi}\hat{f}(-\xi) = \int_0^\infty e^{\xi x}\, x\, f(x)\, dx > 0.$$

If $\hat{f}(0) < 1$ (f a defective density), then $R > 0$; if $\hat{f}(0) > 1$ (f an excessive density), then $R < 0$. Let $f_1(x)$ and $f_2(x)$ be two functions; we write

$$f_1(x) \sim f_2(x) \qquad \text{for } x \to \infty \tag{6.8}$$

if

$$\lim_{x\to\infty} \frac{f_1(x)}{f_2(x)} = 1.$$

Then (6.7) can be restated as

$$Z(x) \sim \frac{\hat{z}(-R)}{-\hat{f}'(-R)}\, e^{-Rx} \qquad \text{for } x \to \infty. \tag{6.9}$$

Remarks (i) The reader who is familiar with complex analysis might recognize that the right-hand side of (6.9) can be interpreted as a *residue*. As pointed out in Remark (v) of Section 2, a renewal equation can solved by the method of Laplace transforms. From (6.1) we have

$$\hat{Z} = \hat{f}\hat{Z} + \hat{z}, \tag{6.10}$$

and hence

$$\hat{Z} = \frac{\hat{z}}{1 - \hat{f}}. \tag{6.11}$$

Consequently,

$$Z(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\hat{z}(\xi)}{1 - \hat{f}(\xi)} e^{\xi x}\, d\xi. \tag{6.12}$$

The residue of the integrand at the simple pole $\xi = -R$ is:

$$\lim_{\xi \to -R} (\xi - -R)\frac{\hat{z}(\xi)}{1 - \hat{f}(\xi)} e^{\xi x} = \frac{\hat{z}(-R)}{-\hat{f}'(-R)} e^{-Rx}. \tag{6.13}$$

(ii) With f being a proper density function, formula (6.3) can be obtained by applying the *Final Value Theorem* of Laplace Transforms (Spiegel [37, p. 6, Theorem 1-17]). The theorem states that, for a sufficiently regular function Z,

$$\lim_{x \to \infty} Z(x) = \lim_{\xi \to 0} \xi\, \hat{Z}(\xi). \tag{6.14}$$

Here

$$\lim_{\xi \to 0} \xi\, \hat{Z}(\xi) = \lim_{\xi \to 0} \xi \frac{\hat{z}(\xi)}{1 - \hat{f}(\xi)}$$
$$= \frac{\hat{z}(0)}{-\hat{f}'(0)},$$

because $\hat{f}(0) = 1$.

7. Large Initial Surplus

We now apply the key renewal theorem to find an asymptotic expression for ϕ satisfying (2.36),

$$\phi = \phi * g + h,$$

where g and h are defined by (2.33) and (2.34), respectively. Since $\rho \geq 0$, we have

$$\hat{g}(0) \leq \frac{\lambda}{c} p_1 < 1,$$

which means that (2.36) is a defective renewal equation. Thus we seek $R > 0$ such that

$$1 = \hat{g}(-R) = \int_0^\infty e^{Rx} g(x)\, dx, \tag{7.1}$$

which is equation (2.57). Hence

$$R = -\xi_2,$$

where ξ_2 is the negative root of Lundberg's fundamental equation (2.21). Note that both ρ (or ξ_1) and R (or $|\xi_2|$) are increasing functions of δ and do not depend on the penalty function w. When confusion may arise, we write $\rho(\delta)$ for ρ and $R(\delta)$ for R. We observe that $\rho(0) = 0$, and R(0) is the adjustment coefficient in classical risk theory.

It follows from the key renewal theorem that

$$\phi(u) \sim \frac{\hat{h}(-R)}{-\hat{g}'(-R)} e^{-Ru} \qquad \text{for } u \to \infty. \tag{7.2}$$

By (2.60),

$$\hat{h}(-R) = \frac{\lambda}{c(R+\rho)} \int_0^\infty \int_0^\infty (e^{Ru} - e^{-\rho u}) \, w(u, y) \, p(u+y) \, dy \, du. \tag{7.3}$$

By (2.59),

$$\hat{g}'(-R) = \frac{1}{\rho + R}\left[1 + \frac{\lambda}{c}\hat{p}'(-R)\right]. \tag{7.4}$$

Hence

$$\phi(u) \sim \frac{\lambda \displaystyle\int_0^\infty \int_0^\infty w(x, y) \, (e^{Rx} - e^{-\rho x}) \, p(x+y) \, dx \, dy}{-\lambda \hat{p}'(-R) - c} e^{-Ru} \qquad \text{for } u \to \infty. \tag{7.5}$$

Now, consider the special case where $w(x, y) \equiv 1$ and $\delta = 0$. Then $\phi = \psi$, and the renewal equation (2.36) is

$$\psi = \psi * g + h, \tag{7.6}$$

with

$$g(x) = \frac{\lambda}{c}[1 - P(x)] \tag{7.7}$$

and

$$\begin{aligned} h(x) &= \frac{\lambda}{c}\int_x^\infty [1 - P(y)] dy \\ &= \int_x^\infty g(y) \, dy. \end{aligned} \tag{7.8}$$

Equation (7.6) is the same as Exercise 12.11 in *Actuarial Mathematics* [4]. Because R is the solution of (2.49), we have

$$\begin{aligned} \int_0^\infty \int_0^\infty (e^{Rx} - 1) \, p(x+y) \, dx \, dy &= \int_0^\infty (e^{Rx} - 1) \, [1 - P(x)] \, dx \\ &= \frac{c}{\lambda} - \int_0^\infty [1 - P(x)] \, dx \\ &= \frac{c}{\lambda} - p_1. \end{aligned} \tag{7.9}$$

Hence (7.5) simplifies as

$$\psi(u) \sim \frac{c - \lambda p_1}{-\lambda\hat{p}'(-R) - c} e^{-Ru} \qquad \text{for } u \to \infty. \tag{7.10}$$

Seal [34, p. 131] points out that the asymptotic formula (7.10) was first published by the Swedish actuary F. Lundberg in 1926; the special case of (7.10) for constant $\{X_j\}$ was derived by A.K. Erlang in 1909 in the context of telephone calls.

We can also obtain an asymptotic formula for $\int_0^\infty e^{-\delta t}\,\psi(u, t)\,dt$, $\delta > 0$. By (3.4),

$$\int_0^\infty e^{-\delta t}\,\psi(u, t)\,dt = \frac{\phi(u)}{\delta}, \tag{7.11}$$

where $w(x, y) \equiv 1$. It follows from (2.62) and (2.46) that

$$\begin{aligned}\hat{h}(-R) &= \frac{1}{c(R+\rho)}\left\{\frac{\delta}{R} + c + \frac{\delta}{\rho} - c\right\}.\\ &= \frac{\delta}{c(R+\rho)}\left(\frac{1}{R} + \frac{1}{\rho}\right).\end{aligned} \tag{7.12}$$

Hence

$$\phi(u) \sim \frac{\delta}{-\lambda\hat{p}'(-R) - c}\left(\frac{1}{R} + \frac{1}{\rho}\right) e^{-Ru} \quad \text{for } u \to \infty \tag{7.13}$$

[which can also be derived by applying (2.47) to (7.5)]. Substituting (7.13) in (7.11) yields

$$\int_0^\infty e^{-\delta t}\,\psi(u, t)\,dt \sim \frac{1}{-\lambda\hat{p}'(-R) - c}\left(\frac{1}{R} + \frac{1}{\rho}\right) e^{-Ru} \quad \text{for } u \to \infty. \tag{7.14}$$

Lundberg's asymptotic formula (7.10) is, of course, a special case of (7.13):

$$\begin{aligned}\psi(u) &= \lim_{\delta \to 0} \phi(u)\\ &\sim \lim_{\delta \to 0} \frac{1}{-\lambda\hat{p}'(-R(\delta)) - c}\left[\frac{\delta}{R(\delta)} + \frac{\delta}{\rho(\delta)}\right] e^{-R(\delta)u}, \quad \text{for } u \to \infty,\end{aligned} \tag{7.15}$$

which, by (2.48), is (7.10). A more interesting way to retrieve (7.10) is by applying the Final Value Theorem (6.14):

$$\begin{aligned}\psi(u) &= \lim_{t \to \infty} \psi(u, t)\\ &= \lim_{\delta \to 0} \delta \int_0^\infty e^{-\delta t}\,\psi(u, t)\,dt,\end{aligned}$$

which, via (7.14), is (7.15) again.

Since $w(x, y)$ can be an arbitrary function, comparing (2.11) with (7.5) yields the asymptotic formula

$$f(x, y \mid u) \sim \frac{\lambda(e^{Rx} - e^{-\rho x})\, p(x+y)}{-\lambda\hat{p}'(-R) - c} e^{-Ru} \qquad \text{for } u \to \infty, \tag{7.16}$$

which generalizes Dufresne and Gerber [13, (24)]. Because (5.13) is a renewal equation for $f(x, y \mid u)$ (as a function of u), it can also be used to derive (7.16); here

$$h(u) = f(x, y \mid 0)\, e^{\rho u} I(x > u), \tag{7.17}$$

and

$$\begin{aligned} \hat{h}(-R) &= f(x, y \mid 0) \int_0^\infty e^{Ru} e^{\rho u} I(x > u)\, du \\ &= f(x, y \mid 0) \frac{e^{(R+\rho)x} - 1}{R + \rho} \\ &= \frac{\lambda(e^{Rx} - e^{-\rho x}) p(x+y)}{c(R + \rho)}. \end{aligned} \tag{7.18}$$

Example For the exponential individual claim example in Section 3, we have

$$-R = \xi_2 = \frac{\lambda + \delta - c\beta - \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}{2c}, \tag{7.19}$$

and the adjustment coefficient is

$$R(0) = \frac{c\beta - \lambda}{c}. \tag{7.20}$$

From (4.13)

$$\hat{g}(\xi) = \frac{\lambda\beta}{c(\beta + \rho)(\beta + \xi)}, \tag{7.21}$$

and hence

$$-\hat{g}'(-R) = \frac{\lambda\beta}{c(\beta + \rho)(\beta - R)^2}. \tag{7.22}$$

Now, let us consider the particular case where $w(x, y) = w(y)$, a function not depending on x. Then

$$\begin{aligned} \omega(x) &= \int_0^\infty w(y)\, p(x+y)\, dy \\ &= \beta e^{-\beta x} \int_0^\infty w(y)\, e^{-\beta y}\, dy \\ &= \beta e^{-\beta x}\, \hat{w}(\beta), \end{aligned} \tag{7.23}$$

and

$$
\begin{aligned}
h(x) &= \frac{\lambda}{c}\int_0^\infty e^{-\rho z}\,\omega(u+z)\,dz \\
&= \frac{\lambda}{c}\beta\hat{w}(\beta)\int_0^\infty e^{-\rho z}e^{-\beta(x+z)}\,dz \\
&= \frac{\lambda\,\beta\,\hat{w}(\beta)\,e^{-\beta x}}{c(\beta+\rho)}.
\end{aligned} \tag{7.24}
$$

Hence

$$
\hat{h}(\xi) = \frac{\lambda\,\beta\,\hat{w}(\beta)}{c(\beta+\rho)(\beta+\xi)}. \tag{7.25}
$$

It follows from (7.25) and (7.22) that

$$
\frac{\hat{h}(-R)}{-\hat{g}'(-R)} = \hat{w}(\beta)\,(\beta - R). \tag{7.26}
$$

Thus, with $w(x, y) = w(y)$, and $P(x) = 1 - e^{-\beta x}$,

$$
\phi(u) \sim \hat{w}(\beta)\,(\beta - R)\,e^{-Ru} \qquad \text{for } u \to \infty. \tag{7.27}
$$

In the next section, we shall see that (7.27) is in fact an equality valid for all $u \geq 0$.

Furthermore, (7.16) is

$$
f(x, y \mid u) \sim \frac{\lambda(e^{Rx} - e^{-\rho x})\,\beta e^{-\beta(x+y)}}{\lambda\beta(\beta - R)^{-2} - c}\,e^{-Ru} \qquad \text{for } u \to \infty. \tag{7.28}
$$

Because ρ and $-R$ are the roots of (4.11), we have

$$
\begin{aligned}
c(\beta+\rho)(\beta - R) &= c(-\beta)^2 + (c\beta - \delta - \lambda)(-\beta) - \beta\delta \\
&= \lambda\beta.
\end{aligned} \tag{7.29}
$$

It follows from (7.29) and some algebra that (7.28) can be rewritten as

$$
f(x, y \mid u) \sim \frac{\lambda\beta(\beta - R)}{c(R + \rho)}\,[e^{Rx} - e^{-\rho x}]\,e^{-\beta(x+y)}e^{-Ru} \qquad \text{for } u \to \infty. \tag{7.30}
$$

Applying (4.19) to (7.30) yields

$$
f(x \mid u) \sim \frac{\lambda(\beta - R)}{c(R + \rho)}\,[e^{Rx} - e^{-\rho x}]\,e^{-\beta x}e^{-Ru} \qquad \text{for } u \to \infty. \tag{7.31}
$$

It turns out that (7.30) and (7.31) are exact for $0 < x \leq u$; see (9.40) below. ||||

To conclude this section we look at the Laplace transform of T, given that ruin occurs:

$$E[e^{-\delta T} \mid T < \infty, U(0) = u] = \frac{E[e^{-\delta T} I(T < \infty) \mid U(0) = u]}{E[I(T < \infty) \mid U(0) = u]}$$

$$= \frac{\phi(u)}{\psi(u)}, \qquad (7.32)$$

where $w \equiv 1$. Consider $\frac{\hat{h}(-R)}{-\hat{g}'(-R)}$ as a function of δ, and write it as $C(\delta)$. It follows from (7.2) that, for $u \to \infty$,

$$\frac{\phi(u)}{\psi(u)} \sim \frac{C(\delta)e^{-R(\delta)u}}{C(0)e^{-R(0)u}} = \frac{C(\delta)}{C(0)}e^{-[R(\delta) - R(0)]u}. \qquad (7.33)$$

If $\delta > 0$, then $R(\delta) > R(0)$, and hence

$$\lim_{u \to \infty} e^{-[R(\delta) - R(0)]u} = 0,$$

which means that

$$\lim_{u \to \infty} E[e^{-\delta T} \mid T < \infty, U(0) = u] = 0. \qquad (7.34)$$

Thus, for each $t > 0$,

$$\lim_{u \to \infty} \Pr[T \leq t \mid T < \infty, U(0) = u] = 0, \qquad (7.35)$$

which implies that, for a large initial surplus u and given that ruin occurs, it occurs late. The result (7.35) is compatible with the observation that the conditional expectation

$$E[T \mid T < \infty, U(0) = u]$$

is essentially a linear function in u in some cases; see (8.43) below, Gerber [19, p. 138, Example 3.2], and Seal [34, p. 114].

8. Martingales

Further insight can be provided to the reader who has some familiarity with *martingales*. Let ξ be a number. Because $\{U(t)\}_{t\geq 0}$ is a stochastic process with stationary and independent increments, a process of the form

$$\{e^{-\delta t + \xi U(t)}\}_{t\geq 0} \qquad (8.1)$$

is a martingale if and only if, for each $t > 0$, its expectation at time t is equal to its initial value, i.e., if and only if

$$E[e^{-\delta t + \xi U(t)} \mid U(0) = u] = e^{-\delta 0 + \xi u}$$

$$= e^{\xi u}. \qquad (8.2)$$

Since

$$E[e^{-\delta t + \xi U(t)} \mid U(0) = u] = \exp(-\delta t + \xi u + \xi c t + \lambda t[\hat{p}(\xi) - 1]),$$

the martingale condition is that

$$0 = -\delta + c\xi + \lambda[\hat{p}(\xi) - 1],$$

which is (2.21), Lundberg's fundamental equation. Thus, for (8.1) to be a martingale, the coefficient of U(t) in (8.1) is either $\xi_1 = \rho \geq 0$ or $\xi_2 = -R < 0$.

With such a ξ, (8.2) holds for each fixed t, $t \geq 0$. However, if we replace t by a stopping time which is a random variable, then there is no guarantee that (8.2) will hold. Fortunately, it holds in two important cases, as we shall see in this and the next paragraph. If the stopping time is T, the time of ruin, the *optional sampling theorem* is applicable to the martingale with $\xi = -R$. For $0 \leq t < T$,

$$\delta t + RU(t) \geq 0,$$

and hence

$$0 < e^{-\delta t - RU(t)} \leq 1.$$

With $\{e^{-\delta t - RU(t)};\ t < T\}$ being bounded, the optional sampling theorem is applicable and we obtain

$$E[e^{-\delta T - RU(T)} \mid U(0) = u] = e^{-Ru}. \tag{8.3}$$

Furthermore, it follows from (2.6) that, even if $\delta = 0$,

$$E[e^{-\delta T - RU(T)} I(T = \infty) \mid U(0) = u] = 0.$$

Consequently, we can rewrite (8.3) as

$$e^{-Ru} = E[e^{-\delta T - RU(T)} I(T < \infty) \mid U(0) = u], \qquad \delta \geq 0, u \geq 0. \tag{8.4}$$

The above is a proof by martingale theory of a generalization of Theorem 12.1 in *Actuarial Mathematics* [4].

We now show that the quantity $e^{-\rho(x-u)}$, which appears throughout this paper (usually with u = 0), has a probabilistic interpretation. For $x > U(0) = u$, let

$$T_x = \min\{t \mid U(t) = x\} \tag{8.5}$$

be the first time when the surplus reaches the level x. We can use equality to define the stopping time T_x because the process $\{U(t)\}$ is skip-free (jump-free) upward. Then, for $0 \le t \le T_x$,

$$e^{-\delta t + \rho U(t)} \le e^{\rho x}. \tag{8.6}$$

Hence we can apply the optional sampling theorem to the martingale $\{e^{-\delta t + \rho U(t)}\}$ to obtain

$$\begin{aligned} e^{-\delta 0 + \rho u} &= E[e^{-\delta T_x + \rho U(T_x)} \mid U(0) = u] \\ &= E[e^{-\delta T_x} \mid U(0) = u]\, e^{\rho x}, \end{aligned}$$

or

$$e^{-\rho(x-u)} = E[e^{-\delta T_x} \mid U(0) = u]. \tag{8.7}$$

With δ interpreted as a force of interest, the quantity $e^{-\rho(x-u)}$ is the expected discounted value of a payment of 1 due at the time when $U(t) = x$ for the first time. We note that (8.7) remains valid even if u is negative. The required condition is $x > u$; the condition $u \ge 0$ is not needed anywhere in the derivation. Formula (8.7) was probably first given by Kendall [25, (14)], although he did not provide a complete proof. It can also be found in Cox and Miller [6, p. 245, (184)], Gerber [21, (11)], Prabhu ([30], [31, p. 79, Theorem 5(i); p. 105, #4]), and Takács [38, p. 88, Theorem 8].

Formula (8.7) can be used to give an alternative proof of the important formula (4.3). For $x > u = U(0)$, let $\pi_1(x, t \mid u)$, $t > 0$, denote the probability density function of the random variable T_x. Hence (8.7) is

$$\int_0^\infty e^{-\delta t}\, \pi_1(x, t \mid u)\, dt = e^{-\rho(x-u)}. \tag{8.8}$$

The differential $\pi_1(x, t \mid u)dt$ is the probability that the surplus process upcrosses level x between t and t+dt and that then this happens for the first time. For $U(0) = u \ge 0$, $x > 0$, let $\pi_2(x, t \mid u)$, $t > 0$, be the function defined by the condition that $\pi_2(x, t \mid u)dt$ is the probability that ruin does not occur by time t and that there is an upcrossing of the surplus process at level x between t and t+dt. It can be proved by duality, a notion to be discussed in the next section, that

$$\pi_1(x, t \mid 0) = \pi_2(x, t \mid 0), \qquad x > 0, t > 0. \tag{8.9}$$

Now, $f(x, y, t \mid u)dtdxdy$ can be interpreted as the probability of the event that "ruin" does not take place by time t, that the surplus process upcrosses through level x between time t and time t+dt, but does not attain level x+dx, i.e., that there is a claim within $\frac{dx}{c}$ time units after T_x, and that the size of this claim is between x+y and x+y+dy. Thus

$$f(x, y, t \mid u)\, dt\, dx\, dy = [\pi_2(x, t \mid u)\, dt]\, [\lambda \tfrac{dx}{c}]\, [p(x + y)\, dy], \tag{8.10}$$

from which it follows that

$$f(x, y, t \mid u) = \tfrac{\lambda}{c}\, p(x + y)\, \pi_2(x, t \mid u). \tag{8.11}$$

This formula is particularly useful if $u = 0$: It follows from (8.9) that

$$f(x, y, t \mid 0) = \tfrac{\lambda}{c}\, p(x + y)\, \pi_1(x, t \mid 0). \tag{8.12}$$

If we multiply (8.12) by $e^{-\delta t}$, integrate from $t = 0$ to $t = \infty$, and apply (8.8) with $u = 0$, we obtain (4.3),

$$f(x, y \mid 0) = \tfrac{\lambda}{c}\, p(x + y)\, e^{-\rho x},$$

once again.

Remarks (i) For $x > u = U(0) \geq 0$, the functions $\pi_1(x, t \mid u)$ and $\pi_2(x, t \mid 0)$ can be expressed in terms of $\pi_3(x, t \mid u)$, the *passage time density* of the surplus process at the level x. The differential $\pi_3(x, t \mid u)dt$ is the probability that the surplus process upcrosses level x between t and t+dt. This is the same as the probability that the surplus at time t is between x–dx and x with $dx = c\, dt$. Hence, we have

$$\pi_3(x, t \mid u) = c\, f_{S(t)}(u + ct - x), \tag{8.13}$$

where

$$f_{S(t)}(s) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}\, p^{*n}(s) \tag{8.14}$$

is the probability density function of S(t), the aggregate claims up to time t. The following version of the *ballot theorem,*

$$\pi_1(x, t \mid 0) = \tfrac{x}{ct}\pi_3(x, t \mid 0), \qquad x > 0, t > 0, \tag{8.15}$$

was first given by Kendall [25, (17)]; see also Cox and Miller [6, p. 251, #12], Dinges [12], Keilson [24], Prabhu [31, p. 81, Theorem 6], Seal [35, p. 47] and Takács [38, p. 87, Theorem 6]. For $x > u$ and $t > 0$, because

$$\pi_1(x, t \mid u) = \pi_1(x - u, t \mid 0) \tag{8.16}$$

and

$$\pi_3(x, t \mid u) = \pi_3(x - u, t \mid 0), \tag{8.17}$$

we have

$$\pi_1(x, t \mid u) = \frac{x - u}{ct} \pi_3(x, t \mid u). \tag{8.18}$$

Gerber [20, Theorem 2] has given a proof of

$$\pi_1(x, t \mid 0) = \frac{x}{ct} \pi_3(x, t \mid 0) = \pi_2(x, t \mid 0) \tag{8.19}$$

by martingales. The second equality of (8.19) is equivalent to equation (2.1) on page 112 of Gerber [19].

(ii) The differential $\pi_2(x, t \mid u)dt$ can be interpreted as the probability that ruin does not occur by time t and that the surplus at time t is between x–dx and x, where $dx = c\,dt$. By (3.6),

$$\begin{aligned} \sigma(u, t) &= \Pr[T > t \mid U(0) = u] \\ &= \int_0^\infty \pi_2(x, t \mid u) \frac{dx}{c}. \end{aligned} \tag{8.20}$$

Hence, it follows from the second equality of (8.19) and from (8.13) that

$$\begin{aligned} \sigma(0, t) &= \int_0^\infty \frac{x}{ct} \pi_3(x, t \mid 0) \frac{dx}{c} \\ &= \frac{1}{ct} \int_0^\infty x\, f_{S(t)}(ct - x)\, dx, \end{aligned} \tag{8.21}$$

which is a result first given by Prabhu [30, (4.6)]; see also Seal [34, (4.8)]. Let

$$F_{S(t)}(s) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} P^{*n}(s) \tag{8.22}$$

be the probability distribution function of S(t). Then integrating the right-hand side of (8.21) by parts and noting that $F_{S(t)}(s) = 0$ for $s < 0$ yields

$$\sigma(0, t) = 0 - 0 + \frac{1}{ct}\int_0^\infty F_{S(t)}(ct - x)\,dx$$

$$= \frac{1}{ct}\int_0^{ct} F_{S(t)}(ct - x)\,dx$$

$$= \frac{1}{ct}\int_0^{ct} F_{S(t)}(s)\,ds, \tag{8.23}$$

which is the same as Gerber [19, p. 113, (2.4)], Panjer and Willmot [27, Theorem 11.7.3], and Seal [35, p. 48, (6)].

(iii) By (8.20), (8.9), and (8.8), the Laplace transform of the survival function $\sigma(0, t)$ is

$$\int_0^\infty e^{-\delta t}\,\sigma(0, t)\,dt = \int_0^\infty e^{-\delta t}\Big[\int_0^\infty \pi_2(x, t \mid 0)\,\frac{dx}{c}\Big]dt$$

$$= \frac{1}{c}\int_0^\infty \int_0^\infty e^{-\delta t}\,\pi_1(x, t \mid 0)\,dt\,dx$$

$$= \frac{1}{c}\int_0^\infty e^{-\rho x}\,dx$$

$$= \frac{1}{c\rho}, \tag{8.24}$$

which is the same as (3.9).

(iv) For $u \geq 0$, $t > 0$, consider the probability

$$\Pr[U(t) \geq 0 \mid U(0) = u] = F_{S(t)}(u + ct).$$

By conditioning on whether or not ruin occurs before time t and distinguishing according to the time τ when the surplus process upcrosses the level 0 for the last time, we have the following equation for the survival function $\sigma(u, t)$,

$$F_{S(t)}(u + ct) = \sigma(u, t) + \int_0^t \pi_3(0, \tau \mid u)\,\sigma(0, t-\tau)\,d\tau, \tag{8.25}$$

which, in the context of Risk Theory, was first given by Prabhu [30, (3.3)]. With (8.13), formula (8.25) is the same as Gerber [19, p. 114, (2.13)], Panjer and Willmot [27, Theorem 11.7.5], Seal [34, (4.16)] and Seal [35, p. 44, (1)]. As $t \to \infty$, (8.25) becomes

$$1 = \sigma(u, \infty) + \int_0^\infty \pi_3(0, \tau \mid u)\,\sigma(0, \infty)\,d\tau,$$

or

$$\psi(u) = [1 - \psi(0)]\int_0^\infty \pi_3(0, \tau \mid u)\, d\tau \tag{8.26}$$

$$= \frac{c - \lambda p_1}{c}\int_0^\infty \pi_3(0, \tau \mid u)\, d\tau, \qquad u \geq 0. \tag{8.27}$$

One may view formula (8.27) as another version of the ballot theorem. Seah [33, p. 426] has pointed out that (8.27) "is not practical for computing."

(v) It follows from (8.26) that, for $x \leq u = U(0)$,

$$\int_0^\infty \pi_3(x, \tau \mid u)\, d\tau = \int_0^\infty \pi_3(0, \tau \mid u - x)\, d\tau$$

$$= \frac{\psi(u - x)}{1 - \psi(0)}. \tag{8.28}$$

For the case $x > u = U(0)$, because $\{U(t)\}$ has a positive drift, the surplus will reach level x with probability 1. Hence, for $x > u$,

$$\int_0^\infty \pi_3(x, \tau \mid u)\, d\tau = 1 + \int_0^\infty \pi_3(x, \tau \mid x)\, d\tau \tag{8.29}$$

$$= 1 + \frac{\psi(x - x)}{1 - \psi(0)}$$

$$= \frac{1}{1 - \psi(0)} \tag{8.30}$$

by (8.28). Furthermore, (8.29) can be generalized as

$$\int_0^\infty \pi_3(x, \tau \mid u)\, e^{-\delta\tau} d\tau = e^{-\rho(x-u)}\Big[1 + \int_0^\infty \pi_3(x, \tau \mid x)\, e^{-\delta\tau} d\tau\Big]$$

$$= e^{-\rho(x-u)}\Big[1 + \int_0^\infty \pi_3(0, \tau \mid 0)\, e^{-\delta\tau} d\tau\Big], \quad x > u,\ \delta \geq 0. \tag{8.31}$$

Further results are given in Remark (vi) of Section 9.

Example Again, consider the case where the individual claim amount random variable has an exponential distribution, $p(x) = \beta e^{-\beta x}$. Then R is given by (7.19). Applying (4.19) to (8.4) yields

$$e^{-Ru} = \Big[\int_0^\infty e^{Ry} p(y) dy\Big]\, E[e^{-\delta T} I(T < \infty) \mid U(0) = u]$$

$$= \hat{p}(-R)\, E[e^{-\delta T} I(T < \infty) \mid U(0) = u]$$

$$= \frac{\beta}{\beta - R} E[e^{-\delta T} I(T < \infty) \mid U(0) = u]. \tag{8.32}$$

Hence

$$E[e^{-\delta T} I(T < \infty) \mid U(0) = u] = \frac{\beta - R}{\beta} e^{-Ru}. \tag{8.33}$$

Formula (8.33) should be compared with the first line of (12.3.8) in *Actuarial Mathematics* [4] which is only for $\delta = 0$. To reconcile (8.33) for $u = 0$ with (4.9) we need to show that, for $\delta > 0$,

$$\frac{R}{\beta} = \frac{\delta}{c\rho}. \tag{8.34}$$

Equation (8.34) holds because the product of the two roots of the quadratic equation (4.11) is $\frac{-\beta\delta}{c}$. As a further check, we want to see that (8.33) with $u = 0$ is consistent with (4.14); here we need the identity

$$\frac{\beta - R}{\beta} = \frac{\lambda}{c(\beta + \rho)}, \tag{8.35}$$

which is true because of (7.29). It follows from (4.13) and (8.35) that

$$g(y) = (\beta - R)e^{-\beta y} = \frac{p(y)}{\hat{p}(-R)}. \tag{8.36}$$

In the particular case where $w(x, y) = w(y)$, a function not depending on x, we can apply (4.19) and (8.33) to obtain an explicit expression for $\phi(u)$:

$$\begin{aligned} \phi(u) &= E[e^{-\delta T} w(|U(T)|)\, I(T < \infty) \mid U(0) = u] \\ &= \left[\int_0^\infty w(y)p(y)dy\right] E[e^{-\delta T} I(T < \infty) \mid U(0) = u] \\ &= \left[\int_0^\infty w(y)e^{-\beta y}dy\right](\beta - R)e^{-Ru} \\ &= \hat{w}(\beta)(\beta - R)e^{-Ru}. \end{aligned} \tag{8.37}$$

This shows that the asymptotic formula (7.27) is actually an exact formula, and

$$\phi(u) = \phi(0)e^{-Ru}. \tag{8.38}$$

Furthermore, for $\delta > 0$,

$$\begin{aligned} E[e^{-\delta T}w(|U(T)|) \mid T < \infty, U(0) = u] &= \frac{E[e^{-\delta T} w(|U(T)|)\, I(T < \infty) \mid U(0) = u]}{E[I(T < \infty) \mid U(0) = u]} \\ &= \frac{\phi(u)}{\psi(u)} \\ &= \frac{\hat{w}(\beta)[\beta - R(\delta)]}{\beta[\beta - R(0)]}\, e^{-[R(\delta) - R(0)]u}. \end{aligned} \tag{8.39}$$

Because of (8.35) [or (7.29)], we can rewrite (8.39) in terms of $\rho(\delta)$ and $\rho(0)$. Noting that $\rho(0) = 0$, we have

$$E[e^{-\delta T}w(|U(T)|) \mid T < \infty, U(0) = u] = \frac{\hat{w}(\beta)}{\beta + \rho(\delta)} \exp\left(\frac{\lambda}{c}\left[\frac{1}{1 + \rho(\delta)/\beta} - 1\right]u\right). \tag{8.40}$$

Differentiating (8.39) with respect to δ and then setting $\delta = 0$ yields

$$E[T\, w(|U(T)|) \mid T < \infty, U(0) = u] = \frac{\hat{w}(\beta)R'(0)}{\beta[\beta - R(0)]} + \frac{\hat{w}(\beta)R'(0)}{\beta}u, \tag{8.41}$$

which is a linear function in u. By (7.20)

$$\beta - R(0) = \frac{\lambda}{c}.$$

From (7.19)

$$R'(0) = \frac{\lambda}{c(c\beta - \lambda)}. \tag{8.42}$$

Hence

$$\begin{aligned} E[T\, w(|U(T)|) \mid T < \infty, U(0) = u] &= \frac{\hat{w}(\beta)R'(0)}{\beta}\Big[\frac{c}{\lambda} + u\Big] \\ &= \frac{\hat{w}(\beta)\,\lambda}{c\beta(c\beta - \lambda)}\Big[\frac{c}{\lambda} + u\Big]. \end{aligned} \tag{8.43}$$

9. Generalization of Dickson's Formula

In [7] Dickson gives the following result for the case $\delta = 0$:

$$f(x \mid u) = \begin{cases} f(x \mid 0)\dfrac{1 - \psi(u)}{1 - \psi(0)}, & x > u \geq 0, \quad (9.1) \\[2ex] f(x \mid 0)\dfrac{\psi(u - x) - \psi(u)}{1 - \psi(0)}, & 0 < x \leq u. \quad (9.2) \end{cases}$$

Here,

$$f(x \mid 0) = \frac{\lambda}{c}[1 - P(x)]; \tag{9.3}$$

see (4.5). The purpose of this section is to generalize (9.1) and (9.2) to the case where $\delta \geq 0$.

A first question is how to extend the definition of $\psi(u)$ for $\delta > 0$. It turns out that the appropriate definition is

$$\psi(u) = E[e^{-\delta T + \rho U(T)}\, I(T < \infty) \mid U(0) = u], \qquad u \geq 0. \tag{9.4}$$

Thus $\psi(u) = \phi(u)$ with $w(x, y) = e^{-\rho y}$; see (2.10). [Compare (9.4) with (8.4).] Then Dickson's formula can be generalized as

$$f(x \mid u) = \begin{cases} f(x \mid 0)\,\dfrac{e^{\rho u} - \psi(u)}{1 - \psi(0)}, & x > u \geq 0, \qquad (9.5) \\ f(x \mid 0)\,\dfrac{e^{\rho x}\psi(u-x) - \psi(u)}{1 - \psi(0)}, & 0 < x \leq u, \qquad (9.6) \end{cases}$$

with

$$f(x \mid 0) = \frac{\lambda}{c} e^{-\rho x}[1 - P(x)] \tag{9.7}$$

according to (4.5). Hence, as a function of x, $f(x \mid u)$ has a discontinuity of amount

$$f(u \mid 0)\, e^{\rho u} = \frac{\lambda}{c}[1 - P(u)] \tag{9.8}$$

at $x = u$. This is the same result as (5.14). [Compare formulas (9.5) and (9.6) with (5.21).]

To prove (9.5) and (9.6) we need some more concepts. We begin by extending the definition of the stopping time T_x as given by (8.5). For a real number x, we now let T_x denote the time of the first upcrossing of the surplus through the level x; we set $T_x = \infty$ if the surplus never upcrosses through the level x. For $x > U(0)$, this is the same as (8.5). For $x < U(0)$, the surplus has to drop below the level x before it can ever upcross through x. We call the stopping time T_0 the *time of recovery;* it is the first time when the surplus reaches zero after ruin. It follows from (8.7) that, for $a < b$,

$$E[e^{-\delta(T_b - T_a)} \mid T_a < T_b] = e^{-\rho(b-a)}. \tag{9.9}$$

Hence

$$\begin{aligned} E[e^{-\delta T_0}\, I(T < \infty) \mid U(0) = u] &= E[e^{-\delta T} e^{-\rho[0 - U(T)]}\, I(T < \infty) \mid U(0) = u] \\ &= \psi(u). \end{aligned} \tag{9.10}$$

This formula shows that $\psi(u)$ can be interpreted as the expected present value of a payment of 1 that is made at the time of recovery, if ruin takes place.

For $a \leq u < b$,

$$\Pr[T_a < \infty \mid U(0) = u] < 1 \tag{9.11}$$

and

$$\Pr[T_b < \infty \mid U(0) = u] = 1 \tag{9.12}$$

because the surplus process $\{U(t)\}$ has a positive drift. We define the stopping time

$$T_{a,b} = \min(T_a, T_b), \tag{9.13}$$

and consider the functions

$$A(a, b \mid u) = E[e^{-\delta T_{a,b}} I(U(T_{a,b}) = a) \mid U(0) = u]$$

$$= E[e^{-\delta T_a} I(T_a < T_b) \mid U(0) = u], \qquad (9.14)$$

and

$$B(a, b \mid u) = E[e^{-\delta T_{a,b}} I(U(T_{a,b}) = b) \mid U(0) = u]$$

$$= E[e^{-\delta T_b} I(T_a > T_b) \mid U(0) = u]. \qquad (9.15)$$

With δ interpreted as a force of interest, $A(a, b \mid u)$ is the expected present value of a payment of 1 which is made when the surplus upcrosses the level a for the first time, provided that the surplus has not reached the level b in the meantime. Similarly, $B(a, b \mid u)$ is the expected present value of a payment of 1 which is made when the surplus reaches the level b for the first time, provided that the surplus has not dropped below the level a in the meantime. Note that, for each constant k,

$$A(a, b \mid u) = A(a+k, b+k \mid u+k) \qquad (9.16)$$

and

$$B(a, b \mid u) = B(a+k, b+k \mid u+k). \qquad (9.17)$$

It follows from (9.10) that, for $u \geq a$,

$$A(a, \infty \mid u) = \lim_{b \to \infty} A(a, b \mid u)$$

$$= \lim_{b \to \infty} A(0, b - a \mid u - a)$$

$$= E[e^{-\delta T_0} I(T_0 < \infty) \mid U(0) = u - a]$$

$$= \psi(u - a). \qquad (9.18)$$

Similarly, it follows from (9.12) and (8.7) that, for $u < b$,

$$B(-\infty, b \mid u) = \lim_{a \to -\infty} B(a, b \mid u)$$

$$= e^{-\rho(b-u)}. \qquad (9.19)$$

Note that, with $\delta = 0$ and $0 \leq u < b$, $A(0, b \mid u)$ is the probability of ruin from an initial surplus u in the presence of an absorbing upper barrier at b.

For $a' < a \leq u < b < b'$, by considering whether $T_a < T_b$ or $T_a > T_b$, we obtain the identities

$$A(a, b' \mid u) = A(a, b \mid u) + B(a, b \mid u)A(a, b' \mid b) \tag{9.20}$$

and

$$B(a', b \mid u) = A(a, b \mid u)B(a', b \mid a) + B(a, b \mid u). \tag{9.21}$$

With $a = 0$, $b = x$, $b' = \infty$ and because of (9.18), (9.20) becomes

$$\psi(u) = A(0, x \mid u) + B(0, x \mid u)\psi(x). \tag{9.22}$$

With $a' = -\infty$, $a = 0$, $b = x$, $b' = \infty$ and because of (9.19), (9.21) becomes

$$e^{-\rho(x-u)} = A(0, x \mid u)e^{-\rho x} + B(0, x \mid u). \tag{9.23}$$

For $0 \le u < x$, formulas (9.22) and (9.23) are two linear equations for $A(0, x \mid u)$ and $B(0, x \mid u)$; their solution is

$$A(0, x \mid u) = \frac{e^{\rho x}\psi(u) - e^{\rho u}\psi(x)}{e^{\rho x} - \psi(x)} \tag{9.24}$$

and

$$B(0, x \mid u) = \frac{e^{\rho u} - \psi(u)}{e^{\rho x} - \psi(x)}. \tag{9.25}$$

With $\delta = 0$, Segerdahl [36] denotes $A(0, x \mid u)$ and $B(0, x \mid u)$ as $\xi(u, x)$ and $\chi(u, x)$, respectively. Formulas (9.24) and (9.25) extend Dickson's [7] formulas (1.3) and (1.4) to the general case of $\delta \ge 0$.

To prove (9.5) let $0 \le u < x$. If ruin should occur with $U(0) = 0$ such that the surplus immediately before ruin is x, then the surplus must attain the level u prior to ruin. Hence

$$f(x \mid 0) = B(0, u \mid 0)\, f(x \mid u), \tag{9.26}$$

or

$$\begin{aligned} f(x \mid u) &= f(x \mid 0)\frac{1}{B(0, u \mid 0)} \\ &= f(x \mid 0)\frac{e^{\rho u} - \psi(u)}{1 - \psi(0)}, \end{aligned} \tag{9.27}$$

which is (9.5).

Formula (9.6) is more intricate because the condition $U(0) = u \ge x = U(T\text{-}) > 0$ means that the surplus is to drop below the level x some time before ruin occurs. Its proof is based on the notion of *duality*, which as pointed out by Feller [17, p. 395] enables one "to prove in an elementary way theorems that would otherwise require deep analytic methods." We shall derive the identity

$$B(0, u \mid 0)\, f(x \mid u) \int_0^{\infty} \frac{p(x + y)}{1 - P(x)} e^{-\rho y}\, dy \;=\; g(x)\, A(-u, 0 \mid -x)\, e^{-\rho u}, \tag{9.28}$$

valid for $0 < x \leq u$. Solving for $f(x \mid u)$ and using (4.4) and (9.16) we get

$$f(x \mid u) \;=\; \frac{\lambda}{c}[1 \;-\; P(x)]e^{-\rho u}\, \frac{A(0, u \mid u - x)}{B(0, u \mid 0)}. \tag{9.29}$$

Applying (9.24) and (9.25) to (9.29) yields

$$f(x \mid u) \;=\; \frac{\lambda}{c}[1 \;-\; P(x)] \frac{\psi(u - x) \;-\; e^{-\rho x}\psi(u)}{1 \;-\; \psi(0)}, \tag{9.30}$$

which is indeed (9.6).

To prove the identity (9.28), we multiply it by dx; the expression on the left-hand side can be interpreted as

$$E[e^{-\delta T_0}\, I(T_u < T < \infty,\; x \leq U(T\text{-}) \leq x{+}dx) \mid U(0) = 0], \tag{9.31}$$

while the expression on the right-hand side is

$$E[e^{-\delta T_0}\, I(-x{-}dx \leq U(T) \leq -x,\; \min_{T < t < T_0} U(t) < -u) \mid U(0) = 0]. \tag{9.32}$$

The equality of (9.31) and (9.32) can be explained by duality. A dual process $\{U^*(t)\}$ of the process $\{U(t)\}$ with $U(0) = 0$ is defined as follows: If $T = \infty$, we set $U^*(t) = U(t)$, and if $T < \infty$, we set

$$U^*(t) \;=\; \begin{cases} -U(T_0 - t) & \text{for } 0 \leq t \leq T_0 \\ U(t) & \text{for } t > T_0 \end{cases}. \tag{9.33}$$

See Figures 3.a and 3.b. In other words, suppose that $\{U(t)\}$ has n jumps before time T_0, and that the jump of size X_i occurs at time t_i, $t_i < T_0$, $i = 1, \ldots, n$; then $\{U^*(t)\}$ has the same n jumps before time T_0, except that the jump of size X_i occurs at time $T_0 - t_i$, $i = 1, \ldots, n$. This is a measure-preserving correspondence, and hence the process $\{U^*(t)\}$ follows the same probability law as the process $\{U(t)\}$. That is, if a certain event in terms of $\{U(t)\}$ is translated as an event that is formulated in terms of $\{U^*(t)\}$, the probabilities, or, as in the case of (9.31) and (9.32), the contingent expectations, are identical.

Incidentally, this duality also explains the symmetric formula (4.7), which is for the case $\delta = 0$.

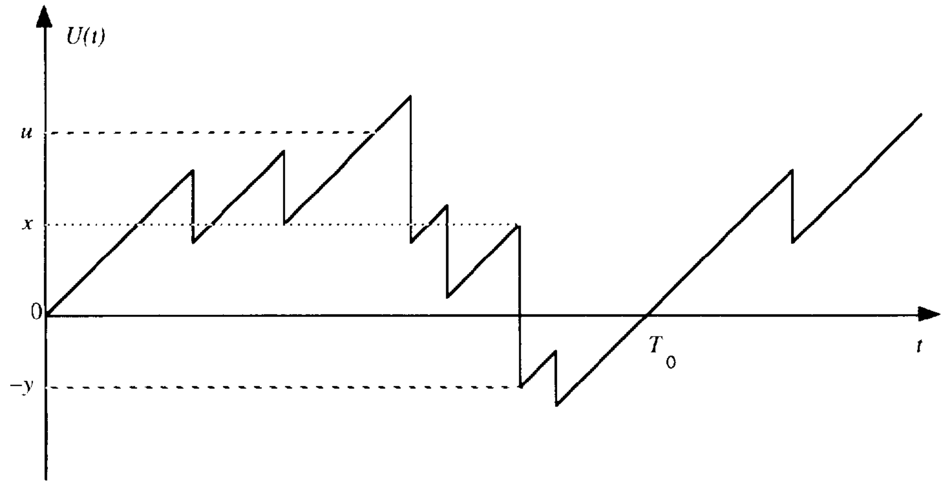

Figure 3.a. A sample path of the process {U(t)} which contributes to (9.31)

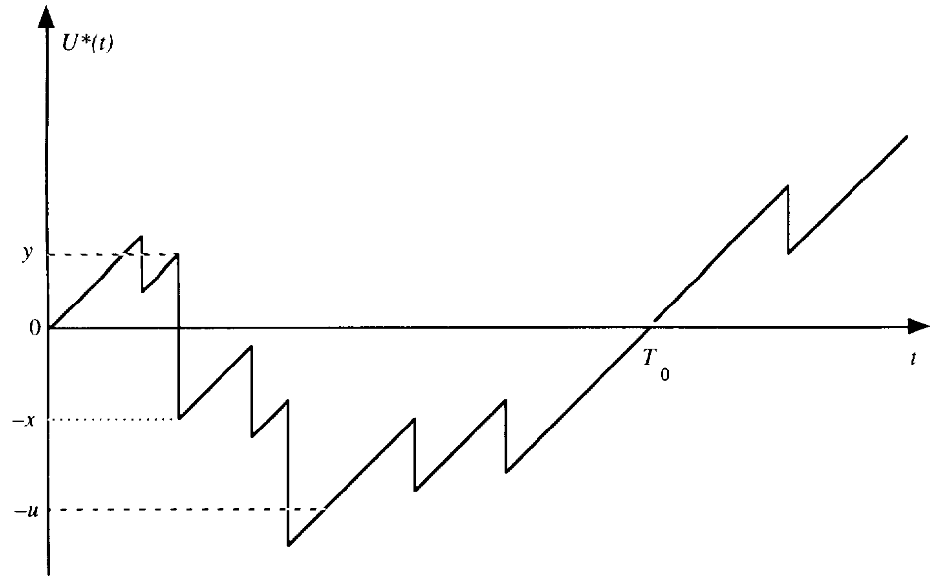

Figure 3.b. The dual sample path which contributes to (9.32)

Using (2.43) we obtain from (9.5) and (9.6) the formula

$$f(x, y \mid u) = \begin{cases} f(x, y \mid 0) \dfrac{e^{\rho u} - \psi(u)}{1 - \psi(0)}, & x > u \geq 0, \quad (9.34) \\ f(x, y \mid 0) \dfrac{e^{\rho x}\psi(u-x) - \psi(u)}{1 - \psi(0)}, & 0 < x \leq u, \quad (9.35) \end{cases}$$

with

$$f(x, y \mid 0) = \frac{\lambda}{c} e^{-\rho x} p(x+y) \tag{9.36}$$

according to (4.3).

Example One consequence of (9.34) and (9.35) is that there is an explicit formula for $f(x, y \mid u)$ whenever there is an explicit expression for the function $\psi(u)$. This is the case for an exponential claim amount distribution,

$$p(x) = \beta e^{-\beta x}, \qquad x \geq 0.$$

Here we have, for $u \geq 0$,

$$\begin{aligned} \psi(u) &= E[e^{-\delta T + \rho U(T)} I(T < \infty) \mid U(0) = u] \\ &= \frac{\beta - R}{\beta + \rho} e^{-Ru} \end{aligned} \tag{9.37}$$

according to (8.37) [with $w(y) = e^{-\rho y}$]. Then

$$\psi(u - x) = e^{Rx}\psi(u). \tag{9.38}$$

Hence, by (9.5) and (9.6) we obtain

$$f(x \mid u) = \begin{cases} \dfrac{\lambda}{c(R+\rho)} e^{-(\rho+\beta)x} [(\beta+\rho)e^{\rho u} - (\beta - R)e^{-Ru}], & x > u \geq 0, \quad (9.39) \\ \dfrac{\lambda(\beta - R)}{c(R+\rho)} e^{-(\rho+\beta)x} [e^{(R+\rho)x} - 1] e^{-Ru}, & 0 < x \leq u. \quad (9.40) \end{cases}$$

To determine $f(x, y \mid u)$ we apply (4.19). We may use the formula

$$\int_0^\infty f(x \mid u)\, dx = E[e^{-\delta T} I(T < \infty) \mid U(0) = u]$$

as a check for the validity of (9.39) and (9.40). After some calculation the left-hand side simplifies as

$$\frac{\lambda}{c(\beta+\rho)} e^{-Ru}, \tag{9.41}$$

while the right-hand side is

$$\frac{\beta - R}{\beta} e^{-Ru} \tag{9.42}$$

by (8.33). These two terms are the same because of (8.35). It is amusing to note that the integral of expression (9.40) from $x = 0$ to $x = \infty$ is also (9.41). ||||

Remarks (i) With $w(x, y) = e^{-\rho y}$,

$$h(u) = \int_u^\infty e^{-\rho(z-u)} g(z)\, dz \tag{9.43}$$

by (2.34), (2.16) and (4.4). It follows from (2.36) [with $\phi(u) = \psi(u)$] that

$$\begin{aligned} \psi(u) &= (\psi * g)(u) + h(u) \\ &= \int_0^u \psi(u-z)\, g(z)\, dz + \int_u^\infty e^{-\rho(z-u)} g(z)\, dz, \end{aligned} \tag{9.44}$$

which generalizes Exercise 12.11 of *Actuarial Mathematics* [4]. With $u = 0$, (9.44) becomes

$$\psi(0) = \int_0^\infty e^{-\rho z} g(z)\, dz = \hat{g}(\rho); \tag{9.45}$$

recall that $\hat{g}(-R) = 1$.

(ii) As an alternative proof, we would like to show that (9.5) and (9.6) satisfy the renewal equation (5.13), or equivalently, with the definition

$$\varphi(u) = \begin{cases} e^{\rho u}, & x > u \geq 0, \\ e^{\rho x} \psi(u - x), & 0 < x \leq u, \end{cases} \tag{9.46}$$

that

$$\varphi(u) - \psi(u) = [(\varphi - \psi) * g](u) + [1 - \psi(0)]\, e^{\rho u}\, I(x > u) \tag{9.47}$$

holds. A direct verification of (9.47) seems difficult. However, we can confirm its validity by means of Laplace transforms. Taking Laplace transforms of (9.44) yields

$$\hat{\psi}(\xi) = \hat{\psi}(\xi)\hat{g}(\xi) + \hat{h}(\xi), \tag{9.48}$$

where

$$\begin{aligned} \hat{h}(\xi) &= \int_0^\infty e^{-\xi u} \left[\int_u^\infty e^{-\rho(z-u)} g(z)\, dz \right] du \\ &= \frac{\hat{g}(\xi) - \hat{g}(\rho)}{\rho - \xi} \end{aligned} \tag{9.49}$$

by changing the order of integration. Hence

$$\hat{\psi}(\xi) = \frac{\hat{h}(\xi)}{1 - \hat{g}(\xi)}$$
$$= \frac{\hat{g}(\xi) - \hat{g}(\rho)}{[1 - \hat{g}(\xi)](\rho - \xi)}. \tag{9.50}$$

From (9.46)

$$\hat{\phi}(\xi) = \frac{e^{(\rho - \xi)x} - 1}{\rho - \xi} + e^{(\rho - \xi)x}\,\hat{\psi}(\xi). \tag{9.51}$$

Thus

$$\hat{\phi}(\xi) - \hat{\psi}(\xi) = \frac{e^{(\rho - \xi)x} - 1}{\rho - \xi}\,\frac{1 - \hat{g}(\rho)}{1 - \hat{g}(\xi)}$$
$$= \frac{e^{(\rho - \xi)x} - 1}{\rho - \xi}\,\frac{1 - \psi(0)}{1 - \hat{g}(\xi)} \tag{9.52}$$

by (9.45). We now see that (9.47) holds.

(iii) As we pointed out earlier, it follows from our generalization of Dickson's formula that there is an explicit formula for $f(x \mid u)$ [and hence $f(x, y \mid u)$] whenever there is an explicit expression for the function $\psi(u)$. If $\hat{\psi}(\xi)$ is a rational function, then, by locating its poles (singularities), we can determine $\psi(u)$. It follows from (9.50) that $\hat{\psi}(\xi)$ is a rational function if and only if $\hat{g}(\xi)$ is a rational function; by (2.54) $\hat{g}(\xi)$ is a rational function if and only if $\hat{p}(\xi)$ is a rational function. It also follows from (9.50) that the singularities of $\hat{\psi}(\xi)$ are exactly the roots of the equation

$$\hat{g}(\xi) = 1. \tag{9.53}$$

We should clarify that here the functions $\hat{\psi}(\xi)$, $\hat{g}(\xi)$ and $\hat{p}(\xi)$ are defined on the whole complex plane by analytic continuation. Consider the example where $p(x) = \beta e^{\beta x}$; although the integral

$$\int_0^\infty e^{-\xi x}\, p(x)\, dx$$

is not defined for complex numbers ξ with $\mathrm{Re}(\xi) \le -\beta$, the rational function $\frac{\beta}{\beta + \xi}$ is. Consequently, while (2.57) has at most one solution, (9.53) can have multiple solutions. Now, let $-r_1, -r_2, \ldots, -r_m$ be the distinct roots of (9.53) and $n_1, n_2, \ldots, n_m$ be their multiplicities. Then it follows from *Heaviside's expansion formula* (Spiegel [37, p. 73]) that

$$\psi(u) = \sum_{k=1}^{m} \frac{1}{(n_k - 1)!} \lim_{\xi \to -r_k} \frac{d^{n_k - 1}}{d\xi^{n_k - 1}} [(\xi + r_k)^{n_k} \hat{\psi}(\xi) e^{\xi u}], \tag{9.54}$$

where $\hat{\psi}(\xi)$ is given by (9.50). In the special case where all poles of $\hat{\psi}(\xi)$ are simple, i.e., $n_1 = n_2 = ... = n_m = 1$, then (9.54) simplifies as

$$\begin{aligned} \psi(u) &= \sum_{k=1}^{m} \lim_{\xi \to -r_k} [(\xi + r_k) \hat{\psi}(\xi) e^{\xi u}] \\ &= \sum_{k=1}^{m} \frac{\hat{h}(-r_k)}{-\hat{g}'(-r_k)} e^{-r_k u} \\ &= \sum_{k=1}^{m} \frac{\hat{g}(-r_k) - \hat{g}(\rho)}{-\hat{g}'(-r_k)(\rho + r_k)} e^{-r_k u}. \end{aligned} \tag{9.55}$$

By (9.53) and (9.45),

$$\hat{g}(-r_k) - \hat{g}(\rho) = 1 - \psi(0).$$

Similar to (2.59), we have

$$-\hat{g}'(-r_k)(\rho + r_k) = -\frac{\lambda}{c}\hat{p}'(-r_k) - 1.$$

Hence (9.55) simplifies as

$$\psi(u) = [1 - \psi(0)] \sum_{k=1}^{m} \frac{e^{-r_k u}}{-\frac{\lambda}{c}\hat{p}'(-r_k) - 1}. \tag{9.56}$$

Putting u = 0 in (9.56) and rearranging, we obtain

$$1 - \psi(0) = \frac{1}{1 + \sum_{k=1}^{m} \frac{1}{-\frac{\lambda}{c}\hat{p}'(-r_k) - 1}}, \tag{9.57}$$

which can be substituted in (9.56) yielding

$$\psi(u) = \frac{c}{1 + \sum_{k=1}^{m} \frac{c}{-\lambda\hat{p}'(-r_k) - c}} \sum_{k=1}^{m} \frac{e^{-r_k u}}{-\lambda\hat{p}'(-r_k) - c}. \tag{9.58}$$

Consider the case where p(x) is a mixture of exponential distributions,

$$p(x) = \sum_{j=1}^{n} A_j \beta_j e^{-\beta_j x}, \qquad x \geq 0, \tag{9.59}$$

where

$$0 < \beta_1 < \beta_2 < ... < \beta_n,$$

and

$$\sum_{j=1}^{n} A_j = 1.$$

Then

$$\hat{p}(\xi) = \sum_{j=1}^{n} \frac{A_j \beta_j}{\beta_j + \xi}, \tag{9.60}$$

and Lundberg's fundamental equation (2.21) becomes

$$\delta + \lambda - c\xi = \lambda \sum_{j=1}^{n} \frac{A_j \beta_j}{\beta_j + \xi}. \tag{9.61}$$

The nonnegative solution of (9.61) is ρ and the negative solutions are the poles of $\hat{\psi}(\xi)$. We now impose the condition that $A_j > 0$, $j = 1, 2, \dots, n$. Then (9.61) has n distinct negative roots $\{-r_k\}$ with

$$0 < r_1 = R < \beta_1 < r_2 < \beta_2 < \dots < r_n < \beta_n.$$

(Inequalities (12.6.15) of *Actuarial Mathematics* [4] are for the case $\delta = 0$; see also Figure 12.7 of *Actuarial Mathematics* [4].) It follows from

$$\hat{p}'(\xi) = -\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)^2} \tag{9.62}$$

and (9.58) that, given the roots $\{-r_k\}$, we have an explicit formula for $\psi(u)$ [and hence explicit formulas for both $f(x \mid u)$ and $f(x, y \mid u)$]. On the other hand, by (2.54) and (9.60),

$$\begin{aligned} \hat{g}(\xi) &= \frac{\lambda}{c(\rho - \xi)}[\hat{p}(\xi) - \hat{p}(\rho)] \\ &= \frac{\lambda}{c} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)(\beta_j + \rho)}, \end{aligned} \tag{9.63}$$

from which we obtain

$$h(\xi) = \frac{\hat{g}(\xi) - \hat{g}(\rho)}{\rho - \xi} = \frac{\lambda}{c} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)(\beta_j + \rho)^2} \tag{9.64}$$

and

$$-\hat{g}'(\xi) = \frac{\lambda}{c} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)^2(\beta_j + \rho)}. \tag{9.65}$$

It follows from (9.55), (9.64) and (9.65) that we have the following alternative formula for $\psi(u)$,

$$\psi(u) = \sum_{k=1}^{n} \frac{\displaystyle\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j - r_k)(\beta_j + \rho)^2}}{\displaystyle\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j - r_k)^2(\beta_j + \rho)}} e^{-r_k u}. \tag{9.66}$$

Note that in the special case n = 1 we obtain again (9.37).

(iv) Substituting the asymptotic expression of $\psi(u)$,

$$\psi(u) \sim Ce^{-Ru} \qquad \text{for } u \to \infty, \tag{9.67}$$

in (9.35) yields

$$\begin{aligned} f(x, y \mid u) &\sim f(x, y \mid 0) \frac{C}{1 - \psi(0)} [e^{\rho x} e^{-R(u-x)} - e^{-Ru}] \\ &= \frac{\lambda C}{c[1 - \psi(0)]} (e^{Rx} - e^{-\rho x})\, p(x + y)\, e^{-Ru} \qquad \text{for } u \to \infty. \end{aligned} \tag{9.68}$$

Because

$$\begin{aligned} C &= \frac{\hat{g}(-R) - \hat{g}(\rho)}{-\hat{g}'(-R)(\rho + R)} \\ &= \frac{1 - \psi(0)}{-\hat{g}'(-R)(\rho + R)} \\ &= \frac{1 - \psi(0)}{-\frac{\lambda}{c}\hat{p}'(-R) - 1}, \end{aligned} \tag{9.69}$$

the asymptotic formulas (9.68) and (7.16) are the same.

(v) The expression

$$E[v^T \bar{a}_{\overline{T_0 - T|}}\, I(T < \infty) \mid U(0) = u] \tag{9.70}$$

is the expected present value of a continuous annuity at a rate of 1 per unit time between the time of ruin and the time of recovery for a given initial surplus u. Because

$$v^T \bar{a}_{\overline{T_0 - T|}} = \frac{1}{\delta}(e^{-\delta T} - e^{-\delta T_0}), \tag{9.71}$$

(9.70) is

$$\frac{1}{\delta}\Big[\int_0^\infty f(x \mid u)\, dx - \psi(u)\Big]. \tag{9.72}$$

We note that

$$\lim_{\delta \to 0} E[v^T \bar{a}_{\overline{T_0 - T|}}\, I(T < \infty) \mid U(0) = u] = E[(T_0 - T)\, I(T < \infty) \mid U(0) = u]. \tag{9.73}$$

Alternatively, (9.70) is

$$E[v^T \frac{1 - e^{\rho U(T)}}{\delta} I(T < \infty) \mid U(0) = u] \tag{9.74}$$

because of (9.9). This yields

$$\begin{aligned} E[(T_0 - T) I(T < \infty) \mid U(0) = u] &= \lim_{\delta \to 0} \frac{\rho}{\delta} E[|U(T)| I(T < \infty) \mid U(0) = u] \\ &= \frac{1}{c - \lambda p_1} E[|U(T)| I(T < \infty) \mid U(0) = u] \end{aligned} \tag{9.75}$$

by (2.48). Formula (9.75) is intuitively clear because $c - \lambda p_1$ is the drift of $\{U(t)\}$. For related results see Egídio dos Reis [16].

(vi) Recall the function $\pi_3(x, t \mid u)$, the passage time density of the surplus process at the level x, an explicit formula for which is given by (8.13) and (8.14). Similarly to (8.31), we have, for $x \le u$ and $\delta \ge 0$,

$$\begin{aligned} \int_0^\infty \pi_3(x, \tau \mid u) e^{-\delta\tau} d\tau &= \psi(x - u)\Big[1 + \int_0^\infty \pi_3(x, \tau \mid x) e^{-\delta\tau} d\tau\Big] \\ &= \psi(x - u)\Big[1 + \int_0^\infty \pi_3(0, \tau \mid 0) e^{-\delta\tau} d\tau\Big]. \end{aligned} \tag{9.76}$$

Putting $x = u = 0$ in (9.76) and solving for the integral yields

$$\int_0^\infty \pi_3(0, \tau \mid 0) e^{-\delta\tau} d\tau = \frac{\psi(0)}{1 - \psi(0)}. \tag{9.77}$$

Applying (9.77) to (8.31) and (9.76), we obtain

$$\int_0^\infty \pi_3(x, \tau \mid u) e^{-\delta\tau} d\tau = \begin{cases} \dfrac{e^{-\rho(x-u)}}{1 - \psi(0)}, & x > u \\ \dfrac{\psi(u - x)}{1 - \psi(0)}, & x \le u \end{cases}. \tag{9.78}$$

The right-hand side of (9.78) can be written as a pair of infinite series using the geometric series formula

$$\frac{1}{1 - \psi(0)} = \sum_{n=0}^\infty \psi(0)^n;$$

the j-th term of either series represents the contribution of the jth upcrossing at the level x to the integral.

10. Optimal Dividends

We now consider a problem that is due to Bruno de Finetti, has been treated by Karl Borch and others, and can be found in the textbooks of Bühlmann [5, Section 6.4] and Gerber [19, Section 10.1]. Here the surplus model is modified in that dividends are paid to the shareholders of the insurance company. We assume that the dividends are paid according to a *barrier strategy* corresponding to a barrier at the level b. Thus whenever the surplus is on the barrier b, dividends are paid continuously, at a rate of c so that the surplus stays on the barrier, until the next claim occurs and the surplus falls below b. If the surplus is below b, no dividends are being paid. Evidently, ruin will occur with certainty in this model. For $0 \le u \le b$, let V(u, b) denote the expected present value of the dividend payments until ruin.

Since no dividends are paid, unless the surplus reaches the level b before ruin occurs, we have, for $0 \le u \le b$,

$$V(u, b) = B(0, b \mid u)\, V(b, b), \tag{10.1}$$

or, by (9.25)

$$V(u, b) = \frac{e^{\rho u} - \psi(u)}{e^{\rho b} - \psi(b)}\, V(b, b). \tag{10.2}$$

To determine V(b, b), we need a boundary condition at u = b. To obtain it, we compare two situations, one with initial surplus b, and the other with initial surplus $u = b - \varepsilon$, $\varepsilon > 0$. If ε is sufficiently small, we can be "almost sure" that in the second situation the surplus will reach the barrier before a claim occurs; hence the dividends paid in the second case will be "essentially" the dividends paid in the first case reduced by ε, from which it follows that

$$V(b - \varepsilon, b) \approx V(b, b) - \varepsilon. \tag{10.3}$$

In the limit, this means that

$$\left.\frac{\partial V}{\partial u}\right|_{u=b} = 1. \tag{10.4}$$

Differentiating (10.2) with respect to u and then setting u = b yields

$$1 = \frac{\rho e^{\rho b} - \psi'(b)}{e^{\rho b} - \psi(b)} V(b, b).$$

Hence

$$V(u, b) = \frac{e^{\rho u} - \psi(u)}{\rho e^{\rho b} - \psi'(b)}, \qquad 0 \le u \le b. \tag{10.5}$$

This formula should be compared with (1.13) in Chapter 10 of Gerber [19]. In Section 10.1 of [19], the function B(0, b | u) is denoted as W(u, b).

Let $\tilde{b}$ be the optimal barrier, i.e., the value of b that maximizes the expected present value of the dividends. In view of (10.5), $\tilde{b}$ is the value that minimizes the denominator, i.e., $\tilde{b}$ satisfies

$$\rho^2 e^{\rho b} - \psi''(b) = 0. \tag{10.6}$$

An equivalent condition is that

$$\left.\frac{\partial^2 V}{\partial u^2}\right|_{u=b} = 0; \tag{10.7}$$

this follows from the explicit form of (10.5).

Example In the case of an exponential claim amount distribution, there is an explicit expression for $\psi(u)$. Substituting (9.37) into (10.5) yields

$$V(u, b) = \frac{(\beta + \rho)e^{\rho u} - (\beta - R)e^{-Ru}}{\rho(\beta + \rho)e^{\rho b} + R(\beta - R)e^{-Rb}}. \tag{10.8}$$

The optimal value $\tilde{b}$ is obtained from the condition that

$$\rho^2(\beta + \rho)e^{\rho b} - R^2(\beta - R)e^{-Rb} = 0. \tag{10.9}$$

Thus

$$\tilde{b} = \frac{1}{\rho + R} \ln \frac{R^2(\beta - R)}{\rho^2(\beta + \rho)} \tag{10.10}$$

is the optimal barrier. ||||

11. Concluding Remarks

This paper studies the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. Motivated by the problem of pricing American options on stocks with jumps, we incorporate the time of ruin in the classical model by discounting. New results are derived, many of which have a probabilistic interpretation, and additional insight is gained for existing results in the classical model. Our next goal is to treat the option pricing problem mentioned in the Introduction.

The results presented can be generalized in various directions. For example, several formulas can be extended to the case where the compound Poisson process is replaced by a more general process with positive, independent and stationary increments, such as the *gamma process* or the *inverse Gaussian process.* In the references there are some recently published related papers.

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