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Randomly Compounded Interest

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The "amount of 1" (i.e., the amount that \$ 1 is worth after 1 year) for an account earning a nominal interest r compounded n times annually (assuming that the n compounding periods all have the length $1/n$ years) is $(1+r/n)^n$. If the n compounding periods were of possibly different lengths $t_1, \ldots, t_n$ then the correct value would be

$$P_n(r, \mathrm{t}) := \Pi_{i=1}^n (1 + rt_i),$$

where $\mathrm{t} = (t_1, \ldots, t_n)$. Note that $P_n(r, \mathrm{t})$ is a polynomial in r of degree n. It is an easy exercise using the inequality between the arithmetic and geometric means to show that if $t_1, \ldots, t_n$ are nonnegative numbers whose sum is 1, then

$$P_n(r, \mathrm{t}) \leq (1 + r/n)^n. \tag{1}$$

It is worth recalling that the latter quantity increases *monotonically* to e^r, which of course leads to the formula for continuously compounded interest.

Now suppose that the n compounding periods have lengths $T_1, \ldots, T_n$ chosen *at random*. What can we say about $P_n(r, \mathrm{T})$, where $\mathrm{T} = (T_1, \ldots, T_n)$? This depends of course on what is meant by choosing "random lengths" and we will consider two different natural possibilities that give rise to different answers to this question.

Model 1.

Recall that a random variable with a *uniform distribution* on a set S is such that the probability of the value of the random variable being in any particular subset of S is proportional to its size. Thus, if S is the unit interval $[0, 1]$, the probability of finding a uniformly distributed variable in any particular subinterval of S is just the length of the subinterval. Our first model is obtained by choosing $n-1$ points $U_1, \ldots, U_{n-1}$ at random on [0,1], *i.e.* $U_1, \ldots, U_{n-1}$ are independent random variables uniformly distributed on $[0, 1]$. Now arrange these values in ascending order: $U_{(1)}, \ldots, U_{(n-1)}$ (these are called the *order statistics*). Set $T_1 = U_{(1)}$, $T_2 = U_{(2)} - U_{(1)}, \ldots, T_n = 1 - U_{(n-1)}$. Thus, the compounding periods are obtained by " throwing down" $n-1$ points at random on the unit interval, which then divides it into n pieces, and compounding the interest at the end of each of these intervals. It can be shown that this model is equivalent to that obtained by choosing a point of the "surface" $\{\mathrm{T} \in \mathbf{R}^n : T_i \geq 0, \sum T_i = 1\}$ according to a uniform distribution. (See [2] for a discussion.) We now proceed to compute the expected effective interest rate and see what happens when n is allowed to increase without bound.

Let $\Omega_n(\lambda) = \{x \in \mathbf{R}^n : t_i \geq 0, \sum t_i = \lambda\}$ and take $X := (T_1, \ldots, T_n)$ to be uniformly distributed on $\Omega_n(1)$, *i.e.* the probability that X lies in any particular (measureable) subset S of $\Omega_n(1)$ is proportional to the surface measure of S. We denote the surface element by $d\sigma_n$. The following Lemma will be useful in our computations.

Lemma 1. *For $\lambda > 0$, $0 \leq d \leq n$ and $n = 2, 3, \ldots$,*

$$\int_{\Omega_n(\lambda)} t_1 \cdot \ldots \cdot t_d \frac{d\sigma}{\sqrt{n}} = \frac{\lambda^{n-1+d}}{(n-1+d)!}. \tag{2}$$

Proof. An easy computation shows that the surface measure $d\sigma_n$ is given by $\sqrt{n} dt_1 \ldots dt_{n-1}$, where we set $t_n = 1 - t_1 - \ldots t_{n-1}$. It is also easy to see that (2) is correct for $n = 2$, and we proceed by induction. Using Fubini's theorem and the induction hypothesis,

$$\int_{\Omega_n(\lambda)} t_1 \cdot \ldots \cdot t_d \frac{d\sigma_n}{\sqrt{n}} =$$
$$\int_0^\lambda t_1 \int_{\Omega_{n-1}(\lambda - t_1)} t_2 \cdot \ldots \cdot t_d \frac{d\sigma_{n-1}}{\sqrt{n-1}} = \int_0^\lambda t_1 \frac{(\lambda - t_1)^{n+d-2}}{(n+d-2)!} dt_1,$$

and computing the last integral immediately gives the result. Lemma 1 is actually a special case of a result proven by Dirichlet; see [3] p. 258.

When $d = 0$ we obtain from (2) that the measure of $\Omega_n(1)$ is $\frac{\sqrt{n}}{(n-1)!}$, so that the uniform distribution on $\Omega_n(1)$ has density function $(n-1)! d\sigma_n / \sqrt{n}$. We now compute $EP_n(r, T)$, the average (i.e. expected) value of 1 dollar after 1 year in the account. To do this, we first observe that by symmetry, the distribution of $(T_1, \ldots, T_n)$ is the same as that of $(T_{\pi(1)}, \ldots, T_{\pi(n)})$, where $\pi(\cdot)$ is any permutation of the integers $1, 2, \ldots, n$. Such a sequence $(T_1, \ldots, T_n)$ is called *exchangeable*. A consequence of this property is that if $1 < i_1 < \ldots < i_d < n$ then for any continuous function Φ

$$E\Phi(T_{i_1}, \ldots, T_{i_d}) = E\Phi(T_1, \ldots, T_d).$$

Using this exchangeability with $\Phi(t_1, \ldots, t_d) = t_1 \cdot \ldots \cdot t_d$, expanding the expression for $P_n(r, T)$, and applying (2),we obtain

$$M_n := EP_n(r, T) = \sum_{d=0}^{n} \binom{n}{d} E[T_1 \cdot \ldots \cdot T_d] r^d = \sum_{d=0}^{n} \binom{n}{d} \frac{(n-1)!}{(n+d-1)!} r^d.$$

Observe that

$$\binom{n}{d} \frac{(n-1)!}{(n+d-1)!} = \frac{n!(n-1)!}{(n-d)!(n-1+d)!} \frac{1}{d!} \leq \frac{1}{d!}.$$

It is then readily seen, using the dominated convergence theorem, that $\lim_{n\to\infty} M_n = e^r$. But by (1), e^r is the maximum possible value of $Y_n := P_n(r, \mathrm{T})$, so we conclude that Y_n converges in mean to the degenerate random variable e^r. Note that we have not established here that this convergence takes place with probability 1.

Model 2

We now imagine that the banker waits a random amount of time before computing the first interest payment, then waits a random amount of time during the remaining time to make the second payment, and so forth; making a total of n payments by year's end. It should be clear that this model must be different from the first one, since it is rather likely that most of the interest payments are made toward the end of the year.

To formalize the model, suppose that the T_i are random variables chosen by picking T_1 uniformly on [0,1], T_2 uniform on $[0, 1 - T_1]$ and in general, T_i is chosen uniformly on $[0, 1 - T_1 - \ldots - T_{i-1}]$. As before, our goal is to calculate the expectation of the random variable $P_n = P_n(r, \mathrm{T})$ for each n.

The conditional density of T_{k+1} given $T_i = t_i$, $1 \leq i \leq k$ is $(1 - \sum_1^k t_i)^{-1}$ for $1 \leq k \leq n-1$. We obtain a recursion formula by conditioning. Let $p_n(r) = E(P_n)$. The functions $p_n(r)$ are polynomials of degree n; $p_1 = 1 + r$, $p_2 = 1 + r + r^2/6$.

Theorem 1. *If $T_1 = t$, then $E(P_n(r, T)|T_1 = t) = (1 + rt)p_{n-1}(r(1 - t))$.*

Proof. $E(P_n|T_1 = t) = (1 + rt)E((1 + rT_2)...(1 + rT_n)| \sum_2^n T_i = 1 - t))$. Let $S_k = T_k/(1 - t)$ $k = 2, 3, ..., n$. Then the random variables S_k satisfy the same conditions as the original T_i but they are one less in number; $S_k \geq 0$, $\sum S_k = 1$. Let $s_k = t_{k+1}/(1 - t)$. Then S_1 is uniform on (0,1) and S_2 is uniform on $(0, 1 - s_1)$, etc. Now

$$E(P_n|T_1 = t) = (1 + rt)E((1 + r(1 - t)S_2)...(1 + r(1 - t)S_n)) = (1 + rt)E(P_{n-1}(r(1 - t))).$$

which is $(1 + rt)p_{n-1}(r(1 - t))$, as desired.

Corollary. For every n, $p_{n+1}(r) = E((1 + rT)p_n(r(1 - T)))$, where T is uniform on (0,1).

From this formula it is easy to find the polynomials p_n by recursion. We can replace T by $1 - T$ in the formula of the corollary. Then

$$p_{n+1}(r) = E((1 + r(1 - T))p_n(rT)) = (1 + r)E(p_n(rT)) - rE(Tp_n(rT)).$$

Let $p_n(r) = \sum a_k(n)r^k$ where $a_k(n) = 0$ once $k > n$. It follows by equating coefficients and writing the moments for T in the above recursion formula that

$$a_k(n + 1) = a_{k-1}(n)/(k(k + 1)) + a_k(n)/(k + 1)$$

for $k = 1, 2, 3, ...$ The formula gives $p_3(r) = 1 + r + (2/9)r^2 + (1/72)r^3$.

In theory the coefficients $a_k(n)$ are calculable from the recursion.For example,

$$a_2(n) = (1/6)(1/3)^{n-2} + (1/4)(1 - (1/3)^{n-2}).$$

We are interested in what happens when n (the number of compounding periods) goes to infinity.

It is interesting to see the difference in the two models for different interest rates. For rates in a reasonably healthy economy, the following table shows that there is little difference in the two compounding schemes. Note that e^r is the average annual return in model 1, and $w(r) = \sum r^k/(k!)^2$ is the average annual return in model 2. In the event of hyperinflation, there is a big difference indeed!

rate	e^r	$w(r)$
.05	1.051	1.051
0.1	1.105	1.103
0.3	1.350	1.323
1	2.718	2.280
10	22026	90

Theorem 2. *For* $k = 0, 1, ...,$

$$\lim_n a_k(n) = (k!)^{-2}.$$

The proof is based on the recurrences. We break it up into a sequence of computations.

Lemma 2. *For every* k, *for every* n, $a_k(n) \leq (k!)^{-2}$.

Proof. The proof is by induction. The statement is true for all n when $k = 0$. Suppose the statement is true for all n for some $k - 1$. Now $a_k(1) = 0$ when $k > 1$ so the statement for k is true when $n = 0$. Suppose $a_k(n) \leq (k!)^{-2}$ for some n. Then

$$a_k(n+1) = a_{k-1}(n)/k(k+1) + a_k(n)/(k+1) \leq ((k-1)!)^{-2}/k(k+1) + (k!)^{-2}/(k+1) = (k!)^{-2}.$$

The conclusion follows.

Lemma 3. *For* $k \geq 1$ *and for every* n, $a_{k-1}(n) \geq k^2 a_k(n)$.

Proof. The proof is by induction. When $k = 1$, $a_0(n) = 1 \forall n$, and $a_1(n) = 1 \forall n$. Suppose that $a_{k-1}(n) \geq k^2 a_k(n)$ for some k and all n. To prove $a_k(n) \geq (k+1)^2 a_{k+1}(n)$ for all n. When $n = 1$ we want $a_k(1) \geq (k+1)^2 a_{k+1}(1)$ but when $k \geq 2$ $a_k(1) = 0$ and $a_{k+1}(1) = 0$ so $a_k(1) \geq (k+1)^2 a_{k+1}(n)$. Suppose $a_k(n) \geq (k+1)^2 a_{k+1}(n)$ for some n : to prove

$$a_k(n+1) \geq (k+1)^2 a_{k+1}(n+1).$$

The left side of this inequality is

$$a_{k-1}(n)/k(k+1) + a_k(n)/(k+1) \geq k^2 a_k(n)/k(k+1) + a_k(n)/(k+1) - a_k(n),$$

so $a_k(n+1) \geq a_k(n)$. The right side of the desired inequality is

$$(k+1)^2 a_{k+1}(n+1) = (k+1)^2[a_k(n)/((k+1)(k+2)) + a_{k+1}(n)/(k+2)]$$

which is less than or equal to

$$(k+1)^2 a_k(n)/((k+1)(k+2)) + a_k(n)/(k+2) = a_k(n).$$

Therefore,

$$a_k(n+1) \geq a_k(n) \geq (k+1)^2 a_{k+1}(n+1)$$

which was to be proved. Note from the last line, we see that for every k and every n, $a_k(n+1) \geq a_k(n)$.

Proof of Theorem 2. Each sequence $\{a_k(n)\}$ is non-decreasing in n and bounded above, therefore it has a limit. Let $a_k = \lim_n a_k(n)$. From the recurrence $a_k = a_{k-1}/k(k+1) + a_k/(k+1)$, hence $a_k = (k!)^{-2}$ by induction. This completes the proof.

Note that the expected effective interest rate for model 2 is less than that of model 1. The kind of result is typical of the many interesting examples of "paradoxes" that arise in probability theory when one does not carefully specify what is meant by the phrase "at random". [2] provides an excellent discussion of this issue in the geometrical context.

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