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A NOTE ON HEDGING AND THE PUT OPTION

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This note develops the optimality of the put option as a hedging strategy. For an investor who owns a stock, he can buy a put option to hedge against the downside risk of the stock price. In this short note, we ask two questions with the hedge strategy of the put option. First, whether the investor is better off with the hedging strategy? Second, if the investor is better off with the hedging strategy, whether it is the optimal hedging plan? Our result shows that in a risk averse world an investor will maximize his utility with buying the put option as a hedge strategy.

A NOTE ON HEDGING AND THE PUT OPTION [1]

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The options market provides an opportunity for investors who wish to hedge their fund and for speculators who want to bid their fortune. The hedging function of a put option can be seen from a very simple example. Let's assume an investor owns a stock. If he worries about downside risk of the stock price[2], he can hedge against the price going below certain level K, by buying a put option with an exercise price K. However, there are two basic questions we need to answer with the hedging strategy. First, whether the investor is better off with the hedging strategy? Second, if the investor is better off with the hedging strategy, whether it is the optimal hedging plan? In this short note, we are going to answer these two questions in a risk averse world.

A. A Mean-Variance Analysis

Assume an investor owns a stock with random rate of return X. The investor is a risk averse person, that means he has a concave utility function U(x) such that $U'(x) > 0$ and $U''(x) < 0$. Now the investor needs to make a hedging decision. Without hedging, his expected return would be E(X). Also the investor can buy a put option to hedge the return going below a certain level K. By buying a put option, the investor's payoff will be

$$Y = X + \max(K-X, 0) - P$$

here, K is the exercise price, P is the put option price.

To charge an actuarial fair price, P should be $E[\max(K-X,0)]$. In this case, the expected value of X is as same as the expected value of Y, i.e. $E(X) = E(Y)$. How about variance? First, we have the following result:

Proposition 1: $\quad Var(X) \geq Var(Y)$

Proof: Let $\quad E(X \mid X > K) = a, \quad Var(X \mid X > K) = A, \quad P(X > K) = p,$

$\quad E(X \mid X \leq K) = b, \quad Var(X \mid X \leq K) = B, \quad P(X \leq K) = 1-p = q.$

Thus,

$$Var(X) = E(X^2) - E^2(X)$$

$$= P(X>K)\, E(X^2 \mid X>K) + P(X \le K)\, E(X^2 \mid X \le K) -$$

$$(P(X>K)\, E(X \mid X > K) + P(X \le K)\, E(X \mid X \le K))^2$$

$$= P(X>K)\, (Var(X \mid X > K) + E^2(X \mid X > K)) +$$

$$P(X \le K)\, (Var(X \mid X \le K) + E^2(X \mid X \le K)) -$$

$$(P(X>K)\, E(X \mid X > K) + P(X \le K)\, E(X \mid X \le K))^2$$

$$= p(A + a^2) + q(B + b^2) - (pa + qb)^2$$

$$= pA + qB + pa^2 + qb^2 - (pa + qb)^2$$

$$= pA + qB + pq(a-b)^2$$

$$\ge pA + pq(a-K)^2 \qquad (1)$$

The inequality is due to the fact:

$$p \ge 0\,,\ B \ge 0\,, \text{and } b = E[X \mid X \le K] \le K.$$

Also,

$$Var(Y) = Var(X + \max(K-X, 0) - P)$$

$$= Var(K + \max(X-K, 0) - P)$$

$$= Var(\max(X-K, 0))$$

$$= E[(\max(X-K, 0)^2] - E[\max(X-K, 0)]^2$$

$$= p[(a-K)^2 + A] - [p(a-K)]^2$$

$$= pA + pq(a-K)^2 \qquad (2)$$

Compare (1) and (2), we have $Var(X) \ge Var(Y)$. ■

Proposition 1 tells us the hedging strategy of buying the put option will reduce the variance of return and in the meantime keep the mean of return unchanged. So under the mean-variance framework, a risk averse investor is better off with buying an actuarial fair put option.

B. The Optimal Hedging Strategy

As we discussed in section A, for an investor owning a stock with return X, buying a put option will reduce the variance of X. The next question is whether this hedging strategy is the optimal strategy in terms of maximized utility?

Assume H(X) ($H(X) \geq 0$) is any hedging benefit based on X. If the price of the hedging plan is charged as an actuarial fair price, then $P = E[H(X)]$. Now we want to see if the hedging plan of buying the put option is better than any other hedging plan H(X) for a risk averse investor with an uncertain return X ($X > 0$).

Proposition 2: A risk averse investor who owns a stock will maximize his utility by buying a put option with exercise price K in comparison with any other hedging plan H(X), given $P = E[H(X)] = E[\max(K-X,0)]$.

Proof: With any hedging plan H(X), the investor's expected utility is $E[U(X + H(X) - P)]$;

With buying the put option with an exercise price K, the investor's expected utility is $E[U(X + \max(K-X, 0) - P)]$.

By using Taylor expansion formula,

$$U(X + H(X) - P) - U(X + \max(K-X, 0) - P)$$

$$\leq [H(X) - \max(K-X, 0)]\, U'(X+ \max(K-X, 0) - P) \qquad (3)$$

Now, we want to prove:

$$[H(X) - \max(K-X, 0)]\, U'(X + \max(K-X, 0) - P)$$

$$\leq [H(X) - \max(K-X, 0)]\, U'(K - P) \qquad (4)$$

There are three cases we need to consider:

(i) If $H(X) = \max(K-X, 0)$, then (4) is true.

(ii) If $H(X) < \max(K-X, 0)$, then $\max(K-X) > 0$, $K > X$.

Thus, $X + \max(K-X, 0) - P = K - P$

Therefore, (4) also is true.

(iii) If $H(X) > \max(K-X, 0)$, then $X + \max(K-X, 0) = \max(K, X) > K$

Thus, $U'(X + \max(K-X, 0) - P) < U'(K - P)$.

In this case, (4) also is true.

Therefore, in all three possible cases, (4) is true.

Combine (3) and (4):

$U(X+H(X)-P) - U(X+\max(K-X, 0)\_P) \leq [H(X) - \max(K-X, 0)]U'(K-P) \quad (5)$

Put expectation on both sides of (5), and note that $E[H(X)] = E[\max(K-X,0)] = P$, We have:

$$E[U(X + H(X) - P] - E[U(X + \max(K-X, 0) - P)] \leq 0$$

i.e. $$E[U(X + H(X) - P] \leq E[U(X + \max(K-X, 0) - P)] \quad (6)$$

which means the investor's utility reaches maximum with buying an actuarial fair put option. ■

Proposition 2 tells us that buying the put option is the optimal hedging strategy for a risk averse investor with a stock of random return. Here two points need to be mentioned. First, let $H(X) = P$, then from (6), we get

$$E[U(X)] \leq E[U(X + \max(K-X, 0) -P)]$$

which also shows the hedging strategy of buying a put option in general always improves a risk averse investor's utility. Second, proposition 2 of the optimal hedging plan of buying a put option in financial market actually is a mirror image of the optimal insurance with a stop-loss policy proved first by Arrow (1963). Arrow pointed out, under certain regular conditions, a stop-loss insurance is the optimal insurance. The actuarial textbook written by Bowers etc.(1986) provides a detailed discussion about the Arrow theorem.

Note

1. I would like to thank an anonymous reviewer who provides me a great help to improve this paper.

2. The importance of downside risk has been studied for a long time. Practicing managers frequently conceptualize risk in terms of an organization's failure to reach a performance target (March & Shapira, 1987). Discussion of the role of downside risk are included in Aaker & Jacobson (1990), Baird & Thomas (1990), Porter (1985) etc..

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