ACTUARIAL RESEARCH CLEARING HOUSE
1995 VOL. 2

STOCHASTIC ANALYSIS OF THE INTERACTION BETWEEN INVESTMENT AND INSURANCE RISKS

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Fellowship Credit Research Paper 95-1

Approved by Society of Actuaries
Education and Examination Research Paper Committee

Effective June 8, 1995

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ACKNOWLEDGEMENTS

I would like to thank Dr. Harry Panjer for accepting to act as my supervisor for this research paper. His support and valuable comments are gratefully acknowledged. I am also grateful to the Research Paper Committee and the reviewers for their suggestions to improve the initial proposal and the paper itself.

ABSTRACT

A portfolio of different insurance policies such as temporary, endowment and whole-life insurance policies is studied in a stochastic mortality and interest environment. The first two moments of the present value of the benefits of the portfolio are derived. The riskiness of the portfolio as measured by the variance of the present value of the benefits can be divided into an insurance risk and an investment risk in two different ways. One way leads to a more natural interpretation of the two risk components. A simple portfolio is used to illustrate the results.

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1. INTRODUCTION.

Our goal is to extend the theory on valuation of insurance liabilities to an environment where future rates of return on the assets backing those liabilities are stochastic. Although the extensions suggested herein are derived in a specific framework, they are believed to be insightful for other more general situations.

In the actuarial literature, stochastic mortality and interest rates models for one policy only have been proposed by Panjer and Bellhouse (1980, 1981), Devolder (1986), Dufresne (1990), Norberg (1990, 1991) and Beekman and Fuelling (1990, 1991) among others.

Generalizing the results to portfolios is usually a complicated task. Some work has been done in this area, see for example, Waters (1978), Frees (1990), Norberg (1993) and Parker(1994a,b).

In this paper we study, in a stochastic mortality and interest environment, the moments of the present value (or interchangeably, the discounted value) of the cash flows arising from the benefit obligations of a general portfolio of life insurance policies. Here, a general portfolio is one containing different types of policies as opposed to one containing identical policies only.

We focus our attention on the first two moments of the present value of the benefits of a portfolio provide actuaries with useful information when time comes to price or value a portfolio. This kind of information would certainly also be useful when determining a contingency reserve or assessing the solvency of a portfolio of life insurance policies.

Note that we only consider the benefit obligations which would correspond to single premium contracts or paid-up contracts. We do not explicitly include expenses and lapses in our model. One could always assume that the rate of return used in the model is net of expenses. To adequately include expenses and lapses explicitly into the model would require the use of more elaborate processes that would capture the dependence between the rate of return, the expenses and the lapses. This is beyond the scope of this paper as even if a generally accepted multi-factor model existed, it would not necessarily lead to simple results.

When considering valuation models for insurance liabilities, we agree with the views expressed in the overview section of Frees (1990) on financial models. Frees argues that the application of valuation theories in finance (e.g. Capital Asset Pricing Model, Arbitrage Pricing Model, etc.) to the insurance problem is highly suspect. His argumentation is primarily based on the fact that key assumptions of the financial models such as frictionless trading, efficient markets, existence of a secondary market, etc. are not satisfied in the insurance context. We would like to go further and suggest that even if all the key assumptions were satisfied, the financial models would not necessarily be suitable for the valuation of insurance liabilities. They might be acceptable for putting a price tag (or market value) on the liabilities but they would not be acceptable for measuring the riskiness of a portfolio of policies or assessing the solvency of an insurance company. Consequently, for the problem at hand, we find it more appropriate to generalize existing actuarial techniques to the case of random mortality and interest than to use financial models.

The random survivorship group approach (see, for example, Bowers et al (1986)) will be used to model mortality. This will imply, assuming mutual independence of the future lifetimes of the lives insured, that the number of policies payable over the years will have a multinomial distribution. The distribution of each curtate-future-lifetime is given by a usual non-parametric life table. The lives insured may have different mortality rates, for example, the portfolio may have smokers, non-smokers, females and males.

The approach we will follow for the investment component of the model is that assets are invested according to some investment strategy in order to support the liabilities.

The assets can be invested in virtually any investment vehicles available on the market including T-Bills, bonds, common shares, mutual funds, etc... The assets invested in instruments with maturity dates are not necessarily kept until maturity. The investment manager may want to sell some fixed income assets if higher returns become available in the future (perhaps with capital gains or losses) or may have to sell them if, for example, the mortality is higher than expected.

The assets and liabilities may and would typically be partially matched. Note that in practice full matching of insurance liabilities is not possible because of the very long term nature of some insurance policies and the randomness of the liability cash flows. Further, some company may not be interested in achieving the highest degree of matching possible. One reason may be that low-risk investment strategies associated with it usually produce lower expected returns. The degree of matching does not require a different investment model in our framework but would call for different parameters of the process for the rate of return.

It is generally understood that the insurance risk (due to mortality) is one that decreases as the number of policies in the portfolio increases. However, the investment risk (due to the rates of return) does not follow this rule because of the highly dependent investment rates of return involved.

The results of Norberg (1993) and Parker (1994a) are generalized using the approach found in the later. The portfolio under consideration and the assumptions made are described in section 2.

We derive expressions for the first two moments of the present value of the benefits of a portfolio in sections 3 and 4.

In section 5, we then present an alternative expression for obtaining the variance of the present value of the benefits of the portfolio which summarizes useful information needed when only minor modifications to the existing portfolio need to be investigated.

Using the variance as a measure of risk, we suggest two ways of dividing the riskiness of a portfolio into its insurance risk and investment risk components in section 6.

For illustrative purposes, the instantaneous rate of return will be modeled by an Ornstein-Uhlenbeck process. The main results for this process are given in section 7.

A simple, hypothetical, portfolio is then used to illustrate the results in section 8. Finally, we use the results of Parker (1993) to approximate the cumulative distribution of the present value of the benefits of the portfolio chosen for our illustrative purposes.

2. A PORTFOLIO OF POLICIES.

Consider a portfolio of c insurance policies where each policy is being issued to one of c independent lives. The portfolio consists of temporary, endowment, pure endowment and/or whole-life insurance contracts. We propose to study such a portfolio by grouping the policies in m groups with similar characteristics. Here we use the following grouping characteristics: age at issue, term of the policy, face amount and mortality table.

The notation that is used throughout the paper is the following:

c_i: Number of policies in group i. Note that $\sum_{i=1}^{m} c_i = c$.

p_i: Proportion of policies in group i, c_i/c.

n_i: Term of each policy in group i.

b_i: Benefit (or face amount) payable at the end of the year of death for each policy in group i.

e_i: Endowment benefit payable at n_i. If 0, the contract is a temporary one.

$K_{i,j}$: Curtate-future-lifetime of the j^{th} $(j=1,2,\ldots,c_i)$ life insured of group i.

x_i: Age at issue of each life insured in group i.

${}_{k|}q^{(i)}_{x_i}$: Probability of death in the k^{th} year after issue (at age x_i) using the mortality table appropriate for group i.

$Z_{i,j}$: Random variable denoting the present value of the benefit that is payable with respect to the j^{th} ($j=1,2,\ldots,c_i$) policy of group i.

Then $Z_{i,j}$ may be defined as:

$$(1) \qquad Z_{i,j} = \begin{cases} b_i \, e^{-y(K_{i,j}+1)} & K_{i,j}=0,1,\ldots,n_i-1 \\ e_i \, e^{-y(n_i)} & K_{i,j}=n_i,\ldots \end{cases}$$

where

$$(2) \qquad y(k) = \int_0^k \delta_s \, ds$$

i.e. the integral of the total instantaneous rate of return (i.e. interest income plus capital gains and losses), δ_s.

We point out that the discounting function, $e^{-y(k)}$, used here should not be mistaken for the current market price of a k-period zero coupon bond.

In order to study $Z_{i,j}$, we need to make the following assumptions (similar assumptions were made by Frees (1990) and Parker (1994a)):

A1 - The random variables $\{K_{i,j}\}$ are independent and, for i fixed, they are identically distributed.

A2 - Conditional on knowing the values of $\{y(k)\}_{k=1}^{\omega-x}$, the random variables $\{Z_{i,j}\}$ are independent and, for i fixed they are identically distributed.

A3 - The random variables $\{K_{i,j}\}$ and $\{y(k)\}_{k=1}^{\omega-x}$ are mutually independent.

A4 - *The moments of the random discounting function,* $e^{-y(k)}$, *are finite.*

Note that the random variables $\{Z_{i,j}\}$ are not independent since they all use the same random discounting factors.

We will use a general approach to study $Z_{i,j}$ in the sense that all we require for now is that the moments of the present value function, $e^{-y(k)}$, and some products of present value functions (such as $e^{-y(k)-y(t)}$) be known.

The m^{th} moment about the origin of $Z_{i,j}$ may be obtained in the following way:

$$(3) \qquad E\left[Z_{i,j}^{\ m}\right] = E\left[E\left[Z_{i,j}^{\ m} | K_{i,j}\right]\right] = \sum_{k=0}^{n_i-1} P\left(K_{i,j}=k\right) \left(b_i\right)^m E\left[e^{-m\cdot y(k+1)}\right]$$
$$+ P\left(K_{i,j}\geq n_i\right) \left(e_i\right)^m E\left[e^{-m\cdot y(n_i)}\right],$$

where

$$(4) \qquad P\left(K_{i,j}=k\right) = {}_{k|}q_{x_i}^{(i)}.$$

Let Z be the random variable representing the total present value of all the benefits to be paid with respect to the entire portfolio of c policies. Then Z may be defined as

$$(5) \qquad Z = \sum_{i=1}^{m} \sum_{j=1}^{c_i} Z_{i,j}.$$

3. EXPECTED VALUE OF Z.

The expected value of Z is simply the sum of the expected values of all the $Z_{i,j}$. We then have:

$$(6) \qquad E[Z] = E\left[\sum_{i=1}^{m} \sum_{j=1}^{c_i} Z_{i,j}\right] = \sum_{i=1}^{m} \sum_{j=1}^{c_i} E[Z_{i,j}] = \sum_{i=1}^{m} c_i\ E[Z_{i,1}].$$

since, by assumption, $E[Z_{i,j}]$ is the same for all $j=1,2,\ldots,c_i$.

This corresponds to a sum of traditional actuarial present values with expected discounting factors being used instead of the deterministic discount function (see Bowers et al (1986)).

4. SECOND MOMENT OF Z.

The second moment of Z can be obtained by generalizing the corresponding result for a portfolio of identical temporary insurance policies derived by Parker (1994a, section 4). The result is given in the following theorem.

THEOREM: The second moment of Z under assumptions A1 to A4 is given by:

$$(7) \qquad E[Z^2] = \sum_{i=1}^{m} c_i \, E[Z_{i,1}^{\,2}] + \sum_{i=1}^{m} c_i(c_i-1) \, E[Z_{i,1} \, Z_{i,2}] + 2 \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} c_i c_r \, E[Z_{i,1} \, Z_{r,1}],$$

where

$$(8) \qquad E\left[Z_{i,1} \, Z_{i,2}\right] = \sum_{k_1=0}^{n_i-1} \sum_{k_2=0}^{n_i-1} b_i^2 \, E\left[e^{-y(k_1+1)-y(k_2+1)}\right] {}_{k_1|}q_{x_i}^{(i)} \; {}_{k_2|}q_{x_i}^{(i)}$$

$$+2 \sum_{k=0}^{n_i-1} b_i e_i \, E\left[e^{-y(k+1)-y(n_i)}\right] {}_{k|}q_{x_i}^{(i)} \, P(K_{i,2} \geq n_i) + e_i^2 \, E\left[e^{-2y(n_i)}\right] P(K_{i,1} \geq n_i)^2$$

and

$$(9)\qquad E\left[Z_{i,1}\, Z_{r,1}\right] = \sum_{k_1=0}^{n_i-1} \sum_{k_2=0}^{n_r-1} b_i b_r\, E\left[e^{-y(k_1+1)-y(k_2+1)}\right]\, {}_{k_1|}q^{(i)}_{x_i}\, {}_{k_2|}q^{(r)}_{x_r}$$

$$+ \sum_{k=0}^{n_i-1} b_i e_r\, E\left[e^{-y(k+1)-y(n_r)}\right]\, {}_{k|}q^{(i)}_{x_i}\, P(K_{r,1}\geq n_r) + \sum_{k=0}^{n_r-1} b_r e_i\, E\left[e^{-y(k+1)-y(n_i)}\right]\, {}_{k|}q^{(r)}_{x_r}\, P(K_{i,1}\geq n_i)$$

$$+ e_r e_i\, E\left[e^{-y(n_r)-y(n_i)}\right]\, P(K_{i,1}\geq n_i)\, P(K_{r,1}\geq n_r).$$

Proof: To prove (7), we start by expanding Z^2 into a double summation, that is:

$$(10)\qquad E[Z^2] = E\left[\left(\sum_{i=1}^{m}\sum_{j=1}^{c_i} Z_{i,j}\right)^2\right] = E\left[\sum_{i=1}^{m}\sum_{r=1}^{m}\sum_{j=1}^{c_i}\sum_{s=1}^{c_r} Z_{i,j}\, Z_{r,s}\right] = \sum_{i=1}^{m}\sum_{r=1}^{m}\sum_{j=1}^{c_i}\sum_{s=1}^{c_r} E\left[Z_{i,j}\, Z_{r,s}\right].$$

The expected value of $(Z_{i,j}\cdot Z_{r,s})$ depends on whether the two random variables concern the same life insured (when i=r and j=s) or not. We then have

$$(11)\qquad E[Z^2] = \sum_{i=1}^{m}\sum_{j=1}^{c_i}\sum_{s=1}^{c_i} E\left[Z_{i,j}\, Z_{i,s}\right] + \sum_{i=1}^{m}\sum_{\substack{r=1\\ r\neq i}}^{m}\sum_{j=1}^{c_i}\sum_{s=1}^{c_r} E\left[Z_{i,j}\, Z_{r,s}\right].$$

The triple summation is given by

$$(12)\qquad \sum_{i=1}^{m}\sum_{j=1}^{c_i}\sum_{s=1}^{c_i} E\left[Z_{i,j}\, Z_{i,s}\right] = \sum_{i=1}^{m}\sum_{j=1}^{c_i} E\left[Z_{i,j}^{\,2}\right] + \sum_{i=1}^{m}\sum_{j=1}^{c_i}\sum_{\substack{s=1\\ s\neq j}}^{c_i} E\left[Z_{i,j}\, Z_{i,s}\right].$$

Using A1 to A4, we have, for j≠s,

(13) $$E\left[z_{i,j}\ z_{i,s}\right] = E\left[E\left[z_{i,j}\ z_{i,s} \mid \{y(n)\}_{n=1}^{\omega-x}\right]\right] = E\left[E\left[z_{i,j} \mid \{y(n)\}_{n=1}^{\omega-x}\right] \cdot E\left[z_{i,s} \mid \{y(n)\}_{n=1}^{\omega-x}\right]\right]$$

$$= E\left[E\left[z_{i,1} \mid \{y(n)\}_{n=1}^{\omega-x}\right] \cdot E\left[z_{i,2} \mid \{y(n)\}_{n=1}^{\omega-x}\right]\right] = E\left[E\left[z_{i,1}\ z_{i,2} \mid \{y(n)\}_{n=1}^{\omega-x}\right]\right]$$

$$= E\left[z_{i,1}\ z_{i,2}\right].$$

Similarly, one can show that, for $i \neq r$,

(14) $$E\left[z_{i,j}\ z_{r,s}\right] = E\left[z_{i,1}\ z_{r,1}\right].$$

And equation (7) immediately follows by substituting (12), (13) and (14) into (11). Note that

(15) $$E\left[z_{i,1}\ z_{i,2}\right] = E\left[E\left[z_{i,1}\ z_{i,2} \mid K_{i,1}, K_{i,2}\right]\right].$$

Since, by assumption A1, $K_{i,1}$ and $K_{i,2}$ are independent, their joint probability function is the product of their probability functions. Equation (8) therefore follows immediately from equation (15). The derivation of (9) is similar. □

5. VARIANCE OF z.

We will use the variance of z as a measure of the riskiness of the portfolio. The variance of z may be obtained from

(16) $$V[z] = E[z^2] - E[z]^2.$$

This way of obtaining the variance is relatively easy to program to study the riskiness of any portfolio. If the riskiness of different portfolios needs to be determined, the program can be run for each portfolio. In practice, the actuary may want to know the impact on the riskiness of the portfolio of higher or lower sales of a particular contract or of introducing a new insurance product on the market. In this case, the different portfolios to be studied carry many of the same contracts. When using (16) to determine the riskiness of the different portfolios, one could redo the entire calculations. Although this is not very time consuming even for portfolios with many groups (200 to 300 groups), it is more efficient to store some intermediary results and use them in the various calculations. The next formula provides a useful way of summarizing the intermediary results needed for studying different portfolios. It is based on the fact that the variance of a sum is the sum of all variances and all covariances, that is:

$$(17) \qquad V[Z] = \sum_{i=1}^{m} V\left[\sum_{j=1}^{c_i} Z_{i,j}\right] + 2 \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} c_i c_r \operatorname{cov}(Z_{i,1}, Z_{r,1}).$$

Alternatively, using the coefficient of correlation between $Z_{i,1}$ and $Z_{r,1}$ which is

$$(18) \qquad \rho(Z_{i,1}, Z_{r,1}) = \frac{\operatorname{cov}(Z_{i,1}, Z_{r,1})}{\sqrt{V[Z_{i,1}]\ V[Z_{r,1}]}},$$

we can write the variance of Z as

$$(19) \qquad V[Z] = \sum_{i=1}^{m} V\left[\sum_{j=1}^{c_i} Z_{i,j}\right] + 2 \sum_{i=1}^{m-1} \sum_{r=i+1}^{m} c_i c_r\, \rho(Z_{i,1}, Z_{r,1}) \sqrt{V[Z_{i,1}]\ V[Z_{r,1}]}.$$

The useful intermediary results which concern the interaction between the different groups in the portfolio can then be summarized in the covariance terms in (17) or in the correlation terms in (19). Assuming that the variances for the different portfolios that the actuary wants to consider all require the same correlation coefficients, then (19) could be used to find them in an efficient way.

In particular, this approach should be considered when the company wants to evaluate the riskiness of its portfolio with more (or less) contracts of certain types (larger or smaller c_i's) or some new types of contracts (larger m).

For example, assume that a company is considering selling a new contract (group m+1) and expect to sell c_{m+1} of them. Then the riskiness of the new portfolio, say $V[Z^*]$, can be obtained by adding the variance of the additional group and its interaction components to the existing riskiness, $V[Z]$. Algebraically, we have

$$(20) \qquad V[Z^*] = V[Z] + V\left[\sum_{j=1}^{c_{m+1}} Z_{m+1,j}\right] + 2\sum_{i=1}^{m} c_i c_{m+1}\, \rho(Z_{i,1}, Z_{m+1,1}) \sqrt{V[Z_{i,1}]\, V[Z_{m+1,1}]}\,.$$

Unfortunately, although the intermediary results needed can be summarized in covariance or correlation terms, those intermediary results would need to be recalculated every time the mortality assumptions or the parameters of the rate of return process change as they are often quite sensitive to these assumptions. This can be seen in appendix A where some correlation coefficients are presented for different parameters.

It is of interest to consider the variance of the average cost per policy, Z/c, when the number of policies, c, becomes very large while keeping the proportion of policies in each group constant. Recall that the proportion of policies in group i is $p_i = c_i/c$.

It can be shown, under assumptions A1 to A4, that the limiting variance of the average cost per policy as c tends to infinity is:

(21) $$\lim_{c\to\infty} V\left[\frac{Z}{c}\right] = \sum_{i=1}^{m} p_i^2 \cdot E[Z_{i,1} \cdot Z_{i,2}] + 2\sum_{i=1}^{m} \sum_{r=i+1}^{m} p_i \cdot p_r \cdot E[Z_{i,1} \cdot Z_{r,1}] - \left(\sum_{i=1}^{m} p_i \cdot E[Z_{i,1}]\right)^2.$$

We get the above result by substituting (6) and (7) into (16), then dividing by c^2 and taking the limit as c tends to infinity.

This represents the minimum variance one could obtain by selling infinitely more policies in the same proportion that they are found in the existing portfolio. Since at that point there would be no variability due to the times of death of the policyholders, this limiting variance could be considered as a measure of the average investment risk per policy associated with the portfolio. Other ways of looking at the insurance risk (or mortality risk) and the investment risk (or interest risk) are presented in the next section.

6. Investment and Insurance Risks.

Suppose one would like the total riskiness of a finite portfolio, $V[Z]$, to be split into its two components that are the insurance risk and the investment risk. Here Z is a function of two sets of random variables. An efficient way of obtaining the variance of a random variable which is a function of other random variables is to use conditional moments. For example, if Z is a function of X and Y, then it is well known that $V[Z] = E[V[Z|X]] + V[E[Z|X]] = E[V[Z|Y]] + V[E[Z|Y]]$.

This provides us with two natural ways of writing the total variance of Z as the sum of two components. Since Z is a function of $\{K_{i,j}\}$ and $\{y(k)\}$, we may use

(22) $$V[Z] = E[V[Z|\{K_{i,j}\}]] + V[E[Z|\{K_{i,j}\}]],$$

or

(23) $$V[Z] = E[V[Z|\{y(k)\}]] + V[E[Z|\{y(k)\}]].$$

In (22), conditioning on $\{K_{i,j}\}$ corresponds to fixing the benefit cash flows of the portfolio. In the conditional expectation of Z given the benefit cash flows the averaging is done over the stochastic rates of return. So, $V[E[Z|\{K_{i,j}\}]]$ is a measure of the variability in Z caused by the stochastic benefit cash flows in a context where the effect of the random rates of return has been averaged out. This term is therefore a candidate for the insurance risk of the portfolio.

The other term in (22), $E[V[Z|\{K_{i,j}\}]]$, is an average over the benefit cash flows of the variability in Z caused by the stochastic rates of return. It is then a candidate for the investment risk.

In (23), the conditioning is on $\{y(k)\}$ and this corresponds to fixing the future rates of return. By a similar reasoning, we can see that $V[E[Z|\{y(k)\}]]$ could also represent the investment risk and $E[V[Z|\{y(k)\}]]$ could represent the insurance risk.

Note that an approach that one may be tempted to use and consisting of 1) fixing the benefit cash flows and studying the variability due to the rates of return as a measure of the *investment risk and 2) fixing the discounting factors and using the variability* due to the benefit cash flows as a measure of the insurance risk is inappropriate. It is essentially saying that $V[Z] = V[E[Z|\{K_{i,j}\}]] + V[E[Z|\{y(k)\}]]$ which is of course wrong.

Without further knowledge it is not clear which of (22) or (23) should be used to split the total riskiness of the portfolio into an investment risk and an insurance risk. We will see in the coming sections that (23) is perhaps more suitable.

Before looking more closely at each of the four conditional expressions appearing in (22) and (23), we will introduce some definitions and obtain some results that will be useful.

6.1 Cash Flows.

Let n be the maximum term for all the contracts in the portfolio ($n=\max_i n_i$) and CF_r be the random cash flow payable at time r with respect to the portfolio.

Let $D_{i,r}$ be the number of deaths between r-1 and r among the policyholders of group i and let S_{i,n_i} be the number of survivors at time n_i from this same group. Then we have that

$$(24) \qquad CF_r = \sum_{i=1}^{m} D_{i,r}\, b_i\, \mathbf{1}_{(n_i \geq r)} + \sum_{i=1}^{m} S_{i,n_i}\, e_i\, \mathbf{1}_{(n_i=r)}.$$

The indicator functions $\mathbf{1}_{(n_i \geq r)}$ and $\mathbf{1}_{(n_i=r)}$ take the value 1 if $n_i \geq r$ and $n_i=r$ respectively, and 0 otherwise. These functions are present to ensure that no cash flows are generated beyond the term n_i of each policy of the particular group i.

Since $\left\{D_{i,r}\right\}_{r=1}^{n_i} \cup \left\{S_{i,n_i}\right\}$ is multinomial $(c_i;\ q_{x_i}^{(i)},\ {}_{1|}q_{x_i}^{(i)}, \ldots, {}_{n_i-1|}q_{x_i}^{(i)},\ {}_{n_i}p_{x_i}^{(i)})$, we can use the results in section 4.4.2 of London (1988) to find the expected value of the cash flows and their autocovariance. The expected value of the cash flow due at time r is:

$$(25) \qquad E[CF_r] = \sum_{i=1}^{m} b_i c_i\ {}_{r-1|}q_{x_i}^{(i)}\, \mathbf{1}_{(n_i \geq r)} + \sum_{i=1}^{m} e_i c_i\ {}_{n_i}p_{x_i}^{(i)}\, \mathbf{1}_{(n_i=r)},$$

where

$$(26) \qquad {}_{n_i}p_{x_i}^{(i)} = 1 - \sum_{k=0}^{n_i-1} {}_{k|}q_{x_i}^{(i)},$$

The covariance between the cash flow due at time s and the one due at time r is, for $s<r\leq n$,

$$(27) \qquad \operatorname{cov}(CF_s, CF_r) = \sum_{i=1}^{m} b_i^2 \operatorname{cov}(D_{i,s}, D_{i,r}) \, \mathbf{1}_{(n_i \geq r)} + \sum_{i=1}^{m} b_i e_i \operatorname{cov}(D_{i,s}, S_{i,n_i}) \, \mathbf{1}_{(n_i = r)}$$

$$= \sum_{i=1}^{m} -b_i^2 \, c_i \; {}_{s-1|}q_{x_i}^{(i)} \; {}_{r-1|}q_{x_i}^{(i)} \, \mathbf{1}_{(n_i \geq r)} + \sum_{i=1}^{m} -b_i e_i c_i \; {}_{s-1|}q_{x_i}^{(i)} \; {}_{n_i}p_{x_i}^{(i)} \, \mathbf{1}_{(n_i = r)}, \qquad s<r.$$

and for $s=r\leq n$, $\operatorname{cov}(CF_r, CF_r) = V[CF_r]$, which is

$$(28) \qquad V[CF_r] = \sum_{i=1}^{m} b_i^2 \, V[D_{i,r}] \, \mathbf{1}_{(n_i \geq r)} + \sum_{i=1}^{m} e_i^2 \, V[S_{i,n_i}] \, \mathbf{1}_{(n_i = r)} + 2\sum_{i=1}^{m} b_i e_i \operatorname{cov}(D_{i,r}, S_{i,n_i}) \, \mathbf{1}_{(n_i = r)}$$

$$= \sum_{i=1}^{m} b_i^2 \, c_i \; {}_{r-1|}q_{x_i}^{(i)} \left[1 - {}_{r-1|}q_{x_i}^{(i)}\right] \mathbf{1}_{(n_i \geq r)} + \sum_{i=1}^{m} e_i^2 \, c_i \; {}_{n_i}p_{x_i}^{(i)} \left[1 - {}_{n_i}p_{x_i}^{(i)}\right] \mathbf{1}_{(n_i = r)}$$

$$+ 2\sum_{i=1}^{m} -b_i e_i c_i \; {}_{r-1|}q_{x_i}^{(i)} \; {}_{r}p_{x_i}^{(i)} \; \mathbf{1}_{(n_i = r)}.$$

6.2 Conditioning on the times of death.

Now that we have the necessary information about the cash flows we can look at the first suggested way of dividing the total riskiness into insurance and investment risks.

As already argued, the first term of (22) may be chosen to represent the investment risk while the second term could represent the insurance risk. Then the investment risk would be

$$(29) \qquad E[V[Z \mid \{K_{i,j}\}]] = E\left[V\left[\sum_{r=1}^{n} CF_r \, e^{-y(r)} \mid \{K_{i,j}\}\right]\right] = E\left[\sum_{r=1}^{n} \sum_{s=1}^{n} CF_r CF_s \operatorname{cov}\left(e^{-y(r)}, e^{-y(s)}\right)\right]$$

$$= \sum_{r=1}^{n} \sum_{s=1}^{n} E[CF_r \, CF_s] \operatorname{cov}\left(e^{-y(r)}, e^{-y(s)}\right).$$

We can find $E[CF_r CF_s]$ from (25) and (27) (or (28) if s=r) since

$$(30)\qquad E[CF_r\, CF_s] = \mathrm{cov}(CF_r,\, CF_s) + E[CF_r]\; E[CF_s].$$

And the insurance risk term in (22) would be

$$(31)\qquad V[E[Z \mid \{K_{i,j}\}]] = V\left[E\left[\sum_{r=1}^{n} CF_r\, e^{-y(r)} \mid \{K_{i,j}\}\right]\right] = V\left[\sum_{r=1}^{n} CF_r\, E\left[e^{-y(r)}\right]\right]$$

$$= \sum_{r=1}^{n} \sum_{s=1}^{n} E\left[e^{-y(r)}\right] E\left[e^{-y(s)}\right] \mathrm{cov}(CF_r,\, CF_s).$$

Finding the limiting variance of Z/c by dividing (22) by c^2 or dividing the two components (29) and (31) by c^2 and taking the limit as c tends to infinity would of course be equivalent to (21). Note that both the average investment risk per policy, (29) divided by c^2, and the average insurance risk per policy, (31) divided by c^2, depend on c, the size of the portfolio. As c tends to infinity, the average insurance risk tends to 0 whereas the average investment risk tends to the limiting variance of the average cost per policy.

6.3 Conditioning on the rates of return.

Another way of dividing the total variance of the portfolio into an insurance risk and an investment risk is proposed in (23). The first term would correspond to the insurance risk and the second to the investment risk. The insurance risk may be obtained as follows:

$$(32)\qquad E[V[Z\,|\,\{y(k)\}]] = E\left[V\left[\sum_{r=1}^{n} CF_r\, e^{-y(r)}\,|\,\{y(k)\}\right]\right] = E\left[\sum_{r=1}^{n}\sum_{s=1}^{n} e^{-y(r)-y(s)}\, \mathrm{cov}(CF_r,\, CF_s)\right]$$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n} E[e^{-y(r)-y(s)}]\, \mathrm{cov}(CF_r,\, CF_s).$$

The investment risk is given by:

$$(33)\qquad V[E[Z\,|\,\{y(k)\}]] = V\left[E\left[\sum_{r=1}^{n} CF_r\, e^{-y(r)}\,|\,\{y(k)\}\right]\right] = V\left[\sum_{r=1}^{n} e^{-y(r)}\, E[CF_r]\right]$$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n} E[CF_r]\, E[CF_s]\, \mathrm{cov}(e^{-y(r)},\, e^{-y(s)}).$$

Again, the limiting variance as c tends to infinity is the same as (21). The average insurance risk, (32) divided by c^2, tends to 0 as c tends to infinity. An interesting fact is that the average investment risk, (33) divided by c^2, not only tends to the limiting variance but is constant for portfolios of all sizes.

Since, for all c, $V[E[Z/c\,|\,\{y(k)\}]]$ is also the limiting $V[Z/c]$ as $c\to\infty$, adopting this definition for the average investment risk means that this risk cannot be diversified by selling more policies.

Because it seems natural to us that the average investment risk should not depend on the size of the portfolio, we suggest that the two risks, insurance and investment, be defined by (32) and (33) respectively.

Another important feature of (23) is that it allows us to determine the riskiness of the portfolio in a more efficient way. Suppose we have the expected value and autocovariance function of $e^{-y(t)}$, t=1,2,...,n and the same for CF_r, r=1,2,...,n for a portfolio. If we add a m+1[st] group to the portfolio, then generally the discounting factors would not be affected and only the cash flows' dynamic would change. The expected cash flows will become, where * indicates the new portfolio with m+1 groups,

$$(34) \qquad E[CF_r^*] = E[CF_r] + b_{m+1} \; {}_{r-1|}q_{x_{m+1}}^{(m+1)} \mathbf{1}_{(n_{m+1} \geq r)} + e_{m+1} \; {}_{r}p_{x_{m+1}}^{(m+1)} \mathbf{1}_{(n_{m+1}=r)}.$$

This is simply adding the expected cash flows of the new group to the existing group.

The new autocovariance function will become, for s<r,

$$(35) \qquad cov(CF_s^*, CF_r^*) = cov(CF_s, CF_r) - b_{m+1}^2 \; c_{m+1} \; {}_{s-1|}q_{x_{m+1}}^{(m+1)} \; {}_{r-1|}q_{x_{m+1}}^{(m+1)} \mathbf{1}_{(n_{m+1} \geq r)}$$

$$- b_{m+1} \; e_{m+1} \; {}_{r-1|}q_{x_{m+1}}^{(m+1)} \; {}_{r}p_{x_{m+1}}^{(m+1)} \mathbf{1}_{(n_{m+1}=r)}, \quad s<r.$$

And the variance will be

$$(36) \qquad V[CF_r^*] = V[CF_r] + b_{m+1}^2 \; c_{m+1} \; {}_{r-1|}q_{x_{m+1}}^{(m+1)} \left(1 - {}_{r-1|}q_{x_{m+1}}^{(m+1)}\right) \mathbf{1}_{(n_{m+1} \geq r)}$$

$$+ e_{m+1}^2 \; c_{m+1} \; {}_{n_i}p_{x_{m+1}}^{(m+1)} \left(1 - {}_{n_{m+1}}p_{x_{m+1}}^{(m+1)}\right) \mathbf{1}_{(n_{m+1}=r)} - 2 \; b_{m+1} e_{m+1} c_{m+1} \; {}_{r-1|}q_{x_i}^{(i)} \; {}_{r}p_{x_i}^{(i)} \; \mathbf{1}_{(n_{m+1}=r)}.$$

Using the same expected values and autocovariance function of the discounting factors, we can recalculate (32) and (33) in very little time. This would give us the insurance and investment risks respectively. The total riskiness of the portfolio is simply the sum of these two risks.

7. A MODEL FOR THE RATE OF RETURN.

The results about the riskiness of a portfolio of insurance contracts which have been presented so far are fairly general in terms of potential models for the rates of return. They only require that the expected value and autocovariance function of the discounting factors be known and finite.

Selecting a stochastic model for the rate of return is not an easy task. It depends on the investment strategy being followed and on the use one wants to make of it. For example, if all the assets are invested in short term fixed-income securities with no default risk, then a model such as the Cox-Ingersoll-Ross (1985) which does not allow negative rates might be acceptable.

When the goal is to find the market value of some security or to study different investment strategies (for optimal expected return under some constraints or for immunization) then term structure models would be suitable. Such models can be found in Ho and Lee (1986), Heath, Jarrow and Morton (1990) and Ritchken and Boenawan (1990) among others.

Here we consider that the market values have been determined and it is those values that are used to implement the investment strategy which has been adopted. The assets can be invested in many different financial instruments and trading may take place on a daily basis. Thus, negative returns are possible. Assets and liabilities may be partially matched.

Given this context, we choose to model the instantaneous rate of return by an Ornstein-Uhlenbeck process or equivalently, the model suggested by Vasicek (1977), for our illustrations. The main results concerning this process are recalled below for completeness. The reader is referred to Parker (1994a, section 6) for more details.

Let δ_t be defined such that

(37) $$d\delta_t = -\alpha(\delta_t-\delta)\, dt + \sigma\, dW_t$$

where α, σ and δ are constants with $\alpha\geq0$ and $\sigma\geq0$, and W_t is the standard Wiener process (see, for example, Arnold (1974, p.134)).

Note that the parameter α is a friction force bringing the process back towards its long term mean, δ. The diffusion coefficient is σ.

Then δ_t is normally distributed with mean

(38) $$E[\delta_t] = \delta + e^{-\alpha t}\,(\delta_0-\delta)$$

and autocovariance function

(39) $$\operatorname{cov}(\delta_s,\delta_t) = \frac{\sigma^2}{2\alpha}\, e^{-\alpha(t+s)}\,(e^{2\alpha s} - 1) \qquad s\leq t.$$

Consequently, its variance is

(40) $$V[\delta_t] = \operatorname{cov}(\delta_t,\delta_t) = \frac{\sigma^2}{2\alpha}\,(1 - e^{-2\alpha t}).$$

For estimating the parameters of the Ornstein-Uhlenbeck process from past data, the reader is referred to section 6.4 of Pandit and Wu (1988). Note that the past data used here is assumed to reflect the whole investment strategy of the company with respect to the portfolio including the allocation of assets and the degree of asset/liability matching.

To illustrate our point, consider the extreme case of a single benefit cash flow to be paid at time 5 with probability one. If the strategy is to buy (and hold) a 5-year zero coupon bond with no risk of default, then δ_0 and δ should be set at the equivalent instantaneous market yield to maturity of this bond and σ should be set at 0. This portfolio would be fully matched and there would be no investment risk. Our results would indeed produce an investment risk of zero since $\sigma=0$.

The function y(k), defined by (2), is an Ornstein-Uhlenbeck position process with mean

$$(41)\qquad E[y(k)] = E\left[\int_0^k \delta_s \, ds\right] = \int_0^k E[\delta_s] \, ds = \delta \, k + (\delta_0 - \delta) \left(\frac{1-e^{-\alpha k}}{\alpha}\right)$$

and autocovariance function

$$(42)\qquad \mathrm{cov}(y(s),y(k)) = \frac{\sigma^2}{\alpha^2} \min(s,k) + \frac{\sigma^2}{2\alpha^3} \left[-2+2e^{-\alpha s}+2e^{-\alpha k}-e^{-\alpha|k-s|}-e^{-\alpha(k+s)}\right].$$

So the variance of y(k) is

$$(43)\qquad V[y(k)] = \frac{\sigma^2}{\alpha^2} k + \frac{\sigma^2}{2\alpha^3} \left[-3+4e^{-\alpha k}-e^{-2\alpha k}\right].$$

Finally, it is well-known that the expected value of the discounting factor is

$$(44)\qquad E[e^{-y(k)}] = \exp\left\{-E[y(k)] + .5 \, V[y(k)]\right\}.$$

And it can be shown that the expected value of the product of 2 discounting factors is given by

$$(45)\qquad E[e^{-y(s)-y(k)}] = \exp\left\{-E[y(s)]-E[y(k)] + .5\left(V[y(s)]+V[y(k)] + 2 \, \mathrm{cov}(y(s),y(k))\right)\right\}.$$

The autocovariance function of the discounting factor is obtainable from

$$(46)\qquad \mathrm{cov}(e^{-y(s)}, e^{-y(k)}) = E[e^{-y(s)-y(k)}] - E[e^{-y(s)}] \, E[e^{-y(k)}].$$

With these explicit results, we can now use the results of the previous sections to analyze a portfolio.

8. ILLUSTRATIONS.

8.1 A Portfolio.

We will use the simple, hypothetical, portfolio described in table 1 in order to illustrate the results presented in this paper. Note that the face amounts are in thousands of dollars and so are the monetary values presented in this section.

Table 1. Description of the illustrative portfolio.

group i	age x_i	mortality table	face amount b_i	e_i	term n_i	number c_i
1	30	1	50	50	10	1000
2	35	1	100	50	5	2500
3	50	1	150	0	10	2000
4	30	2	50	0	10	1500
5	40	2	100	100	10	500
6	40	3	75	0	5	2500
7	45	4	25	0	5	3000
8	55	2	50	50	10	500

This portfolio of 13500 policies is divided into eight groups with similar characteristics. For example, the 1000 lives insured in group 1 all bought 10-year endowment insurance policies with a face amount of 50 (or $50000), they are all aged 30 and their mortality rates are those of mortality table 1.

The entire portfolio uses four distinct mortality tables. Note that we will use the same mortality table for the first three groups. In our notation, we will then have $q^{(1)}= q^{(2)}= q^{(3)}$. Similarly, we have $q^{(4)}= q^{(5)}= q^{(8)}$. The four mortality tables used here are, in order, the CA 1980-1982 male ultimate (see appendix B) times 1, .9, .8 and .75. One could think of these mortality tables as being male smoker, female smoker, male non-smoker and female non-smoker tables.

8.2 Moments of Z.

The first two moments of Z were calculated with the Ornstein-Uhlenbeck process as the model for the rate of return. Its parameters were arbitrarily chosen to be δ=.06, δ_0=.08, α=.1 and σ=.01.

Some useful intermediary results for the portfolio under consideration are displayed in Table 2. Part A) presents some results for each group. Part B) presents some results about the interaction between the various groups.

Table 2. Summary of intermediary results.

A) Each group.

Group i	$E[Z_{i,1}]$	$E[Z_{i,1}^2]$	$E[Z_{i,1}\ Z_{i,2}]$
1	24.5202	613.127	611.192
2	34.6825	1224.47	1206.31
3	9.3730	951.585	88.200
4	.4627	16.016	.215
5	49.2734	2480.68	2467.23
6	.6463	38.632	.418
7	.3409	6.789	.116
8	25.4843	672.588	658.501

B) Interaction between groups, $E[Z_{i,1} \cdot Z_{r,1}]$.

i	r							
	1	2	3	4	5	6	7	8
1	—	855.40	231.63	11.428	1228.0	15.892	8.3822	634.37
2	855.40	—	326.12	16.096	1718.8	22.448	11.840	888.67
3	231.63	326.12	—	4.3529	465.42	6.0673	3.2002	240.59
4	11.428	16.096	4.3529	—	22.963	.29949	.15797	11.871
5	1228.0	1718.8	465.42	22.963	—	31.934	16.844	1274.6
6	15.892	22.448	6.0673	.29949	31.934	—	.22048	16.513
7	8.3822	11.840	3.2002	.15797	16.844	.22048	—	8.7101
8	634.37	888.67	240.59	11.871	1274.6	16.513	8.7101	—

The values in Table 2 were then used to compute the first two moments of Z using the results of sections 3 and 4. Table 3 presents the first two moments and the standard deviation of the average cost per policy, Z/c, for our illustrative portfolio. It also presents the corresponding results for other portfolios differing only in size (c). Note that they all have the same proportion of contracts in each of the eight groups.

Table 3. Moments of Z/c for different sizes of portfolio.

c	$E[Z/c]$	$E[(Z/c)^2]$	$sd[Z/c]$
10	12.6432	175.094	3.9042
100	12.6432	162.247	1.5476
1000	12.6432	160.962	1.0537
13500	12.6432	160.830	.9890
27000	12.6432	160.824	.9863
67500	12.6432	160.821	.9847
infinity	12.6432	160.819	.9836

From Table 3, it appears that the average cost per policy of our portfolio of size 13500 should have a distribution relatively close to the limiting one (since the standard deviation of .9890 is relatively close to the limiting value of .9836). This suggests that the insurance risk of the portfolio is small compared to its investment risk. We will use this fact to approximate the distribution of Z in section 8.4. Note that the limiting distribution of Z/c is that of a weighted sum of correlated lognormal variables.

The next table presents the correlation coefficients between the present value of the benefit of two contracts of different groups. These values could be used along with those of Table 2A) to obtain Table 3.

Table 4. Correlation between groups, $\rho(Z_{i,1} \cdot Z_{r,1})$.

i	r							
	1	2	3	4	5	6	7	8
1	—	.31053	.01774	.00601	.78993	.00210	.00265	.57209
2	.31053	—	.00764	.00260	.29349	.00114	.00144	.21493
3	.01774	.00764	—	.00014	.01675	.00005	.00007	.01219
4	.00601	.00260	.00014	—	.00567	.00002	.00002	.00413
5	.78993	.29349	.01675	.00567	—	.00198	.00251	.53982
6	.00210	.00114	.00005	.00002	.00198	—	.00001	.00146
7	.00265	.00144	.00007	.00002	.00251	.00001	—	.00184
8	.57209	.21493	.01219	.00413	.53982	.00146	.00184	—

8.3 Investment and Insurance Risks.

We now consider equations (22) and (23) as ways of dividing the total riskiness of the portfolio into two components, namely the insurance and investment risks. The results in Table 5 summarize the two approaches for portfolios of different sizes. Note that Table 5 presents results for the average cost per policy, Z/c, instead of Z.

Table 5. Insurance, Investment and Total Riskiness of portfolios.

Number of contracts	$E[V[\frac{Z}{c}\|\{K_{i,j}\}]]$	$V[E[\frac{Z}{c}\|\{K_{i,j}\}]]$	$V[\frac{Z}{c}]$	$E[V[\frac{Z}{c}\|\{y(k)\}]]$	$V[E[\frac{Z}{c}\|\{y(k)\}]]$
13500	.96761444	.01052335	.97813780	.01057409	.96756371
27000	.96758908	.00526168	.97285075	.00528704	.96756371
67500	.96757385	.00210467	.96967853	.00211482	.96756371
infinity	.96756371	0	.96756371	0	.96756371

The second and third columns correspond to equation (22). For each value of c, they add up to $V[Z/c]$. As indicated earlier, $E[V[\frac{Z}{c}|\{K_{i,j}\}]]$, which would correspond to the investment risk varies with c. What would be the insurance risk, $V[E[\frac{Z}{c}|\{K_{i,j}\}]]$, tends to

0 as c tends to infinity. Here the insurance risk is relatively small compared with the investment risk.

The last two columns correspond to equation (23). They also add up to $V[Z/c]$. The investment risk, $V[E[\frac{Z}{c}|\{y(k)\}]]$, is the same for all values of c and the insurance risk, $E[V[\frac{Z}{c}|\{y(k)\}]]$, tends to 0 as the size of the portfolio approaches infinity.

Using (23), we find that the investment risk of the portfolio is 176 338 486 (i.e. $.96756371 \times 13500^2$). The insurance risk is $.01057409 \times 13500^2$ or 1 927 112.

As mentioned earlier, a method which is occasionally encountered to study the riskiness of a portfolio consists of *i)* fixing the interest process and estimating the variance of Z (or Z/c) and adding this variance to *ii)* the variance of Z (or Z/c) when the times at death are fixed. This essentially considers the sum of $V[E[\frac{Z}{c}|\{K_{i,j}\}]]$ and $V[E[\frac{Z}{c}|\{y(k)\}]]$ as the total riskiness which is unacceptable since the two risk components do not add up to the total riskiness of the portfolio. For our illustrative portfolio, the difference between this approach and (22) or (23) may be considered small but for other portfolios (particularly small ones) the difference could be quite significant.

8.4 Limiting Distribution of Z.

Assuming that the insurance risk is negligible, one is left with the investment risk of the portfolio which can be studied by considering the expected cash flows of the portfolio (see, for example, Frees (1990, proposition 5) and Parker (1994b, section 3). We will now study the moments and the distribution of the present value of those expected cash flows. Since the insurance risk of the illustrative portfolio may be considered small, the following results may be considered good approximations of the corresponding results for the present value of the benefits of the portfolio.

Table 6 presents the expected cash flows for our portfolio. They were obtained by using (25). Note that here n is equal to 10.

Table 6. Expected Cash Flows, $E[CF_r]$.

Time, r	1	2	3	4	5	6	7	8	9	10
$E[CF_r]$	3297	3591	3924	4290	128575	3651	3965	4308	4670	124233

Let ζ be the discounted value of the expected cash flows generated by the portfolio. Then, ζ is given by

$$(47) \qquad \zeta = \sum_{r=1}^{n} E[CF_r]\, e^{-y(r)}.$$

The first two moments about the origin of the present value of these ten cash flows may be obtained from equations (3) and (5) of Parker (1993) which are:

$$(48) \qquad E[\zeta] = E\left[\sum_{r=1}^{10} E[CF_r]\exp\{-y(r)\}\right] = \sum_{r=1}^{10} E[CF_r]\, E\left[\exp\{-y(r)\}\right]$$

and

$$(49) \qquad E[\zeta^2] = E\left[\left(\sum_{r=1}^{10} E[CF_r]\exp\{-y(r)\}\right)^2\right] = E\left[\sum_{r=1}^{10}\sum_{s=1}^{10} E[CF_r]\, E[CF_s]\exp\{-y(r)-y(s)\}\right]$$

$$= \sum_{r=1}^{10}\sum_{s=1}^{10} E[CF_r]\, E[CF_s]\, E\left[\exp\{-y(r)-y(s)\}\right].$$

Note that (48) is also c $E[Z/c]$ and that $E[\zeta^2]-E[\zeta]^2$ gives the same result as (21) and (33).

The expected value is 170684 and the standard deviation is 13279. These values can be checked using the entries of table 3. The expected value is equal to 13500x12.6432 and the standard deviation may be obtained by multiplying the number of policies by the limiting standard deviation of the average cost per policy, that is 13500x.9836. Since the exact standard deviation of Z for our portfolio is 13352 (13500x.9890), the standard deviation is underestimated by 73 when the expected cash flows are studied instead of the random cash flows.

Using the method described in section 5 of Parker (1993), the cumulative distribution of the discounted value of all future expected cash flows was approximated.

Basically, the cumulative distribution function of ζ is obtained from

$$(50) \qquad F_{\zeta}(z) = \int_{-\infty}^{\infty} g_n(z,y)\, dy,$$

where the function $g_n(z,y)$ is approximated by recursion using the integral equation

$$(51) \qquad g_r(z,y) = \int_{-\infty}^{\infty} f_{y(r)}(y \mid y(r-1)=x)\; g_{r-1}\left(z - E[CF_r]\, e^{-y},\, x\right) dx.$$

The starting value being

$$(52) \qquad g_1(z,y) = \begin{cases} \phi\left(\dfrac{y - E[y(1)]}{(V[y(1)])^{.5}}\right) & \text{if } z \geq E[CF_1]e^{-y} \\ 0 & \text{otherwise} \end{cases}$$

where $\phi(\cdot)$ is the probability density function of a zero mean and unit variance normal random variable.

In Figure 1, the cumulative distribution function of ζ is illustrated. The numerical results by using the trapezoidal rule with 41 points to approximate the integral in (51) and using linear interpolation to obtain the values of g_{r-1} at the required 41 points.

Figure 1. Cumulative Distribution of ζ, $F_\zeta(z)$.

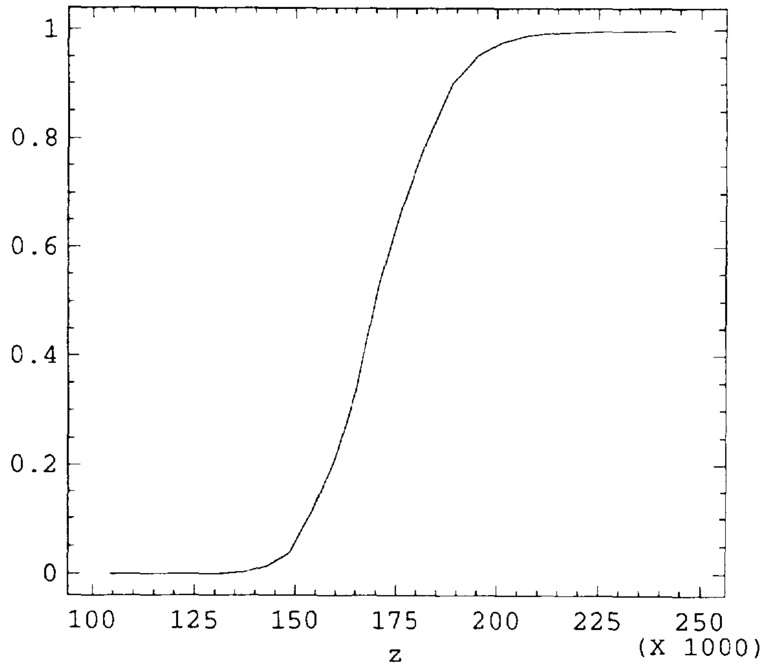

Table 7 presents the probability that the discounted value of the expected cash flows will be smaller than a given value z, $F_\zeta(z)$, for different values of z.

Table 7. Distribution of ζ, the discounted value of the expected cash flows.

z	$F_\zeta(z)$	z	$F_\zeta(z)$
131952.58	0.000559	188942.70	0.900325
137485.60	0.003562	195029.02	0.951095
143018.62	0.013687	201115.34	0.976627
148551.64	0.037561	207201.67	0.989101
154084.67	0.112221	213287.99	0.994144
159617.69	0.207236	219374.31	0.996282
165150.71	0.337487	225460.64	0.997338
170683.73	0.528920	231546.96	0.997950
176770.05	0.680995	237633.28	0.998441
182856.37	0.800400	243719.60	0.998882

The values in Table 7 can be used to estimate the contingency margin that should be added to each single premium in order for the portfolio to be profitable with a given probability. For example, the net single premium of each policy in our portfolio could be loaded by about 14.3% ($\frac{195029}{170683}$ - 1) if we want it to be profitable with a probability of at least .95. Recall that the 14.3% slightly underestimates the true contingency margin that should be added, because the expected cash flows were used instead of the random cash flows. It should also be mentioned that the company could decide to use a smaller premium loading and put aside, perhaps in some kind of appropriated surplus, the difference between 195029 and the total premiums collected from the policyholders.

8.5 Adding some contracts.

Suppose the company wants to sell two more groups of contracts. The characteristics and the sale's forecast are as follows:

Table 8. Additional contracts to the portfolio.

group i	age x_i	mortality table	face amount b_i	e_i	term n_i	number c_i
9	30	4	100	0	20	1000
10	35	3	50	25	10	1000

The moments of Z^* (or Z^*/c) can be obtained from those of Z (or Z/c) and the additional results that are presented in Table 9.

Table 9. Summary of additional intermediary results.

A) Each additional group.

Group i	$E[Z_{i,1}]$	$E[Z_{i,1}^2]$	$E[Z_{i,1}\ Z_{i,2}]$
9	1.6751	85.938	2.861
10	12.5537	168.103	160.138

B) Correlation between groups, $\rho(Z_{i,1}, Z_{r,1})$.

	r									
i	1	2	3	4	5	6	7	8	9	10
9	.02211	.00771	.00045	.00015	.02086	.00005	.00007	.01507	—	.01188
10	.45002	.16724	.00955	.00323	.42461	.00113	.00143	.30754	.01188	—

The expected value of Z^* was obtained by adding $c_i\ E[Z_{i,1}]$, i=9,10, to the expected value of Z found in Table 3. The variance of Z^* was obtained from (19). The moments of Z^*/c are presented in the next table.

Table 10. Moments of Z^*/c.

c	$E[Z^*/c]$	$E[(Z^*/c)^2]$	$sd[Z^*/c]$
15500	11.9298	143.273	.9756
infinity	11.9298	143.265	.9712

The expected cash flows of the new portfolio (see Table 11) were obtained by adding the expected cash flows of the two additional groups of contracts to those of the old portfolio (see Table 6). Note that with the two additional groups, the maximum term becomes n=20.

Table 11. Expected Cash Flows, $E[CF^*_r]$.

Time, r	1	2	3	4	5	6	7	8	9	10
$E[CF^*_r]$	3457	3755	4094	4469	128765	3854	4184	4546	4930	149069
Time, r	11	12	13	14	15	16	17	18	19	20
$E[CF^*_r]$	165	181	200	222	246	273	303	336	372	410

The average riskiness of the new portfolio and its breakdown into average investment and insurance risks according to (22) are

(53) $$V[Z] = E[V[Z|\{K_{i,j}\}]] + V[E[Z|\{K_{i,j}\}]]$$
$$.95171107 = .94335618 + .00835489.$$

And according to (23), they are

(54) $$V[Z] = E[V[Z|\{y(k)\}]] + V[E[Z|\{y(k)\}]]$$
$$.95171107 = .94331078 + .00840029.$$

The cumulative distribution function of the discounted value of the expected cash flows of the old and new portfolios are illustrated in Figure 2.

Figure 2. Cumulative Distribution of ζ and ζ^*.

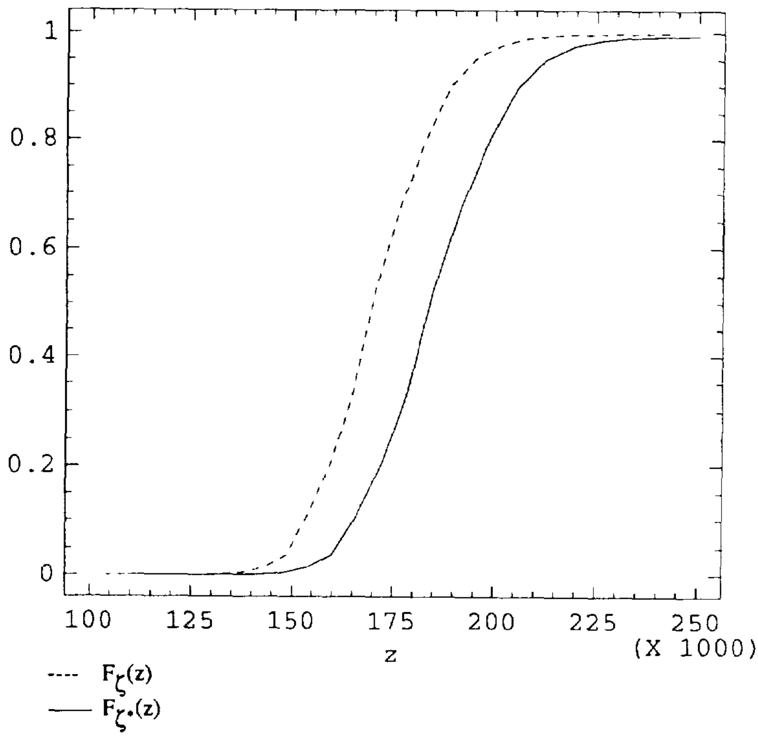

Table 12 presents the probability that the discounted value of the expected cash flows will be smaller than a given value z, $F_{\zeta^*}(z)$, for different values of z.

Again, if we want the new portfolio to be profitable with a probability of about .95, we can use the values in Table 12 to determine the contingency margin that would be needed. We could load the single premium of each contract by 14.9% $\left(\frac{212512}{184913} - 1\right)$. Alternatively, we could load the premiums by a lesser percentage and set aside the difference between 212512 and the premiums collected.

Table 12. Distribution of ζ^*, the discounted value of the expected cash flows*.

z	$F_{\zeta^*}(z)$	z	$F_{\zeta^*}(z)$
141004.37	0.000501	205612.19	0.898259
147276.97	0.003352	212512.05	0.950357
153549.57	0.013155	219411.91	0.975756
159822.18	0.037652	226311.78	0.986637
166094.78	0.109458	233211.64	0.991023
172367.39	0.206725	240111.51	0.992864
178639.99	0.336306	247011.37	0.993777
184912.59	0.524506	253911.23	0.994298
191812.46	0.679079	260811.10	0.994748
198712.32	0.800788	267710.96	0.995563

Adding the two new groups to the existing portfolio would increase the average riskiness per policy from 14.3% to 14.9%. In dollar amount, the contingency margin required for a 95% chance of solvency would increase to 27599 (212512-184912) from 24345 (195029-170684). Recall that these margins underestimate the true ones because we are assuming no insurance risks when in fact there are some.

9. Remarks and Conclusion.

In this paper, we have presented a method for finding the expected value and variance of Z, the present value of the benefits of a portfolio containing different kinds of insurance contracts.

Using correlation coefficients between the present values of the benefits of pairs of contracts, we derived a way of updating the first two moments of Z which can be used when a few group sizes are changed or when some new types of contracts are added to the portfolio.

Two expressions were suggested from splitting the total riskiness of a portfolio (measured by the variance) into an insurance risk and an investment risk. It was suggested that equation (23), the one conditioning on the process for the rate of return, should be preferred. It appears to be more consistent since its average is constant over all portfolios differing only in sizes. The multinomial distribution of the random cash flows generated by the portfolio was used to obtain the two risk components.

Finally, a thorough analysis of a simple portfolio was done to illustrate the results and methods found in the paper including some potential applications.

The methodology used here could be extended to portfolios of level premium insurance contracts and to portfolios of annuities. Another extension would be to include lapses and expenses in the model. For this, the model would need to consider some correlation because lapses and expenses with the process for the rate of return. In this case, beyond the problem of modeling the correlations just mentioned, finding the moments is likely to be considerably harder and approximating the distribution of Z would be even more difficult.

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APPENDIX A

Correlation between insurance contracts.

In this appendix we present the correlation coefficients between the discounted values of the benefits of two insurance contracts. We will consider 5 different contracts, they are

Table A.1. Five Insurance Contracts.

contract	age at issue	type of contract
1	30	5-year temporary
2	30	25-year temporary
3	50	5-year temporary
4	30	5-year endowment
5	30	25-year endowment

The mortality table used is the CA80-82, male (see appendix B).

Two important points are to be taken from the following tables of correlation coefficients. Firstly, they vary with the mortality rates and with the parameters of the process for the rate of return. Secondly, different correlation coefficients vary differently to the same change in mortality rates or parameters of the process for the rate of return.

A change in the mortality rates can be studied by looking at the correlation coefficients between contract 2, 4 or 5 with contract 1 and the corresponding correlation coefficients with contract 3.

Note that for σ tending to 0, all correlation coefficients would tend to 0. This is because we have assumed mutual independence between the times at death of the different lives insured. If σ is 0, we have no more randomness in the future rates of return and therefore no correlation between the discounted values of the benefits for two different contracts.

Table A.2. Correlation coefficients between $z_{i,1}$ and $z_{r,1}$.

i) Ornstein-Uhlenbeck with δ=0.06, δ_0=0.08, α=0.1 and σ=0.01

i \ r	1	2	3	4	5
1	—	0.000065	0.000012	0.001990	0.000720
2	0.000065	—	0.000167	0.031798	0.034156
3	0.000012	0.000167	—	0.005119	0.001861
4	0.001990	0.031798	0.005119	—	0.358930
5	0.000720	0.034156	0.001861	0.358930	—

ii) Ornstein-Uhlenbeck with δ=0.06, δ_0=0.04, α=0.1 and σ=0.01

i \ r	1	2	3	4	5
1	—	0.000072	0.000012	0.002120	0.000837
2	0.000072	—	0.000187	0.035532	0.042340
3	0.000012	0.000187	—	0.005442	0.002160
4	0.002120	0.035532	0.005442	—	0.415199
5	0.000837	0.042340	0.002160	0.415199	—

iii) Ornstein-Uhlenbeck with δ=0.06, δ_0=0.08, α=0.5 and σ=0.01

i \ r	1	2	3	4	5
1	—	0.000012	0.000005	0.001168	0.000159
2	0.000012	—	0.000032	0.009213	0.004248
3	0.000005	0.000032	—	0.002987	0.000407
4	0.001168	0.009213	0.002987	—	0.119874
5	0.000159	0.004248	0.000407	0.119874	—

iv) Ornstein-Uhlenbeck with δ=0.06, δ_0=0.08, α=0.1 and σ=0.03

i \ r	1	2	3	4	5
1	—	0.000540	0.000106	0.006233	0.002348
2	0.000540	—	0.001395	0.092933	0.151530
3	0.000106	0.001395	—	0.016028	0.006072
4	0.006233	0.092933	0.016028	—	0.409221
5	0.002348	0.151530	0.006072	0.409221	—

APPENDIX B

Mortality Table CA80-82, male

x	q_x	x	q_x	x	q_x	x	q_x
0	.01092						
1	.00081	26	.00143	51	.00694	76	.06442
2	.00063	27	.00139	52	.00768	77	.07002
3	.00048	28	.00136	53	.00848	78	.07607
4	.00047	29	.00134	54	.00933	79	.08251
5	.00039	30	.00132	55	.01026	80	.08941
6	.00030	31	.00132	56	.01127	81	.09683
7	.00022	32	.00134	57	.01239	82	.10483
8	.00019	33	.00139	58	.01360	83	.11338
9	.00019	34	.00145	59	.01488	84	.12243
10	.00022	35	.00153	60	.01628	85	.13203
11	.00027	36	.00163	61	.01781	86	.14227
12	.00035	37	.00175	62	.01951	87	.15319
13	.00049	38	.00189	63	.02138	88	.16475
14	.00069	39	.00205	64	.02339	89	.17692
15	.00092	40	.00223	65	.02556	90	.18975
16	.00112	41	.00245	66	.02790	91	.20332
17	.00128	42	.00271	67	.03046	92	.21767
18	.00139	43	.00301	68	.03317	93	.22325
19	.00147	44	.00334	69	.03601	94	.22003
20	.00153	45	.00372	70	.03907	95	.22234
21	.00157	46	.00414	71	.04243	96	.24450
22	.00158	47	.00461	72	.04617	97	.30086
23	.00157	48	.00512	73	.05024	98	.41245
24	.00153	49	.00567	74	.05460	99	.56973
25	.00148	50	.00628	75	.05930	100	.74112
						101	.89506
						102	1.00000

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