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An Investment Actuary's Approach to ALM

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Introduction

This paper is to some extent a sequel to my previous paper "A Bond Manager's Method for ALM" published in Actuarial Research Clearing House Volume 1993.3. It borrows from that paper the concept of benchmark weights for measuring the sensitivity of the present value of a set of cash flows to changes in interest rates. However, a completely different approach to calculating benchmark weights is presented.

As a first step, a new process is introduced for computing spot rates from the prices of a set of benchmark bonds subject to a smoothness constraint. This process leads directly to an equation for benchmark weights which forms the basis for a relatively simple yet potentially very powerful new approach to asset-liability matching.

Computation of Spot Rates - A Textbook Example

In order to discount a single cash flow at a fixed point in time, a spot rate is required. Spot rates can in theory be obtained by observing market quotations for zero coupon bonds. However, the trading volume for such bonds is many times smaller than for regular coupon bonds so the yields observed are not necessarily representative of the market as a whole. Furthermore, zero coupon bonds are rarely available at all of the durations for which spot rates are required. For these reasons, it is common practice to compute spot rates from a series of yields on regular coupon bonds. In theory, this is a relatively simple task. However, most approaches presented in textbooks assume that bonds are available at all of the terms required to do the calculations. To calculate spot rates for each month for the next 30 years, 360 regular coupon bonds are required with maturities from 1 month to 30 years. Assuming all of this data is available, the spot rates can easily be calculated with the following formulas :

$$P_j = \sum_{t=1}^{m} v_t \times {}_jb_t \qquad (1)$$

Where :

- P_j is the quoted price of bond j (including accrued coupon).
- m is the number of durations for which spot rates are required. m must equal the number of bonds available.
- v_t is the discount factor for duration t.
- ${}_jb_t$ is the cash flow at time t on bond j. These cash flows include both coupon and maturity payments and are, of course, zero after the maturity date.

$$i_t = \left(\frac{1}{v_t}\right)^{\frac{1}{t}} - 1 \quad (2)$$

Where :

i_t is the spot rate at duration t.

v_t is the discount factor for duration t.

The 360 equations represented by formula 1 can easily be solved by starting with the bond that has the shortest term and proceeding to the longer term bonds.

Unfortunately, no market in the world has reliable quotations for 360 bonds all maturing exactly one month apart. To cope with this problem, most processes for calculating spot rates work with only a few benchmark bonds and use an interpolation method to generate the additional data required for formulas (1) and (2). There are four problems with this :

- Interpolation is often more of an art than a science. An apparently well chosen method may work quite well with one set of data and poorly with another. Since spot rates are likely to be required on an on-going basis, the method selected will be applied to several different sets of data. What often happens in practice is that after several months of working with a given interpolation method, anomalies arise because of a significant change in the shape of the yield curve. Because the method has already proven to be reliable, such anomalies can go undetected for long periods.

- It is not always clear how an interpolation between two bonds should be done. Should it be based on term to maturity, average term or duration. This can be significant if an interpolation is being done which involves bonds trading both above and below par by substantial amounts. Again anomalies can result.

- Even quotes for benchmark bonds are not 100% reliable. For example, a few large trades late in the day can result in closing prices on one or two benchmark bonds which are not consistent with the other benchmarks. If an interpolation process does not include an element of smoothing, it will generally have difficulty dealing with this kind of situation with the result that interpolated values may contain obvious anomalies.

- The spot rate at any given duration is a function of all of the previously determined rates. If the shorter term rates contain anomalies, their cumulative effect can be quite significant on the long duration spot rates.

To illustrate the difficulties encountered in practice, let's review the method illustrated in Appendix C of my previous paper and apply it to a practical situation. The key elements of the method are as follows :

- Cubic polynomials are fitted to the data in each of 3 areas of the yield curve : up to 1 year, 1 year to 7 years, and beyond 7 years. In each area, 4 bonds are available, so the polynomials can be defined to exactly reproduce the known values. These interpolating polynomials were used to provide coupon rates and yields at six-month intervals for the 30-year period.

- Given the interpolated coupon rates and yields, the prices were calculated.
- Formulas (1) and (2) were applied to determine spot rates at six-month intervals.

This methodology was applied to the following set of benchmark bonds :

Coupon	Benchmark	Price	Yield
4.60%	1 month	100.053	4.00%
4.75%	3 months	100.068	4.50%
5.00%	6 months	100.000	5.00%
5.40%	1 year	99.904	5.50%
6.20%	3 years	100.542	6.00%
7.25%	5 years	100.000	7.25%
7.50%	7 years	100.000	7.50%
7.90%	10 years	100.000	7.90%
8.55%	20 years	100.000	8.55%
8.60%	30 years	100.000	8.60%

TABLE 1 - Benchmark Bonds

(Note that all yields are semi-annual. Coupons are semi-annual except for 1 and 3 month maturities.)

A graph of the resulting spot rates follows :

230

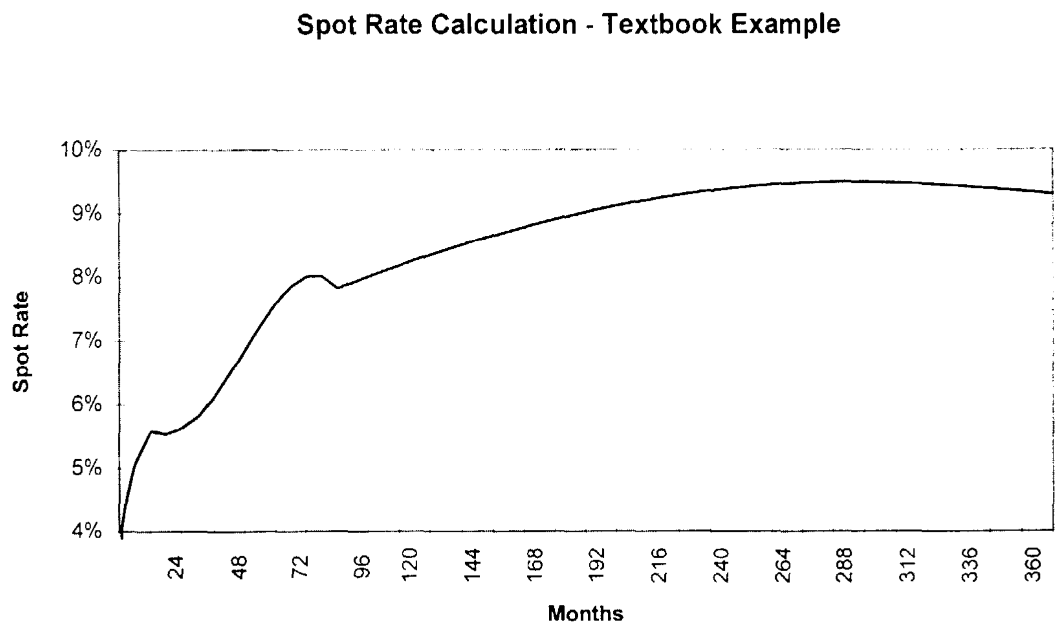
Spot Rate Calculation - Textbook Example
10%
9%
8%
7%
6%
5%
4%
Spot Rate
24
48
72
96
120
144
168
192
216
240
264
288
312
336
360
Months

Figure 1

Examples of the anomalies mentioned above appear quite clearly in the calculated spot rates. The small dip in the short term rates and the bump around 7 years do not seem reasonable given the original data. Of course, efforts could be made to improve the interpolation method. However, there is no guarantee that any interpolation method will necessarily produce a reasonably smooth curve of spot rates. Whatever method is used to interpolate coupons and yields, the resulting spot rates will always need to be checked to make sure that they appear reasonable. The following section presents a method for incorporating a mathematical form of the smoothness requirement into the process of computing the spot rates. Using this approach, smoothness of the results can be better controlled.

Computation of Spot Rates - An Investment Actuary's Approach

The general problem of determining spot rates from a set of benchmark bonds involves two constraints :

- The prices of the benchmark bonds calculated by discounting their cash flows using the computed spot rates must be reasonably close to the quoted prices. It would be convenient if the calculated prices were exactly equal to the quoted prices. However that may not always be possible in view of the second constraint. Furthermore, quoted prices are usually for a bond with a par value of 100 and contain only 3 digits after the decimal. Therefore, it would be unreasonable to require that the calculated price be precisely equal to the quoted price.

- As discussed already, the resulting curve of spot rates should appear reasonably smooth.

These constraints are very much like those encountered when constructing a mortality table. In fact, several of the methods used for graduating mortality data can be applied to this problem. In the body of this paper, we will concentrate only on the Whitaker-Henderson method. A second approach is outlined but not studied in depth in Appendix A.

In order to apply the Whitaker-Henderson method, we need to define both measures of fit and smoothness. The measure of fit can be defined as follows :

$$F = \sum_{j=1}^{n} (E_j - P_j)^2 \qquad (3)$$

Where :

- n is the number of benchmark bonds being used.
- E_j is the estimated price of bond j using the calculated spot rates.
- P_j is the quoted price of bond j (including accrued coupon).

The measure of smoothness is a little more difficult. If the smoothness measure is defined in terms of the spot rates themselves, a direct solution which follows the Whitaker-Henderson approach simply is not possible. Instead, we will define smoothness in terms of the discount factors associated with the spot rates. If the discount factors are smooth, clearly the spot rates to which they are directly related must also be reasonably smooth. The discount factors can be defined as follows :

$$v_t = \frac{1}{(1+i_t)^t} \qquad (4)$$

Where :

v_t is the discount factor for duration t.

i_t is the spot rate for duration t.

The measure of smoothness is defined as it is in the Whitaker-Henderson method using a forward difference operator :

$$S = \sum_{t=1}^{m-z} (\Delta^z v_t)^2 \qquad (5)$$

Where:

m is the number of durations for which spot rates are required.

z is the degree of the difference operator to be used.

v_t is the discount factor for duration t as defined above.

As in the Whitaker-Henderson method, we will seek to minimize the combination of these two measures with a parameter, h, controlling how much emphasis is placed on smoothness versus fit. This leads to the following equation to be minimized :

$$M = \sum_{j=1}^{n} (E_j - P_j)^2 + h \sum_{t=1}^{m-z} (\Delta^z v_t)^2 \qquad (6)$$

In order to perform the minimization, we need to replace E_j with the formula used to calculate the estimated price. This equation is simply the present value of the bond cash flows as follows :

$$E_j = \sum_{t=1}^{m} v_t \times {}_j b_t \qquad (7)$$

Where :

$_jb_t$ is the cash flow at time t on bond j. These cash flows include both coupon and maturity payments and are, of course, zero after the maturity date.

Substituting equation (7) into equation (6) gives us the following function to minimize :

$$M = \sum_{j=1}^{n} [(\sum_{t=1}^{m} (v_t \times {}_jb_t)) - P_j]^2 + h\sum_{t=1}^{m-z} (\Delta^z v_t)^2 \tag{8}$$

This equation can be re-written in matrix form as :

$$M = (BV-P)^T (BV-P) + h\ V^T K_z^T K_z\ V \tag{9}$$

Where :

- B is a matrix containing the cash flows, $_jb_t$, defined previously, with n rows, one for each benchmark bond, and m columns, 1 for each duration.
- V is a vector containing the discount factors, v_t, defined previously, with m rows.
- P is a vector containing the benchmark bond prices (including accrued coupon), P_j, defined previously, with n rows.
- T indicates that the matrix to the left is to be transposed.
- h is the parameter which controls the emphasis on smoothness versus fit.
- K_z is a matrix with m-z rows and m columns containing in the first row the binomial coefficients for a zth difference formula followed by a series of zeros. In each successive row, the coefficients are moved to the right one position and a zero inserted.

The values of v_t which minimize this function can be found by differentiating with respect to each of the values, v_t, and setting the m partial derivatives obtained equal to zero. This process is a little messier than for a standard Whitaker-Henderson graduation. However, if applied to equation (8), it will lead to the following system of equations written in matrix form :

$$B^T (BV-P) + h\ K_z^T K_z\ V = 0 \tag{10}$$

Where :

0 is interpreted to be a vector with m rows containing only zeros.
A factor of 2 has been dropped from the left side of the equation.

With a little algebraic manipulation, the following solution for V is obtained :

$$(B^T B + h K_z^T K_z)^{-1} B^T P = V \tag{11}$$

Which can be written in a simplified form as :

$$N P = V \quad with \quad N = (B^T B + h K_z^T K_z)^{-1} B^T \tag{12}$$

Once the discount factors have been calculated, the spot rates can easily be computed using equation (2).

Note that N is independent of P. Once N has been calculated, it can be used to generate several spot rate discount factors using different benchmark bond prices. By varying P, a whole class of spot rate yield curves can be defined. For this class of yield curves, there is a linear relationship between the spot discount factors and the benchmark bond prices which is defined by equation (12).

The approach described was applied to the problem previously presented using $z=2$ and $h=0.5$. The resulting spot rate yield curve is presented in the following graph :

Spot Rate Calculation - Investment Actuary's Approach

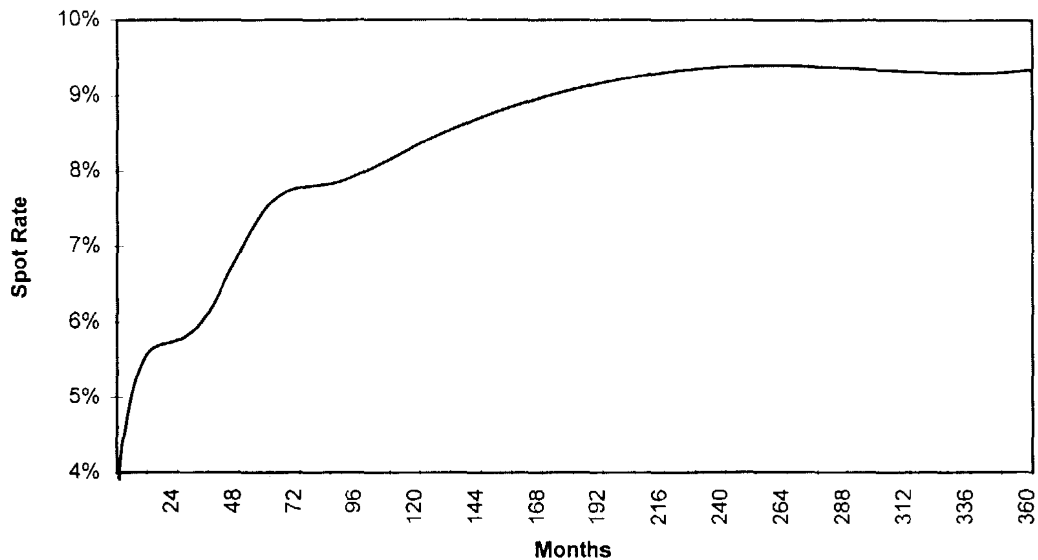

Figure 2

You will note that the yield curve follows the same basic form as in the previous graph. However, the two discontinuities have been smoothed out.

To assist the reader in understanding the steps of this approach, a detailed example using annual coupon bonds with n = 5 and m = 20 is included in Appendix B.

Derivation of Benchmark Weights

For any given set of cash flows, we can write the present value using matrix multiplication as :

$$C\ V \tag{13}$$

Where :

- C is a vector containing m columns, one for each of the cash flows.
- V has been previously defined as a vector of discount factors containing m rows.

Assuming the discount factors have been derived using the investment actuary's approach, previously described, we can use equation (12) to derive the following relationship :

$$C\ N\ P\ -\ C\ V \tag{14}$$

The matrix product (C N) yields a vector of n columns, 1 for each benchmark bond which we will refer to as the benchmark weights. When these weights are multiplied by the price of each benchmark bond, the result is an alternative method for calculating the present value of the cash flows. This result can be summarized in the following equation :

$$W\ P\ -\ C\ V \quad \textit{with} \quad W\ -\ C\ N \tag{15}$$

Where :

- W is a vector of n columns containing the benchmark weights. It is the result of the matrix product of C and N.
- P is a vector of n rows containing the quoted prices of the benchmark bonds.
- C is a vector of m columns containing the cash flows.
- V is a vector of m rows containing the calculated spot rate discount factors.

If we accept the investment actuary's approach for calculating spot rates, this equation defines an exact relationship which allows us, by using the left side only, to determine the present value of a set of cash flows without actually calculating discount factors or spot rates. Furthermore, since W is completely independent of P, we can easily calculate the present value for any given set of values, P. This allows us to quickly determine the value of the cash flows for any yield curve in the class of yield curves defined by equation (12).

A set of cash flows for 360 months for both assets and liabilities is shown in Appendix C. The matrix N determined using the benchmark bonds in Table 1 and the investment actuary's approach with z = 2 and h = 0.5 is shown in Appendix D. The benchmark weights for each set of cash flows appear below :

Benchmark	Assets	Liabilities
1 month	6.792	-0.143
3 months	8.620	10.546
6 months	-6.127	-5.207
1 year	25.215	29.600
3 years	21.215	21.690
5 years	13.165	9.745
7 years	-1.868	-1.051
10 years	9.132	9.967
20 years	9.101	9.806
30 years	2.514	2.086

TABLE 2 - Benchmark Weights (amounts in millions)

The calculation of the present value of the two streams of cash flows using the above benchmark weights is shown in the following 2 tables :

Benchmark	Weights (millions)	Price	Value (millions)
1 month	6.792	100.053	679.5
3 months	8.620	100.068	862.6
6 months	-6.127	100.000	-612.7
1 year	25.215	99.904	2519.1
3 years	21.215	100.542	2133.0
5 years	13.165	100.000	1316.5
7 years	-1.868	100.000	-186.8
10 years	9.132	100.000	913.2
20 years	9.101	100.000	910.1
30 years	2.514	100.000	251.4
		Total	8785.8
		Textbook Method	8790.5

TABLE 3 - Value of Assets

Benchmark	Weights (millions)	Price	Value (millions)
1 month	-0.143	100.053	-14.3
3 months	10.546	100.068	1055.3
6 months	-5.207	100.000	-520.7
1 year	29.600	99.904	2957.2
3 years	21.690	100.542	2180.8
5 years	9.745	100.000	974.5
7 years	-1.051	100.000	-105.1
10 years	9.967	100.000	996.7
20 years	9.806	100.000	980.6
30 years	2.086	100.000	208.6
		Total	8713.6
		Textbook Method	8718.2

TABLE 4 - Value of Liabilities

Note the relatively small difference in the values obtained using the investment actuary's approach versus the textbook method presented earlier. This is due primarily to the two discontinuities that have been removed from the yield curve. Taking assets less liabilities, the difference is reported as 72.3 million using the textbook method spot rates while the investment actuary's rates produce a figure of 72.2 million. Given the small difference between the two, in most practical circumstances, it will not matter which rates are used to determine present values. However, the investment actuary's approach yields equation (15) which provides a basis for a very powerful asset-liability matching strategy.

Asset-Liability Matching with The Investment Actuary's Approach

As a result of equation (15), any two sets of cash flows which have exactly the same set of benchmark weights, W, will have exactly the same present value for any yield curve in the class of yield curves defined by equation (12). If the benchmark weights of our assets are different from the benchmark weights of our liabilities, we can easily buy and sell benchmark bonds until the benchmark weights are the same. Once this is achieved, the present value of our assets will equal the present value of our liabilities for every yield curve in the class of yield curves defined by equation (12). It could be said that we have perfect asset-liability matching without actual cash flow matching. However, the sensitivity of surplus to changes in interest rates is zero only for the class of yield curves defined by equation (12).

We will illustrate how we can match assets to liabilities using the benchmark weights presented in Tables 3 and 4.

First, however, we need to determine what the benchmark weights are for each of our benchmark bonds. Since (C N) defines the benchmark weights for any set of cash flows, we can easily define the benchmark weights for all of our benchmark bonds as follows :

$$BN \tag{16}$$

Where :

B is an n row by m column matrix containing the benchmark bond cash flows as previously defined.

N is the matrix defined in equation (12).

General reasoning might indicate that B N should be the identity matrix. This is not true for the same reason that a Whitaker-Henderson mortality graduation does not necessarily reproduce any of the originally observed mortality rates. However, B N is very nearly an identity matrix for the values of z and h used in this example. B N for this example is shown in Appendix E for your reference.

To determine exactly how much of each benchmark bond to buy/sell in order to make the benchmark weights of the assets equal to those of the liabilities, the following equation is required :

$$W_A + X\ (B\ N) - W_L \tag{17}$$

Where :

W_A is the vector of benchmark weights for the assets.

W_L is the vector of benchmark weights for the liabilities.

X is the vector of exchanges of benchmark bonds that must be executed so that the benchmark weights of the assets will equal those of the liabilities.

B is the matrix of cash flows for the benchmark bonds.

N is defined in equation (12).

The solution for X is easily obtained as follows :

$$X - (W_L - W_A)\ (B\ N)^{-1} \tag{18}$$

It should be noted that the use of B in this equation is not an absolute requirement. We can substitute the matrix of cash flows for any set of n assets to be used to perform the re-matching of the portfolio. In addition, it is also possible to allow more or less than n assets and use equation (17) to develop a set of matching constraints which do not require exact equality. An optimization method such as Simplex can then be used to determine the exchanges which satisfy the constraints and have a minimum cost. This idea is discussed further in the section Areas for Further Study.

Using the fact that the benchmark weights are the product of C and N, equation (18) can be re-written as follows :

$$X = (C_L - C_A)\, N\, (B\, N)^{-1} \tag{19}$$

Where :

C_A is a vector containing the asset cash flows.
C_L is a vector containing the liability cash flows.

The quantity $N(B\,N)^{-1}$ is a matrix containing m rows and n columns. Once calculated it can be used to determine a matching strategy for any set of cash flow differences of length m or less. The matrix for this example is almost identical to the matrix N shown in Appendix E because B N as pointed out earlier is very nearly the identity matrix. The matching strategy for the example is displayed in the following table :

Benchmark	Liability Weights W_L	Asset Weights W_A	Difference $W_L - W_A$	Purchase (Sale) X
1 month	-0.143	6.792	-6.935	-6.935
3 months	10.546	8.620	1.926	1.926
6 months	-5.207	-6.127	0.920	0.920
1 year	29.600	25.215	4.385	4.385
3 years	21.690	21.215	0.475	0.475
5 years	9.745	13.165	-3.420	-3.420
7 years	-1.051	-1.868	0.818	0.818
10 years	9.967	9.132	0.834	0.834
20 years	9.806	9.101	0.706	0.706
30 years	2.086	2.514	-0.427	-0.427
Total	87.040	87.758	-0.718	-0.718

TABLE 5 - Matching Strategy
(amounts in millions)

The original asset cash flows, the liability cash flows and the asset cash flows after the matching strategy has been applied are shown in Appendix C in detail, and summarized by years in the following graphs.

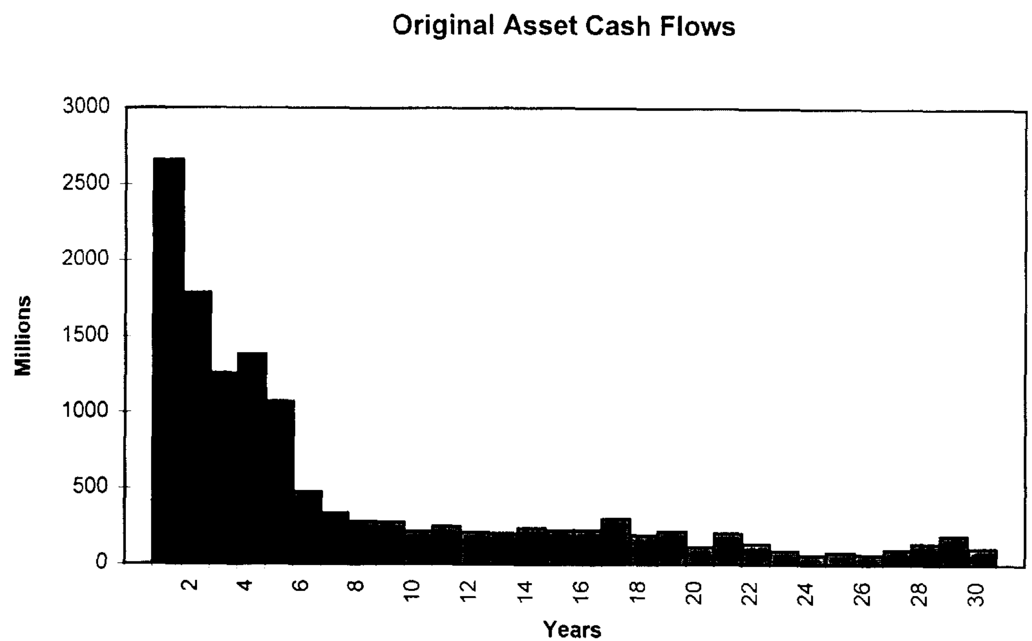
Original Asset Cash Flows
Millions
3000
2500
2000
1500
1000
500
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
Years

Figure 3

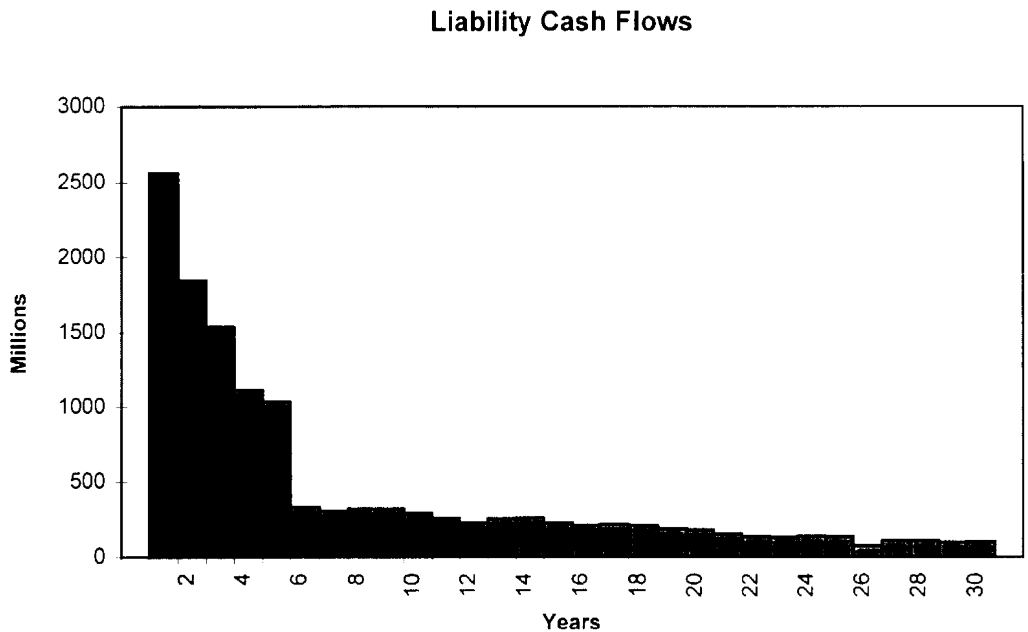
Liability Cash Flows
Millions
3000
2500
2000
1500
1000
500
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
Years

Figure 4

Matched Asset Cash Flows

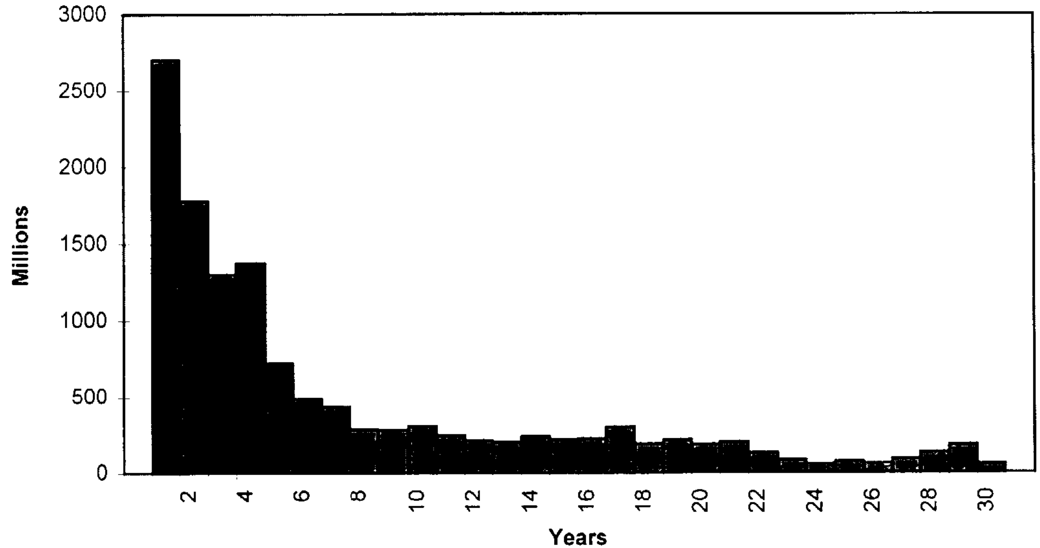

Figure 5

To test the effectiveness of the matching strategy developed, a set of 8 interest scenarios are shown in Appendix F for the bonds presented in Table 1. Using these scenarios and the investment actuary's approach with $z=2$ and $h=0.5$, spot rate yield curves were computed and the following present values obtained :

Scenario	Assets	Liabilities	Difference
Base	8713.6	8713.6	0.0
Flat Decrease	9033.7	9033.7	0.0
Flat Increase	8421.3	8421.3	0.0
Slope Decrease	8737.0	8737.0	0.0
Slope Increase	8692.7	8692.7	0.0
Trough	8884.9	8884.9	0.0
Bump	8550.8	8550.8	0.0
Twist	8855.9	8855.9	0.0

TABLE 6 - Using Investment Actuary's Yield Curves
(amounts in millions)

Table 6 demonstrates that for the class of yield curves produced by the investment actuary's approach with $z=2$ and $h=0.5$, surplus is always zero. Although this does not cover all of the changes in yield curve that could possibly occur, the approach also appears to protect very well against other possible changes in the yield curve. Using the same interest scenarios as for Table 6, the textbook approach described previously was applied with the following results :

Scenario	Assets	Liabilities	Difference
Base	8718.3	8718.2	0.1
Flat Decrease	9038.4	9038.2	0.1
Flat Increase	8426.0	8425.9	0.0
Slope Decrease	8741.3	8741.3	0.0
Slope Increase	8697.7	8697.6	0.1
Trough	8888.1	8888.1	0.0
Bump	8556.8	8556.6	0.2
Twist	8860.3	8860.6	-0.3

TABLE 7 - Using Textbook Yield Curves
(amounts in millions)

It should be clear from these tables that the investment actuary's approach produces a matching strategy that protects the value of surplus against a wide range of changes in the yield curve. However, it should be noted that the example so far has assumed that the cash flows themselves are not sensitive to changes in interest rates. The following section deals with the additional complexity of interest-sensitive cash flows.

Options Handling

In this section we will consider the possibility that the cash flows to be matched might be affected by changes in interest rates. The cash flows C_A and C_L that appear in equation (19) should be projected taking into account any options that might reasonably be exercised in the future. Of course, whether or not these options are exercised depends on the projected level of interest rates. Usually, a base set of interest assumptions is established and used to project C_A and C_L for use in equation (19). However, if an alternate interest scenario is assumed, the cash flows will probably change and equation (19) will become :

$$X' = (C_{L'} - C_{A'}) \, N \, (B \, N)^{-1} \tag{20}$$

Where :

$C_{A'}$ is a vector containing the asset cash flows,
$C_{L'}$ is a vector containing the liability cash flows,
with both sets of cash flows projected using an alternate interest scenario.

Subtracting equation (19) from equation (20) we obtain :

$$(X'-X) = ((C_{L'}-C_L) - (C_{A'}-C_A)) \, N \, (B \, N)^{-1} \tag{21}$$

Equation (21) defines the additional exchanges of benchmark bonds which would have to be executed in order to maintain a matched position given the changes in cash flows caused by the move to the alternate interest scenario. The cost of making these exchanges prior to the change in interest environment is as follows :

$$(X'-X)P \tag{22}$$

In the alternate interest environment, the cost of the exchanges is as follows :

$$(X'-X)P' \tag{23}$$

Where :

P' is the vector of prices of the benchmark bonds in the alternate interest environment (including accrued coupon).

For options which are in the hands of the company, these will typically only be exercised when it is to the financial advantage of the company to do so. If these are the only options in the portfolio, the cost calculated according to equation (23) will be lower than that calculated according to equation (22) due to the gains realized from the exercise of the options.

However, it is normally true that most of the options in an insurance company portfolio are in the hands of the company's policyholders or debtors. These options can be expected to be exercised whenever this would be to the detriment of the company. As a result, the cost of the exchanges according to equation (23) will be greater than that calculated according to equation (22) and the company will suffer a loss in attempting to re-match its portfolio. In addition, this loss could well turn out to be larger than calculated by comparing equation (23) to equation (22) simply because interest rates could move significantly further than predicted by the alternate scenario. In order to control this cost, a relatively simple options strategy can be put in place. To fix the maximum cost of re-matching at that specified by equation (22), all that is required is to buy options on the benchmark bonds at the prices, P, in the amounts specified by the vector X'-X. Where an element of X'-X is positive, this indicates that a purchase may be necessary so a call should be bought to protect against a price increase. Where an element of X'-X is negative, this indicates that a sale may be necessary so a put should be bought to protect against a decrease in price.

This process will be illustrated with a simple example. Consider a fixed-rate mortgage loan where the borrower has the right to increase the regular monthly payment by 10%. When interest rates are high relative to the mortgage rate, borrowers are less likely to make additional payments for two reasons. First, they may have floating rate loans where the rate is rising and it makes more financial sense to make extra payments on these. Secondly, the borrower may have investment opportunities that provide a higher interest return than the rate required on the mortgage. On the other hand, when interest rates are low relative to the mortgage rate, borrowers are more likely to make additional payments.

Suppose we have a mortgage loan portfolio with a book value of 1 billion all with a 25-year amortization and a 5-year term before the fixed interest rate of 8.25% is re-established. Because the renewal of the mortgages will be done at current market rates at the end of 5 years, such a portfolio of loans can be treated as if it matured at the end of 5 years. For asset-liability matching purposes, the cash flows on such a portfolio of loans would then consist of the monthly payments for 5 years and the outstanding balance at the end of the 5 years. Assuming no optional payments are made, the monthly payments amount to 7,792,288 and the balance at the end of the 5 years is 923,909,659. Let's assume a 1% decrease in interest rates causes all borrowers to increase their monthly payments by the maximum of 10%. The new monthly payments are 8,571,517 and the outstanding balance 866,487,403. The increase in the monthly payments is 779,229 and the reduction in the outstanding balance 57,422,256. These cash flow differences can be substituted for $C_{A'} - C_A$ in equation (21) to determine the options which must be bought to protect the company against potential financial losses associated with the increase in mortgage payments. (For the purpose of this example, it is assumed that $C_{L'} - C_L$ is zero.)

The results of this strategy are shown in the following table :

Benchmark	Original Environment	Alternate Environment	Difference (X' - X)
1 month	0.087	0.087	0.001
3 months	0.247	0.312	-0.065
6 months	-0.175	-0.235	0.059
1 year	0.388	0.576	-0.188
3 years	0.527	0.782	-0.255
5 years	9.369	8.934	0.435
7 years	-0.033	-0.046	0.014
10 years	0.004	0.005	-0.002
20 years	0.000	0.000	0.000
30 years	0.000	0.000	0.000
Total	10.415	10.415	0.000

TABLE 8 - Exchanges for Mortgage Option
(in millions)

The cost of making these asset exchanges in various circumstances is shown in the following table :

Benchmark	Difference	Price 1	Price 2	Paid
1 month	0.001	0.1	0.1	0.1
3 months	-0.065	-6.5	-6.5	-6.5
6 months	0.059	5.9	6.0	5.9
1 year	-0.188	-18.7	-18.9	-18.9
3 years	-0.255	-25.6	-26.3	-26.3
5 years	0.435	43.5	45.4	43.5
7 years	0.014	1.4	1.4	1.4
10 years	-0.002	-0.2	-0.2	-0.2
20 years	0.000	0.0	0.0	0.0
30 years	0.000	0.0	0.0	0.0
Total	0.000	-0.1	0.9	-1.1

TABLE 9 - Cost of Exchanges for Mortgage Option
(in millions)

In this table Price 1 indicates the price of the asset exchanges in the original interest rate environment. Price 2 is the price of the asset exchanges after the 1% decrease in interest rates. As can be seen, the exchanges of assets would have resulted in a profit of 0.1 million in the original interest environment and a loss of 0.9 million in the new interest environment if an options strategy had not been put in place. However, because of the options purchased, the assets to be purchased can be acquired at their original prices. In addition, all of the assets to be sold can be sold for more than the original purchase price because of the decline in interest rates. This results in a profit of 1.1 million when the portfolio is re-matched. Of course, a profit of this size might not emerge in another example. However, it should be noted that the asset exchanges will have a cost that is never greater than it would have been using the original yield curve. Thus the strategy will never result in a loss. However, the cost of purchasing the options must be deducted in order to determine the overall financial impact of the strategy.

The process described is very similar to the "delta hedging" technique used extensively in the derivative products industry. It has not been described here in great detail. However, the reader should have a general idea of how the investment actuary's approach might be adapted to deal with interest-sensitive cash flows. Of course, the key issue in dealing with interest-sensitive cash flows is the identification of a relationship which describes how the cash flows react to changes in interest rate environment. In the case of a simple callable bond, for example, such a relationship is relatively easy to define. On the other hand, prepayments on mortgages and surrenders of single premium deferred annuities are much more difficult to predict. However, assuming a relationship can be established, the technique illustrated in this section can be used to define a relatively simple options strategy to protect the company against financial losses caused by changes in the cash flows resulting from changes in interest rate environment.

Appropriateness of the Approach

It should be clear at this point that the use of equation (12) in combination with equation (2) to generate spot rate yield curves provides the basis for an approach to asset-liability matching that is very powerful. Once such a strategy is implemented for a portfolio containing fixed cash flows, the sensitivity of surplus to a change in yield curve is zero for all yield curves in the class defined by equation (12). At the same time, the strategy is simple to implement because it can be executed with exchanges of benchmark bonds only. Furthermore, as described above, this approach can also be used to construct strategies for cash flows which are sensitive to changes in interest rates.

However, it remains to be demonstrated that the investment actuary's approach does result in reasonable yield curves given a careful choice of z and h. Fortunately, the ability to produce reasonable yield curves can quite easily be tested. For any given spot rate yield curve, the prices of each of the benchmark bonds can be easily calculated. Equation (12) can then be applied to produce discount factors and equation (2) to generate spot rates from these discount factors. The spot rates thus obtained can be compared to the original spot rates used to generate the benchmark bond prices. To demonstrate the ability of the approach to reasonably reproduce a given spot rate yield curve, two illustrations will be presented. The benchmark bonds for both illustrations are those shown in Table 1. In the first illustration, the original spot rates are taken to be 8% at all durations. In the second illustration, a saw-tooth pattern is assumed for the original spot rates.

The spot rates computed using the investment actuary's approach with $z = 2$ and $h = 0.5$ are shown below compared to the 8% level result that might be expected.

Benchmark	Original Data	Investment Actuary's Approach	Difference
1 month	8.00%	8.00%	0.00%
3 months	8.00%	8.00%	0.00%
6 months	8.00%	8.00%	0.00%
1 year	8.00%	8.00%	0.00%
3 years	8.00%	8.00%	0.00%
5 years	8.00%	8.00%	0.00%
7 years	8.00%	8.00%	0.00%
10 years	8.00%	8.00%	0.00%
20 years	8.00%	7.99%	0.01%
30 years	8.00%	8.04%	-0.04%

TABLE 10 - Spot Rate Comparison
(level rates)

You will note that the rates obtained are extremely close to the original 8% rates except at the very long durations where they are still very close.

The second test is a little more of a challenge. In this test, the 1-month spot rates are taken to be 7% with spot rates at each benchmark duration then alternating between 8% and 7% in a saw-tooth pattern. Rates between the benchmark durations are obtained by linear interpolation between adjacent rates. This is not a very likely pattern for spot rates. However, this pattern has been chosen to see whether or not the investment actuary's approach can reasonably reproduce it. The results are shown below :

Benchmark	Original Data	Investment Actuary's Approach	Difference
1 month	7.00%	7.00%	0.00%
3 months	8.00%	8.00%	0.00%
6 months	7.00%	7.00%	0.00%
1 year	8.00%	8.00%	0.00%
3 years	7.00%	7.00%	0.00%
5 years	8.00%	7.99%	0.01%
7 years	7.00%	6.99%	0.01%
10 years	8.00%	8.03%	-0.03%
20 years	7.00%	6.78%	0.22%
30 years	8.00%	8.25%	-0.25%

TABLE 11 - Spot Rate Comparison
(saw-tooth pattern)

Again, the rates are very close except at the long durations where they are still reasonably close. The failure to exactly reproduce the spot rates at the long durations is not surprising given the small amount of data available for the long durations in our example. In the first 20 years of the example, there are 95 cash flows being discounted which contribute to all 10 of the benchmark bond prices. In the last 10 years, there are only 20 cash flows being discounted which contribute to only the 30-year benchmark bond price.

Of course, it should be noted that the Whitaker-Henderson method is one of the more flexible graduation methods. If greater accuracy is required, it is relatively easy to add additional benchmark bonds at 15 and 25 years. Introducing weights into the formula and making these larger at the longer durations might also lead to some improvement. Variations in z and h might also produce more satisfactory results. In addition, there are many other variations of Whitaker-Henderson that could be applied such as mixed differences and the polynomial plus exponential form.

It should also be noted that the failure to reproduce a given pattern of rates is not necessarily a failure of the approach. In the test presented here, we have used 360 spot rates to calculate 10 benchmark bond prices and then tried to re-calculate the 360 rates using only the 10 prices. It is hardly surprising that we do not get exactly the original rates. The spot rates produced by the investment actuary's approach are just as valid a solution to the equations presented as the original spot rates. In both examples, the prices calculated using the spot rates produced by the investment actuary's approach are equal to the original prices to 3 decimal digits. Whether the solution produced by the investment actuary's approach or some other method is a better estimate of the true spot rates cannot be determined in a real life problem. We simply cannot determine the true shape of the spot-rate yield curve from only 10 benchmark bond prices. However, the investment actuary's approach does provide a class of solutions that reproduces the benchmark bonds prices very well, provides a smooth yield curve and has the additional advantage of providing an explicit relationship for defining benchmark weights.

Pros and Cons of the Approach for Yield Curve Computation

The great strength of the approach introduced in this paper is that it relies on the Whitaker-Henderson method. The flexibility and power of this method is well known in the field of graduation of mortality tables. The principal advantages of this approach for producing spot rates are as follows :

- The approach can be applied to a wide variety of source data. For example, instead of just 10 benchmark bonds, an entire bond market could be used as input data with appropriate weights based on trading volume. Zero coupon bonds could be included in the matrix B along with regular coupon bonds. Furthermore, the matrix B could consist entirely of mortgages if a spot rate yield curve for mortgages were required. In fact, any fixed interest asset can be incorporated into B.

- There is a great deal of flexibility in the choice of parameters for applying the approach. Smoothness can be measured in terms of difference operators of any degree as well as linear combinations of these. Different weights can be applied to the different benchmark bonds and different values of h can be chosen to change the emphasis on smoothness versus fit.

- The approach provides an automatic balancing between fit and smoothness. This allows us to avoid the issue of any prior interpolation of the source data which can lead to anomalies as illustrated earlier. In addition, anomalies in the source data will tend to be smoothed out if the value of h is large enough.

- Probably the biggest advantage of the approach is that the resulting spot rates are all linear combinations of the benchmark bond prices (see equation 12). This leads to an explicit definition for benchmark weights in equation (15) and to equations (18) and (19) which describe a simple strategy for matching fixed cash flows.

There are, however, two minor disadvantages of the approach :

- The approach does not directly assure smoothness of spot rates. Instead the smoothness constraint is applied to discount factors. It would certainly be more satisfying if a direct solution could be achieved which applied the smoothness constraint to spot rates. However, this indirect approach was necessary in order to ensure a relatively straightforward solution.

- To obtain m spot rates, we must invert an m-square matrix. Even to obtain monthly spot rates for 30 years, such an inversion goes beyond the ability of most spreadsheet software. However, the matrix to be inverted is symmetrical and positive definite. This allows the Choleski factorization method to be used to factor it into an upper and lower triangular matrix. The inverses of these can then be easily obtained and the two multiplied together to get the inverse required. Although this involves more steps, it involves fewer computations than blindly using an inversion technique which can be applied to any matrix. This process is illustrated in the detailed example in Appendix B.

If we are looking for daily discount factors for 30 years, we need to invert a matrix which is 10,950 by 10,950. This probably goes beyond the ability of all but the largest computers. However, we can probably still solve the problem for monthly discount factors and use an interpolation method to obtain the daily factors with little loss of accuracy. In addition the alternate approach presented in Appendix A should be considered since it involves the inversion of much smaller matrices.

Pros and Cons of the Approach for ALM

Asset-liability matching approaches range from exact cash flow matching which perfectly controls risk but is almost impossible to put into practice to duration matching which is much easier to implement but does not always provide adequate protection against real-life changes in the yield curve. All of the methods which lie in between are necessarily a compromise between the need to control risk and the degree of complexity of the implementation process. The investment actuary's approach may be somewhat complex in its derivation. However, the end product is simple and easy to implement because it involves only exchanges of benchmark bonds. At the same time risk is very tightly controlled because for a very broad range of yield curves defined as part of the approach, the value of assets stays exactly equal to the value of liabilities. More specifically the advantages can be stated as follows :

- In order for any method to succeed, it is critical that all parties to the ALM process understand it. Because the investment actuary's approach is built around benchmark bonds and the final strategy in equations (18) and (19) is simply an exchange of benchmark bonds, the approach can be very easily grasped even if all of the details of its derivation are not completely understood.

- It is quite common for investment managers to have a perception about which areas of the yield curve will perform better over the short run. This approach provides a very simple means of tuning the asset portfolio to profit from such anticipated changes. An investment manager who believes that the 3 to 5 year area of the curve is over-priced relative to the rest of the curve can simply reduce the benchmark weights at 3 years and 5 years below the corresponding weights for the liabilities and increase them where the curve is thought to be under-priced.

- Benchmark bonds are by definition heavily traded issues where the bid to offer spread is typically tighter than on non-benchmark issues. Because the ALM strategy developed here can be executed with exchanges of benchmark bonds, it is very easy to implement and will likely turn out to be cheaper than most other strategies.

- As shown previously, the approach can relatively easily be adapted to provide strategies for matching cash flows which are interest-sensitive provided that the relationships between the cash flows and the level of interest rates can be defined.

- The approach offers a number of different possibilities which have not been studied in great depth but are outlined in a later section called Areas for Further Study.

Of course, no approach is likely to be perfect and this one is no exception. There are two principal disadvantages of the approach that could be cited :

- The derivation of the approach is relatively complex. If equation (19) is written only in terms of basic data, we obtain the following :

$$X = (C_L - C_A)\,(B^T B + h\,K_z^T K_z)^{-1}\,B^T\,(B\,(B^T B + h\,K_z^T K_z)^{-1}\,B^T)^{-1} \qquad (24)$$

  Such an equation is likely to scare away even the most seasoned of actuaries. It is therefore essential that discussions of the approach focus on the benchmark weights and the benefits of matching with benchmark weights.

- There may not be enough benchmark bonds in the portfolio to be able to sell the amounts suggested by equation (19). In this case, a solution would be to enter into a swap. Since swaps are normally based on benchmark bonds this is a relatively simple solution. Instead of selling a bond you don't have, you can agree to receive float and pay fixed which amounts to approximately the same thing. It is also possible to find a combination of benchmark bond exchanges plus sales of existing assets that will bring the benchmark weights of the assets back into line with the benchmark weights of the liabilities.

Areas for Further Study

This paper is only an introduction to a new approach to asset-liability matching. It was not intended as an exhaustive study. Its primary purpose was to introduce the key elements of the approach so that other actuaries active in the asset-liability matching field can investigate its potential more fully. The following is a brief list of items that should be studied in more depth :

- As noted earlier the approach to computing spot rates presented in the body of this paper is just one of a family of methods. Appendix A outlines an alternate approach based on cubic splines. It is likely that other approaches can be developed using a number of other graduation methods.

- The matrix N developed in equation (12) can also be obtained by an empirical approach regardless of the method used to determine the spot rates. This can be done by varying one benchmark price at a time and observing the effect on the discount factors generated. The effect on the discount factors divided by the change in the benchmark price gives us one column of the matrix N. Repeating this for each of the n benchmark bonds yields the complete matrix. It would be interesting to apply this approach to a number of methods in current use for calculating spot rates and compare the results to the investment actuary's approach.

- There are many different variations on the basic Whitaker-Henderson approach that could be investigated. If weights are introduced, equation (11) would appear as follows :

$$(B^T W' B + h K_z^T K_z)^{-1} B^T W' P = V \tag{25}$$

Where :

W' is an n-square matrix containing the weights applicable to each of the benchmark bonds along the diagonal and zeros elsewhere. (W' should not be confused with the vector of benchmark weights, W, introduced in equation (15)).

The mixed difference form should also be investigated.

- One approach that offers particular promise is the polynomial plus exponential form because of the fact that v_t is essentially an exponential function. In this form smoothness is measured as follows :

$$S = \sum_{t=1}^{m-z} (\Delta^z v_t - r\Delta^{z-1} v_t)^2 \tag{26}$$

Where:

r is equal to -i / (1 +i) in the case of a level yield curve.

In the case of a non-level yield curve, r can be allowed to vary. In this case, i in the above formula would be the one-period forward rate. As a first approximation, r could be calculated using the forward rates from the most recently determined yield curve. For subsequent iterations the forward rates calculated from the spot rates emerging from the investment actuary's approach could be used. It is uncertain whether or not the iteration process would actually converge to a solution. However, there is a good possibility that this would yield a very powerful method for deriving a spot rate yield curve particularly in the situation where yield curves are being calculated on a frequent basis so a good first approximation is available from the prior yield curve. The one small disadvantage of this approach is that N in equation (12) is no longer independent of P. It would be useful to investigate how much this affects the use of the investment actuary's approach for constructing ALM strategies.

- Additional work is required on selection of parameters. As discussed previously, tests can be conducted using known spot rates and comparing the results of the method to the known rates. This approach can be useful in selecting appropriate parameters. In the tests done in preparing this paper, it was observed that changing the order of the difference operator, z, had a significant effect on the yield curves obtained. However, variations in h had to be relatively large to have a significant impact on the yield curve obtained. Using h=0.5 compared to h=50000, the root mean squared difference between the two yield curves is only 0.02%.

- An interpolation method can be introduced to generate daily discount factors from monthly factors. Assuming that each daily factor is a linear combination of the monthly factors, we can construct a matrix, J, which contains all of the coefficients necessary to perform the interpolations required. For the example in the body of the paper which has 360 months, J would need 10,950 rows and 360 columns. J could then be applied to both sides of equation (12) to give :

$$J\,N\,P = J\,V \tag{27}$$

This can be re-written as follows :

$$N'\,P = V' \tag{28}$$

Which is in the same form as equation (12) except that N' has 10,950 rows instead of 360 rows and V' has length 10,950 instead of 360. This would allow the approach to be applied to daily cash flows. Of course, further study is required to identify the best interpolation method and to evaluate the degree of error that this might introduce.

- As mentioned previously, the technical complexity of the derivation of this approach might make it less acceptable to the non-technical participants in the ALM process. However, there may be a way to derive an empirical approach similar to what was used to produce the bond manager's method in my prior paper which could produce results similar to those shown in this paper. Such an approach might prove more appealing to the less technical participants in the ALM process.

- The approach presented in this paper does not assume any correlation among the price movements of the benchmark bonds. In practice, there is certainly some correlation. It may be possible to introduce an element of correlation into the ALM strategy produced by equation (19).

- The ALM strategy suggested by equation (19) is quite specific and offers little flexibility to the asset manager. It should be possible to develop a range for X which is a function of the risk tolerance of the company.

- As pointed out earlier, it is not an absolute requirement that matching be done in such a way that the benchmark weights of the assets are exactly equal to the benchmark weights of the liabilities. Equation (17) can be re-written in the form of a series of constraints instead of exact equalities. In addition, the matrix B, can be replaced with a matrix of cash flows for any set of assets available for re-balancing the portfolio. An optimization approach such as Simplex can then be used to find the exchanges of assets that satisfy the matching constraints and result in the lowest cost subject to the additional constraint that the present value of the asset cash flows is unchanged after the exchanges. If some of the available assets are under-priced or over-priced relative to the benchmark bonds, it will often be possible to satisfy the matching constraints at a negative cost. In other words, asset-liability matching risk can sometimes be reduced at a profit.

This problem can also be constructed to allow for a purchase price which is different from the sale price to reflect the bid-offer spread which exists in most marketsor to make specific allowance for commissions and other transaction costs. This approach is very similar to what has become known as "Rich Cheap Analysis" except that asset-liability matching constraints have been integrated into the process. This approach could prove to be very powerful. However, it needs to be developed more thoroughly before it can be put into practice.

- The benchmark weights for a series of cash flows clearly change as time elapses. It seems likely that these do not change unless a real cash flow is received or a cash flow occurs on a benchmark bond. When these events occur, it is likely that only minor re-tuning of the portfolio would be required in practice. Assuming the portfolio is open to new business, it is quite likely that this retuning could easily be done as part of the investment of the new business cash flows. This issue needs to be investigated in more detail.

- Periodically it will be necessary to add, delete or replace the benchmark bonds being used. Assuming the portfolio is being matched using equation (19), the introduction of new benchmark bonds will almost certainly result in the need to re-tune the portfolio. The degree of retuning required in practical circumstances should be investigated further.

Conclusion

This paper was intended only as an introduction to two new processes which could prove very useful in asset-liability matching work. Equation (12) summarizes the process for developing a spot rate yield curve from a series of regular coupon bonds (or any other assets for that matter). Equation (19) introduces a new approach to matching assets to liabilities which depends directly on equation (12). Equation (21) indicates how this approach might be adapted to matching problems where cash flows are interest sensitive.

None of the ideas in this paper have been explored in great depth. It is hoped that actuaries working in the asset-liability matching field will be able to make use of the ideas presented here and study some of the unexplored issues more thoroughly.

Appendix A - Alternate Solution

In this alternate solution, we will derive a vector of discount factors such that the calculated price of all of the benchmark bonds using these discount factors is exactly equal to the market price (plus accrued coupon). This condition is expressed in equation (1) in the body of the paper which we will re-write in matrix form :

$$P = B\ V \tag{29}$$

Where :

- P is a vector containing the bond prices, P_i, defined in the body of the paper, with n rows.
- B is a matrix containing the cash flows, ${}_ib_t$, defined in the body of the paper, with n rows, one for each benchmark bond, and m columns, 1 for each duration.
- V is a vector containing the discount factors, v_t, defined in the body of the paper, with m rows.

Because m is generally much greater than n, there are multiple solutions to this system of equations. However, we will further assume that there are n pivotal values in V and that the remaining values in V are obtained by interpolation. This gives us the following equation :

$$V = J\ V'' \tag{30}$$

Where :

- J is a matrix with m rows and n columns containing all of the coefficients necessary to perform the interpolations required to generate the full vector V from the vector of pivotal values V".
- V" is a vector of length n containing the pivotal values of the discount factors. The durations for the pivotal values might coincide with the terms of the benchmark bonds. However, this is not a requirement.

Equation (30) can now be substituted into equation (29) to give :

$$P = B\ J\ V'' \tag{31}$$

The solution for this system of equations can be easily obtained yielding the following :

$$(B\ J)^{-1}\ P = V'' \tag{32}$$

Finally equation (32) can be substituted into equation (30) to obtain the full vector, V, as follows :

$$J\ (B\ J)^{-1}\ P = V \tag{33}$$

This can be written in the same form as equation (12) as follows :

$$N\ P = V \quad with \quad N = J\ (B\ J)^{-1} \tag{34}$$

In this form, the equation can form the basis for the ALM strategies outlined in the main paper.

The implementation of this approach will be illustrated using cubic splines with 2 arcs. The 2 arcs can be represented by 2 polynomials :

$$p_0\ (t) = a_1 + a_2 t + a_3 t^2 + a_4 t^3 \tag{35}$$

$$p_1\ (t) = a_1 + a_2 t + a_3 t^2 + a_4 t^3 + a_5 (t-u)^3 \tag{36}$$

Where :

- t is duration (in this example measured in months).
- a_i are 5 constants to be determined.
- u is the point where the polynomial arcs join.

Using the method of least squares, the coefficients a_1, a_2 ... a_5 can be determined. The solution is most easily written in matrix form as follows :

$$A = (Y^T\ Y)^{-1}\ Y^T\ V'' \tag{37}$$

Where :

- A is the vector of coefficients a_1, a_2 ... a_5 with 5 rows.
- V" is the vector of pivotal values introduced in equation (30) with n rows.
- Y is a matrix with n rows and 5 columns defined as follows :

$$
\begin{vmatrix}
1 & d_1 & d_1^2 & d_1^3 & 0 \\
1 & d_2 & d_2^2 & d_2^3 & 0 \\
. & . & . & . & . \\
. & . & . & . & . \\
1 & d_k & d_k^2 & d_k^3 & 0 \\
1 & d_{k+1} & d_{k+1}^2 & d_{k+1}^3 & (d_{k+1}-u)^3 \\
. & . & . & . & . \\
. & . & . & . & . \\
1 & d_n & d_n^2 & d_n^3 & (d_n-u)^3
\end{vmatrix} \tag{38}
$$

Where :

- d_j are the durations of the n pivotal values.
- d_k is the longest pivotal value duration less than or equal to u.
- u as defined previously, is the point where the polynomial arcs join.

Given the coefficients of the polynomial arcs, the values, V, can be determined using the following matrix equation :

$$
V = D\,A \tag{39}
$$

Where :

- D is a matrix with m rows and 5 columns defined as follows :

$$
\begin{vmatrix}
1 & 1 & 1^2 & 1^3 & 0 \\
1 & 2 & 2^2 & 2^3 & 0 \\
. & . & . & . & . \\
. & . & . & . & . \\
1 & g & g^2 & g^3 & 0 \\
1 & (g+1) & (g+1)^2 & (g+1)^3 & (g+1-u)^3 \\
1 & (g+2) & (g+2)^2 & (g+2)^3 & (g+2-u)^3 \\
. & . & . & . & . \\
. & . & . & . & . \\
1 & m & m^2 & m^3 & (m-u)^3
\end{vmatrix} \tag{40}
$$

Where :

- g is the largest duration less than or equal to u (note that u is not necessarily an integer).

Using the definition of A from equation (37), equation (39) can be re-written as follows :

$$V = D\,(Y^T\,Y)^{-1}\,Y^T\,V'' \tag{41}$$

This is in the same form as equation (30) so J can easily be written as :

$$J = D\,(Y^T\,Y)^{-1}\,Y^T \tag{42}$$

Finally, this can be substituted into equation (34) giving us :

$$N\,P = V \qquad with \qquad N = D\,(Y^T\,Y)^{-1}\,Y^T\,(B\,D\,(Y^T\,Y)^{-1}\,Y^T)^{-1} \tag{43}$$

This is in the same form as equation (12) in the body of the paper which allows it to be used in the same way to construct asset-liability matching strategies.

The solution where more arcs are used is identical to the above except that Y and D are redefined. Any good text on graduation should provide more detail on how this is done. A special case arises where (n-3) polynomial arcs are used. In this case, the n pivotal values in V" will be exactly reproduced in V in equation (41). Y is then an n-square matrix whose inverse exists so J reduces to :

$$J = D\,Y^{-1} \tag{44}$$

Given this equation for J and recognizing the fact that (B D) is an n-square matrix, equation (43) becomes :

$$N\,P = V \qquad with \qquad N = D\,(B\,D)^{-1} \tag{45}$$

Clearly the number of arcs used will significantly affect the values of the elements of the matrix N. In addition, the location of the points where the various arcs join will affect the values obtained. Significant additional study is required to identify the specific approach which will yield the most satisfactory results over a range of practical situations.

Appendix B - Detailed Example

A set of 5 benchmark bonds with annual coupons will be used to illustrate how the investment actuary's approach can be applied to produce spot rates. The bonds used are shown in the following table :

Coupon	Month of Maturity	Price	Yield
6.000%	12	99	7.07%
6.500%	60	100	6.50%
7.000%	120	101	6.86%
7.500%	168	102	7.27%
8.000%	240	103	7.70%

The calculation of N for equation (12) using $z=2$ and $h=0.5$ is shown step by step on the following pages.

Benchmark Bond Cash Flows - B

106.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6.5000	6.5000	6.5000	6.5000	106.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000	107.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7.5000	7.5000	7.5000	7.5000	7.5000	7.5000	7.5000	7.5000	7.5000	7.5000	7.5000	7.5000	7.5000	107.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000	108.0000

$B^T B$

11447.50	211.50	211.50	211.50	861.50	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
211.50	211.50	211.50	211.50	861.50	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
211.50	211.50	211.50	211.50	861.50	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
211.50	211.50	211.50	211.50	861.50	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
861.50	861.50	861.50	861.50	11511.50	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
169.25	169.25	169.25	169.25	169.25	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
169.25	169.25	169.25	169.25	169.25	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
169.25	169.25	169.25	169.25	169.25	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
169.25	169.25	169.25	169.25	169.25	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
869.25	869.25	869.25	869.25	869.25	869.25	869.25	869.25	869.25	11569.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	11620.25	64.00	64.00	64.00	64.00	64.00	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	864.00
864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	11664.00

K_z

1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1

$K_z^T K_z$

1	-2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-2	5	-4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-4	6	-4	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-4	5	-2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-2	1

$B^T B + .5 K_z^T K_z$

11448.00	210.50	212.00	211.50	861.50	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
210.50	214.00	209.50	212.00	861.50	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
212.00	209.50	214.50	209.50	862.00	169.25	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
211.50	212.00	209.50	214.50	859.50	169.75	169.25	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
861.50	861.50	862.00	859.50	11514.50	167.25	169.75	169.25	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
169.25	169.25	169.25	169.75	167.25	172.25	167.25	169.75	169.25	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
169.25	169.25	169.25	169.25	169.75	167.25	172.25	167.25	169.75	869.25	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
169.25	169.25	169.25	169.25	169.25	169.75	167.25	172.25	167.25	869.75	120.25	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
169.25	169.25	169.25	169.25	169.25	169.25	169.75	167.25	172.25	867.25	120.75	120.25	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
869.25	869.25	869.25	869.25	869.25	869.25	869.25	869.75	867.25	11572.25	118.25	120.75	120.25	870.25	64.00	64.00	64.00	64.00	64.00	864.00
120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.75	118.25	123.25	118.25	120.75	870.25	64.00	64.00	64.00	64.00	64.00	864.00
120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.75	118.25	123.25	118.25	870.75	64.00	64.00	64.00	64.00	64.00	864.00
120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.25	120.75	118.25	123.25	868.25	64.50	64.00	64.00	64.00	64.00	864.00
870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.25	870.75	868.25	11623.25	62.00	64.50	64.00	64.00	64.00	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.50	62.00	67.00	62.00	64.50	64.00	64.00	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.50	62.00	67.00	62.00	64.50	64.00	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.50	62.00	67.00	62.00	64.50	864.00
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.50	62.00	67.00	62.00	864.50
64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.50	62.00	66.50	863.00
864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.00	864.50	863.00	11664.50

L such that : $L L^T = B^T B + .5 K_z^T K_z$

106.9953	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.9674	14.4958	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.9814	14.1835	3.0663	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.9767	14.3566	0.6380	2.0183	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8.0518	58.3381	6.0686	1.0780	89.4894	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.5818	11.4611	1.1602	0.6643	-5.8315	1.6116	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.5818	11.4611	1.1602	0.4166	-5.8006	-1.2768	1.2628	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.5818	11.4611	1.1602	0.4166	-5.8062	0.2542	-1.1744	1.3100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.5818	11.4611	1.1602	0.4166	-5.8062	-0.0560	0.4917	-0.9529	1.4167	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8.1242	58.8629	5.9587	2.1396	-29.8200	-0.2878	0.4916	1.5054	-0.0012	84.3179	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.1239	8.1429	0.8243	0.2960	-4.1252	-0.0398	0.0680	0.1555	0.5126	-5.9185	1.6040	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.1239	8.1429	0.8243	0.2960	-4.1252	0.0398	0.0680	0.1555	0.1596	-5.8889	1.2910	1.2220	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.1239	8.1429	0.8243	0.2960	-4.1252	-0.0398	0.0680	0.1555	0.1596	-5.8948	0.2457	-1.2746	1.1854	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8.1335	58.9306	5.9655	2.1421	29.8543	-0.2882	0.4921	1.1250	1.1552	-42.6606	0.4778	0.6457	-0.3429	73.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.5982	4.3339	0.4387	0.1575	-2.1955	-0.0212	0.0362	0.0827	0.0850	3.1374	-0.0351	0.0174	0.4883	-5.4887	1.6117	0.0000	0.0000	0.0000	0.0000	0.0000
0.5982	4.3339	0.4387	0.1575	-2.1955	-0.0212	0.0362	0.0827	0.0850	-3.1374	-0.0351	0.0174	0.0665	-5.4564	-1.2529	1.2708	0.0000	0.0000	0.0000	0.0000
0.5982	4.3339	0.4387	0.1575	-2.1955	-0.0212	0.0362	0.0827	0.0850	-3.1374	-0.0351	0.0174	0.0665	-5.4633	0.2749	-1.1868	1.2751	0.0000	0.0000	0.0000
0.5982	4.3339	0.4387	0.1575	-2.1955	-0.0212	0.0362	0.0827	0.0850	-3.1374	-0.0351	0.0174	0.0665	-5.4633	-0.0353	0.4746	-1.0329	1.3478	0.0000	0.0000
0.5982	4.3339	0.4387	0.1575	-2.1955	-0.0212	0.0362	0.0827	0.0850	-3.1374	-0.0351	0.0174	0.0665	-5.4633	-0.0353	0.0812	0.5615	-1.0015	1.1331	0.0000
8.0751	58.5073	5.9227	2.1267	-29.6399	-0.2861	0.4886	1.1169	1.1469	-42.3543	0.4744	0.2349	0.8978	73.7541	0.4768	1.0957	2.2868	2.8267	1.6995	1.7341

L^{-1}

0.00935	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.00127	0.06899	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.00017	-0.31910	0.32613	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.00008	-0.38984	-0.10310	0.49547	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	-0.01864	-0.02087	-0.00597	0.01117	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	-0.16761	-0.26781	-0.22584	0.04043	0.62049	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.00002	-0.45940	-0.63229	-0.41922	0.09221	0.62738	0.79191	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.00004	-0.65889	-0.86344	-0.51603	0.12435	0.44204	0.70994	0.76336	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.00003	-0.54891	-0.69428	-0.38072	0.09904	0.10416	0.20272	0.51348	0.70587	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	-0.00845	-0.00964	-0.00380	0.00133	-0.00943	-0.01729	-0.01362	0.00001	0.01186	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.06120	0.08788	0.06302	-0.01295	-0.12211	-0.23095	-0.28832	-0.22552	0.04376	0.62344	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	-0.01335	0.00792	0.03944	-0.00210	-0.25898	-0.48815	-0.53441	-0.33041	0.10338	0.65863	0.81830	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-0.00001	-0.10749	-0.09730	0.00135	0.01237	-0.38718	-0.72879	-0.75183	-0.40352	0.16106	0.57895	0.87986	0.84357	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	-0.00944	-0.01025	-0.00322	0.00140	-0.01608	-0.03021	-0.02854	-0.01161	0.00706	0.00097	-0.00311	0.00396	0.01370	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	-0.02960	-0.03788	-0.02139	0.00542	0.01021	0.01959	0.03730	0.04416	-0.00183	-0.16560	-0.28597	-0.24208	0.04665	0.62046	0.00000	0.00000	0.00000	0.00000	0.00000
-0.00001	-0.10216	-0.11741	-0.04768	0.01629	-0.10508	-0.19695	-0.16474	-0.03408	0.04915	-0.18116	-0.35252	-0.26579	0.10481	0.61171	0.78689	0.00000	0.00000	0.00000	0.00000
-0.00001	-0.16148	-0.18096	-0.06629	0.02491	-0.21484	-0.40323	-0.36234	-0.11866	0.09696	-0.15073	-0.33680	-0.22220	0.14618	0.43556	0.73236	0.78423	0.00000	0.00000	0.00000
-0.00001	-0.15741	-0.17387	-0.05966	0.02382	-0.23602	-0.44321	-0.40883	-0.15102	0.10503	-0.07293	-0.20802	-0.10859	0.13186	0.13464	0.28415	0.60098	0.74194	0.00000	0.00000
-0.00001	-0.13461	-0.14664	-0.04698	0.02000	-0.22353	-0.41993	-0.39500	-0.15799	0.09839	-0.00201	-0.07980	-0.00478	0.10410	-0.12132	-0.16815	0.14255	0.65576	0.88255	0.00000
-0.00001	-0.06463	-0.06316	-0.00833	0.00829	-0.18372	-0.34567	-0.34942	-0.17739	0.07744	0.21157	0.30711	0.30789	0.01948	-0.89085	-1.76136	-2.15351	-1.85204	-0.86491	0.57666

$(L^{-1})^T L^{-1} = (B^T B + .5 K_z^T K_z)^{-1}$

0.00009	0.00006	0.00003	0.00001	-0.00001	-0.00002	-0.00002	-0.00002	-0.00001	-0.00001	0.00000	0.00000	-0.00001	-0.00001	-0.00001	-0.00001	-0.00001	-0.00001	0.00000	0.00000
0.00006	1.33360	1.33133	0.62210	-0.20059	-0.57858	-0.63841	-0.49979	-0.27553	-0.07159	0.01294	0.01703	-0.02268	-0.07185	-0.09848	-0.10690	-0.10124	-0.08536	-0.06290	-0.03727
0.00003	1.33133	1.93763	1.02646	-0.26211	-0.85545	-0.95140	-0.73702	-0.38869	-0.07202	0.05809	0.06254	-0.00140	-0.08009	-0.12350	-0.13842	-0.13130	-0.10819	-0.07479	-0.03642
0.00001	0.62210	1.02646	0.90202	-0.15560	-0.64410	-0.72539	-0.55259	-0.27003	-0.01328	0.09090	0.09232	0.03784	-0.02865	-0.06622	-0.08044	-0.07660	-0.05964	-0.03426	-0.00480
-0.00001	-0.20059	-0.26211	-0.15560	0.03782	0.12690	0.14135	0.10923	0.05702	0.00955	-0.00991	-0.01052	-0.00086	0.01100	0.01758	0.01987	0.01886	0.01544	0.01048	0.00478
-0.00002	-0.57858	-0.85545	-0.64410	0.12690	1.41382	1.63871	1.20073	0.47612	-0.18173	-0.44235	-0.43554	-0.28335	-0.10013	0.00749	0.05410	0.05346	0.01856	-0.03838	-0.10595
-0.00002	-0.63841	-0.95140	-0.72539	0.14135	1.63871	2.69023	2.16965	0.90092	-0.34182	-0.83421	-0.82141	-0.53400	-0.18798	0.01526	0.10323	0.10196	0.03598	-0.07164	-0.19933
-0.00002	-0.49979	-0.73702	-0.55259	0.10923	1.20073	2.16965	2.38644	1.14415	-0.34148	-0.93207	-0.92003	-0.58037	-0.17072	0.06870	0.17070	0.16631	0.08479	-0.04639	-0.20150
-0.00001	-0.27553	-0.38869	-0.27003	0.05702	0.47612	0.90092	1.14415	0.91768	-0.16751	-0.60130	-0.59648	-0.35317	-0.05883	0.11173	0.18238	0.17568	0.11289	0.01400	-0.10230
-0.00001	-0.07159	-0.07202	-0.01328	0.00955	-0.18173	-0.34182	-0.34148	-0.16751	0.07726	0.17394	0.17091	0.11370	0.04494	0.00437	-0.01342	0.01359	-0.00098	0.01986	0.04466
0.00000	0.01294	0.05809	0.09090	-0.00991	-0.44235	-0.83421	-0.93207	-0.60130	0.17394	1.29070	1.29064	0.68320	-0.05444	-0.47727	-0.64598	-0.61794	-0.44726	-0.18476	0.12200
0.00000	0.01703	0.06254	0.09232	-0.01052	-0.43554	-0.82141	-0.92003	-0.59648	0.17091	1.29064	1.90722	1.09750	-0.12932	-0.83168	-1.11067	-1.06188	-0.77545	-0.33605	0.17710
-0.00001	-0.02268	-0.00140	0.03784	-0.00086	-0.28335	-0.53400	-0.58037	-0.35317	0.11370	0.68320	1.09750	0.99686	-0.08039	-0.69790	-0.94425	-0.90325	-0.65393	-0.27052	0.17755
-0.00001	-0.07185	-0.08009	-0.02865	0.01100	-0.10013	-0.18798	-0.17072	-0.05883	0.04494	-0.05444	-0.12932	-0.08039	0.06332	0.14449	0.17517	0.16676	0.13001	0.07502	0.01124
-0.00001	-0.09848	-0.12350	-0.06622	0.01758	0.00749	0.01526	0.06870	0.11173	0.00437	-0.47727	-0.83168	-0.69790	0.14449	1.77534	2.42811	2.32366	1.67024	0.66344	-0.51372
-0.00001	-0.10690	-0.13842	-0.08044	0.01987	0.05410	0.10323	0.17070	0.18238	-0.01342	-0.64598	-1.11067	-0.94425	0.17517	2.42811	4.36695	4.51425	3.36267	1.37503	-1.01571
-0.00001	-0.10124	-0.13130	-0.07660	0.01886	0.05346	0.10196	0.16631	0.17568	-0.01359	-0.61794	-1.06188	-0.90325	0.16676	2.32366	4.51425	5.63414	4.52777	1.98841	-1.24186
-0.00001	-0.08536	-0.10819	-0.05964	0.01544	0.01856	0.03598	0.08479	0.11289	-0.00098	-0.44726	-0.77545	-0.65393	0.13001	1.67024	3.36267	4.52777	4.41056	2.18060	-1.06801
0.00000	-0.06290	-0.07479	-0.03426	0.01048	-0.03838	-0.07164	-0.04639	0.01400	0.01986	-0.18476	-0.33605	-0.27052	0.07502	0.66344	1.37503	1.98841	2.18060	1.52696	-0.49877
0.00000	-0.03727	-0.03642	-0.00480	0.00478	-0.10595	-0.19933	-0.20150	-0.10230	0.04466	0.12200	0.17710	0.17755	0.01124	-0.51372	-1.01571	-1.24186	-1.06801	-0.49877	0.33254

$N = (B^T B + 0.5 K_Z^T K_Z)^{-1} B^T$				
0.00943	0.00000	0.00000	0.00000	0.00000
0.00613	0.00325	-0.00066	0.00017	-0.00001
0.00310	0.00601	-0.00103	0.00027	-0.00002
0.00059	0.00785	-0.00084	0.00022	-0.00002
-0.00118	0.00834	0.00015	-0.00004	0.00000
-0.00197	0.00710	0.00219	-0.00057	0.00005
-0.00206	0.00477	0.00460	-0.00107	0.00009
-0.00173	0.00201	0.00677	-0.00128	0.00011
-0.00120	-0.00061	0.00811	-0.00093	0.00008
-0.00073	-0.00253	0.00808	0.00021	-0.00002
-0.00051	-0.00323	0.00618	0.00237	-0.00020
-0.00048	-0.00312	0.00325	0.00487	-0.00034
-0.00054	-0.00256	0.00009	0.00712	-0.00030
-0.00062	-0.00190	-0.00257	0.00851	0.00004
-0.00065	-0.00148	-0.00405	0.00853	0.00079
-0.00064	-0.00125	-0.00456	0.00743	0.00187
-0.00061	-0.00117	-0.00432	0.00548	0.00320
-0.00055	-0.00119	-0.00355	0.00290	0.00472
-0.00047	-0.00128	-0.00243	-0.00009	0.00635
-0.00039	-0.00141	-0.00114	-0.00327	0.00805

The discount factors obtained by multiplying N times P, and the corresponding spot rates are shown in the following table :

Month	Discount Factor	Spot Rate
12	0.93396	7.0707%
24	0.88157	6.5052%
36	0.82976	6.4183%
48	0.77905	6.4408%
60	0.72997	6.4974%
72	0.68298	6.5612%
84	0.63781	6.6355%
96	0.59421	6.7230%
108	0.55193	6.8266%
120	0.51076	6.9494%
132	0.47047	7.0952%
144	0.43167	7.2517%
156	0.39492	7.4082%
168	0.36076	7.5541%
180	0.32966	7.6785%
192	0.30102	7.7922%
204	0.27430	7.9060%
216	0.24896	8.0310%
228	0.22451	8.1795%
240	0.20050	8.3664%

Appendix C : Cash Flow Tables

Original Asset Cash Flows by Year and Month (in millions, rounded)

Year	1	2	3	4	5	6	7	8	9	10	11	12	Total
1	737.3	184.6	130.6	149.1	142.3	194.0	139.4	233.4	298.4	162.4	143.1	141.0	2,656.6
2	121.5	163.7	173.3	126.2	167.2	135.4	141.5	216.3	130.3	129.3	113.8	165.0	1,785.5
3	158.9	101.1	83.2	115.3	106.3	125.3	115.3	167.8	111.6	120.0	0.0	53.0	1,260.7
4	131.8	116.1	95.3	122.7	143.8	92.5	119.0	94.0	92.4	144.4	119.9	108.1	1,383.9
5	123.2	54.0	69.6	104.6	133.2	89.8	55.8	127.3	110.1	81.7	79.3	39.0	1,072.5
6	102.4	49.8	82.2	47.1	21.5	31.8	21.1	20.1	42.3	22.6	20.3	6.5	473.9
7	19.7	20.7	49.5	24.9	23.1	34.8	13.4	20.8	33.9	49.8	14.2	23.7	335.5
8	19.5	21.2	30.4	24.4	24.3	24.6	11.1	21.9	19.2	41.9	16.0	16.6	278.9
9	26.8	12.1	15.0	12.9	20.7	24.4	12.5	54.0	21.3	19.4	18.4	28.8	275.1
10	17.7	12.6	32.8	15.1	12.4	22.1	10.5	11.6	14.1	26.6	10.4	18.4	214.3
11	16.8	18.4	24.4	13.1	20.6	34.7	13.9	5.2	11.4	21.1	9.1	45.9	245.8
12	13.3	4.8	22.5	15.7	26.3	23.9	14.0	4.9	17.5	28.1	14.2	13.5	210.8
13	10.9	4.3	22.6	14.9	12.2	13.4	9.9	7.1	25.1	26.9	16.2	25.7	202.2
14	11.0	7.6	10.2	12.6	8.6	47.0	15.8	19.4	17.1	19.4	27.0	26.3	235.9
15	12.6	9.4	20.9	11.1	8.2	17.5	20.0	23.3	48.1	17.0	4.8	10.8	218.8
16	12.4	8.6	14.9	38.2	13.6	13.1	10.2	1.7	37.0	19.3	11.8	26.7	223.4
17	11.8	8.1	60.9	24.9	5.7	22.2	5.7	14.9	62.6	46.6	7.0	11.0	298.3
18	15.3	4.9	11.9	17.1	3.1	13.8	11.4	19.5	30.9	5.0	26.9	12.4	190.4
19	25.9	8.8	10.8	19.1	8.7	13.5	29.7	6.7	12.6	45.6	7.3	8.4	216.3
20	7.2	6.5	11.6	8.3	6.9	8.8	0.0	14.9	14.9	8.7	0.0	8.3	116.1
21	7.0	6.5	7.6	6.4	1.8	10.4	1.1	8.3	111.7	11.6	0.2	13.8	207.5
22	0.2	17.7	0.0	3.1	0.3	12.6	6.3	0.0	1.0	9.1	0.9	62.6	135.9
23	0.0	3.9	7.3	16.8	12.5	5.2	0.0	2.0	6.0	8.7	0.9	5.4	91.6
24	0.0	7.7	2.0	1.1	2.5	10.4	4.2	1.7	0.9	1.6	3.4	4.2	63.7
25	8.2	3.1	2.7	0.1	5.9	1.6	5.1	4.9	11.5	3.9	0.1	8.9	81.0
26	4.0	2.1	6.1	0.0	0.0	4.3	2.4	7.4	5.4	0.1	0.0	9.1	66.8
27	0.0	0.0	4.1	3.2	2.0	10.3	0.0	1.3	0.0	39.5	5.0	5.3	97.7
28	5.9	14.8	5.3	3.8	16.9	11.6	0.0	0.0	19.8	0.0	0.0	33.5	139.6
29	4.5	46.5	2.5	0.0	0.0	7.0	2.7	0.0	0.0	0.5	2.0	94.8	189.4
30	35.2	25.5	0.0	2.8	2.4	0.0	3.4	0.0	0.0	2.4	4.6	0.0	106.3

Appendix C : Cash Flow Tables

Liability Cash Flows by Year and Month (in millions, rounded)

Year	1	2	3	4	5	6	7	8	9	10	11	12	Total
1	79.0	160.4	205.5	208.1	243.4	202.9	229.3	306.4	392.4	190.9	175.4	167.1	2,561.8
2	126.7	143.6	204.3	160.0	139.8	118.7	142.4	151.5	235.6	131.4	143.8	147.9	1,847.7
3	124.8	133.2	202.3	154.8	125.6	117.1	135.8	147.2	156.1	78.3	72.2	85.1	1,535.5
4	74.8	91.9	127.2	115.9	124.6	77.6	83.6	96.8	113.7	59.6	67.9	75.9	1,113.5
5	74.4	80.5	91.5	95.1	86.9	80.7	95.5	101.7	123.0	80.5	59.4	60.2	1,034.4
6	29.6	27.6	29.0	19.8	25.9	25.9	35.2	29.5	28.0	24.0	25.1	21.1	326.7
7	26.0	22.0	28.3	26.8	23.3	25.4	36.6	22.9	18.4	18.5	22.5	24.7	302.4
8	28.6	23.0	20.2	27.2	20.6	28.3	38.4	25.0	21.2	27.4	26.2	25.0	319.3
9	27.6	21.2	24.8	29.7	25.5	28.7	31.2	23.7	24.0	23.9	26.7	23.3	319.3
10	21.0	24.1	27.9	25.3	24.3	23.5	31.1	21.8	19.4	26.9	16.6	16.4	288.3
11	24.0	22.5	22.2	18.5	19.5	15.7	22.7	23.1	18.4	19.2	18.4	20.2	255.2
12	17.1	15.3	19.8	17.6	14.7	21.6	19.7	20.2	16.9	17.6	18.3	15.4	226.1
13	20.0	18.6	14.6	14.1	22.5	24.5	26.1	18.8	21.0	20.9	19.8	19.7	253.5
14	18.8	18.0	16.4	20.2	20.9	21.6	26.0	23.0	20.0	18.1	21.2	18.3	256.6
15	18.3	15.6	21.6	16.7	13.6	21.3	16.8	15.5	13.9	21.9	15.3	18.2	223.8
16	21.1	17.2	10.7	13.3	16.2	20.1	21.9	13.1	16.1	12.8	11.4	16.9	206.9
17	20.1	16.0	14.0	18.4	10.7	22.7	19.2	18.1	16.9	14.2	13.5	13.8	214.6
18	14.7	18.3	10.2	15.2	19.2	16.4	17.1	21.4	15.0	13.7	14.4	13.9	207.6
19	18.1	15.0	17.4	13.4	12.5	13.5	16.0	9.3	8.1	12.6	17.3	14.8	186.9
20	17.3	8.8	17.2	15.3	14.4	16.5	12.1	10.6	17.5	14.4	8.0	8.0	180.0
21	10.7	10.4	10.8	16.1	11.8	10.8	8.3	10.8	10.9	6.6	15.4	8.4	152.2
22	5.4	13.8	10.3	11.2	10.6	12.2	7.3	12.4	6.8	9.0	5.7	5.7	132.6
23	10.9	8.7	12.0	4.9	7.2	8.2	8.8	7.4	13.7	11.0	8.8	5.6	130.1
24	13.4	3.9	12.0	12.2	14.8	9.3	5.7	8.2	5.6	11.9	11.2	4.9	137.0
25	13.2	10.9	7.3	11.1	10.6	8.7	5.7	7.7	6.4	2.7	11.9	11.2	132.4
26	4.0	4.6	2.5	8.8	10.9	3.7	4.2	3.9	0.0	0.0	0.0	5.0	73.7
27	34.8	2.2	5.0	3.7	0.0	2.2	29.8	0.0	1.0	0.0	0.0	1.1	107.0
28	36.9	3.6	0.0	0.0	0.0	0.7	34.4	4.6	0.0	0.0	0.0	0.0	108.1
29	31.9	0.0	1.0	4.2	0.0	0.0	27.8	3.1	0.0	0.0	1.0	0.0	97.9
30	29.3	1.0	3.7	3.8	0.0	0.0	27.3	0.0	3.0	2.3	1.0	0.0	101.4

Appendix C : Cash Flow Tables

Matched Asset Cash Flows by Year and Month (in millions, rounded)

Year	1	2	3	4	5	6	7	8	9	10	11	12	Total
1	41.1	184.6	325.5	149.1	142.3	296.8	139.4	233.4	298.4	162.4	143.1	587.9	2,705.1
2	121.5	163.7	173.3	126.2	167.2	132.0	141.5	216.3	130.3	129.3	113.8	161.6	1,778.7
3	158.9	101.1	83.2	115.3	106.3	121.9	115.3	167.8	111.6	120.0	0.0	97.2	1,301.4
4	131.8	116.1	95.3	122.7	143.8	87.6	119.0	94.0	92.4	144.4	119.9	103.3	1,374.2
5	123.2	54.0	69.6	104.6	133.2	85.0	55.8	127.3	110.1	81.7	79.3	-307.9	720.8
6	102.4	49.8	82.2	47.1	21.5	39.3	21.1	20.1	42.3	22.6	20.3	14.1	488.9
7	19.7	20.7	49.5	24.9	23.1	42.4	13.4	20.8	33.9	49.8	14.2	112.9	432.3
8	19.5	21.2	30.4	24.4	24.3	29.0	11.1	21.9	19.2	41.9	16.0	21.1	287.9
9	26.8	12.1	15.0	12.9	20.7	28.8	12.5	54.0	21.3	19.4	18.4	33.3	284.1
10	17.7	12.6	32.8	15.1	12.4	26.6	10.5	11.6	14.1	26.6	10.4	106.4	306.7
11	16.8	18.4	24.4	13.1	20.6	35.9	13.9	5.2	11.4	21.1	9.1	47.1	248.2
12	13.3	4.8	22.5	15.7	26.3	25.0	14.0	4.9	17.5	28.1	14.2	14.7	213.1
13	10.9	4.3	22.6	14.9	12.2	14.6	9.9	7.1	25.1	26.9	16.2	26.8	204.6
14	11.0	7.6	10.2	12.6	8.6	48.2	15.8	19.4	17.1	19.4	27.0	27.4	238.3
15	12.6	9.4	20.9	11.1	8.2	18.7	20.0	23.3	48.1	17.0	4.8	12.0	221.1
16	12.4	8.6	14.9	38.2	13.6	14.2	10.2	1.7	37.0	19.3	11.8	27.9	225.8
17	11.8	8.1	60.9	24.9	5.7	23.4	5.7	14.9	62.6	46.6	7.0	12.1	300.7
18	15.3	4.9	11.9	17.1	3.1	15.0	11.4	19.5	30.9	5.0	26.9	13.6	192.7
19	25.9	8.8	10.8	19.1	8.7	14.7	29.7	6.7	12.6	45.6	7.3	9.6	218.6
20	7.2	6.5	11.6	8.3	6.9	10.0	0.0	14.9	14.9	8.7	0.0	80.1	189.1
21	7.0	6.5	7.6	6.4	1.8	8.6	1.1	8.3	111.7	11.6	0.2	12.0	203.8
22	0.2	17.7	0.0	3.1	0.3	10.8	6.3	0.0	1.0	9.1	0.9	60.8	132.2
23	0.0	3.9	7.3	16.8	12.5	3.3	0.0	2.0	6.0	8.7	0.9	3.6	87.9
24	0.0	7.7	2.0	1.1	2.5	8.6	4.2	1.7	0.9	1.6	3.4	2.3	60.0
25	8.2	3.1	2.7	0.1	5.9	-0.2	5.1	4.9	11.5	3.9	0.1	7.0	77.3
26	4.0	2.1	6.1	0.0	0.0	2.4	2.4	7.4	5.4	0.1	0.0	7.2	63.1
27	0.0	0.0	4.1	3.2	2.0	8.5	0.0	1.3	0.0	39.5	5.0	3.4	94.0
28	5.9	14.8	5.3	3.8	16.9	9.7	0.0	0.0	19.8	0.0	0.0	31.6	136.0
29	4.5	46.5	2.5	0.0	0.0	5.2	2.7	0.0	0.0	0.5	2.0	92.9	185.8
30	35.2	25.5	0.0	2.8	2.4	-1.8	3.4	0.0	0.0	2.4	4.6	-44.6	59.9

Appendix D : $N = (B^T B + h K_z^T K_z)^{-1} B^T$

1 mo	3 mo	6 mo	1 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
0.009962	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.004265	0.006196	-0.000577	0.000040	-0.000002	0.000000	0.000000	0.000000	0.000000	0.000000
0.000000	0.009882	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
-0.001401	0.008551	0.002886	-0.000200	0.000008	-0.000002	0.000000	0.000000	0.000000	0.000000
-0.001035	0.004493	0.006623	-0.000290	0.000011	-0.000003	0.000001	0.000000	0.000000	0.000000
0.000000	0.000000	0.009756	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.000607	-0.002636	0.010828	0.000941	-0.000036	0.000009	-0.000002	0.000000	0.000000	0.000000
0.000866	-0.003759	0.010253	0.002370	-0.000079	0.000021	-0.000005	0.000001	0.000000	0.000000
0.000856	-0.003714	0.008443	0.004126	-0.000114	0.000030	-0.000007	0.000001	0.000000	0.000000
0.000655	-0.002843	0.005813	0.006047	-0.000124	0.000032	-0.000007	0.000001	0.000000	0.000000
0.000343	-0.001490	0.002775	0.007971	-0.000091	0.000024	-0.000005	0.000001	0.000000	0.000000
0.000000	0.000000	-0.000256	0.009737	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
-0.000296	0.001284	-0.002869	0.011182	0.000167	-0.000043	0.000010	-0.000001	0.000000	0.000000
-0.000547	0.002373	-0.005085	0.012326	0.000403	-0.000103	0.000023	-0.000003	0.000000	0.000000
-0.000755	0.003279	-0.006928	0.013187	0.000700	-0.000177	0.000040	-0.000005	0.000000	0.000000
-0.000924	0.004013	-0.008421	0.013785	0.001053	-0.000262	0.000059	-0.000007	0.000000	0.000000
-0.001057	0.004586	-0.009586	0.014137	0.001453	-0.000355	0.000080	-0.000009	0.000000	0.000000
-0.001154	0.005009	-0.010448	0.014264	0.001895	-0.000454	0.000102	-0.000012	0.000001	0.000000
-0.001220	0.005295	-0.011029	0.014184	0.002371	-0.000554	0.000125	-0.000015	0.000001	0.000000
-0.001256	0.005453	-0.011352	0.013915	0.002874	-0.000654	0.000147	-0.000017	0.000001	0.000000
-0.001266	0.005496	-0.011439	0.013476	0.003398	-0.000751	0.000169	-0.000020	0.000001	0.000000
-0.001252	0.005434	-0.011312	0.012885	0.003937	-0.000842	0.000190	-0.000022	0.000001	0.000000
-0.001216	0.005277	-0.010994	0.012160	0.004482	-0.000924	0.000208	-0.000025	0.000001	0.000000
-0.001161	0.005038	-0.010508	0.011320	0.005029	-0.000994	0.000224	-0.000027	0.000001	0.000000
-0.001089	0.004727	-0.009876	0.010384	0.005569	-0.001050	0.000237	-0.000028	0.000001	0.000000
-0.001003	0.004355	-0.009119	0.009368	0.006097	-0.001089	0.000246	-0.000029	0.000001	0.000000
-0.000906	0.003933	-0.008260	0.008292	0.006605	-0.001107	0.000250	-0.000030	0.000001	0.000000
-0.000800	0.003471	-0.007320	0.007173	0.007088	-0.001103	0.000249	-0.000030	0.000001	0.000000
-0.000686	0.002980	-0.006322	0.006028	0.007539	-0.001074	0.000243	-0.000029	0.000001	0.000000
-0.000569	0.002471	-0.005286	0.004876	0.007951	-0.001016	0.000230	-0.000027	0.000001	0.000000
-0.000450	0.001954	-0.004235	0.003734	0.008318	-0.000927	0.000210	-0.000025	0.000001	0.000000
-0.000332	0.001440	-0.003191	0.002621	0.008634	-0.000805	0.000182	-0.000022	0.000001	0.000000
-0.000217	0.000940	-0.002173	0.001552	0.008891	-0.000646	0.000146	-0.000017	0.000001	0.000000
-0.000107	0.000463	-0.001204	0.000546	0.009084	-0.000449	0.000101	-0.000012	0.000001	0.000000
-0.000005	0.000021	-0.000305	-0.000380	0.009207	-0.000209	0.000047	-0.000006	0.000000	0.000000
0.000087	-0.000376	0.000503	-0.001209	0.009252	0.000074	-0.000017	0.000002	0.000000	0.000000
0.000166	-0.000719	0.001200	-0.001923	0.009214	0.000405	-0.000092	0.000011	0.000000	0.000000
0.000233	-0.001010	0.001790	-0.002529	0.009099	0.000777	-0.000175	0.000021	-0.000001	0.000000
0.000288	-0.001251	0.002281	-0.003033	0.008912	0.001186	-0.000265	0.000031	-0.000001	0.000000
0.000333	-0.001447	0.002679	-0.003441	0.008659	0.001626	-0.000358	0.000042	-0.000002	0.000000
0.000369	-0.001599	0.002989	-0.003760	0.008347	0.002092	-0.000455	0.000054	-0.000002	0.000000
0.000395	-0.001712	0.003218	-0.003996	0.007981	0.002579	-0.000551	0.000065	-0.000003	0.000000
0.000412	-0.001788	0.003373	-0.004155	0.007567	0.003081	-0.000644	0.000076	-0.000004	0.000001
0.000422	-0.001831	0.003458	-0.004244	0.007111	0.003594	-0.000734	0.000087	-0.000004	0.000001
0.000424	-0.001842	0.003480	-0.004268	0.006618	0.004112	-0.000817	0.000097	-0.000004	0.000001
0.000421	-0.001825	0.003446	-0.004234	0.006094	0.004631	-0.000892	0.000106	-0.000005	0.000001
0.000411	-0.001783	0.003361	-0.004148	0.005545	0.005145	-0.000957	0.000113	-0.000005	0.000001
0.000396	-0.001720	0.003230	-0.004015	0.004976	0.005650	-0.001009	0.000120	-0.000005	0.000001
0.000377	-0.001637	0.003061	-0.003843	0.004394	0.006139	-0.001046	0.000124	-0.000006	0.000001
0.000354	-0.001537	0.002859	-0.003637	0.003804	0.006609	-0.001067	0.000126	-0.000006	0.000001
0.000328	-0.001424	0.002629	-0.003402	0.003211	0.007055	-0.001070	0.000127	-0.000006	0.000001
0.000300	-0.001301	0.002378	-0.003145	0.002620	0.007471	-0.001052	0.000125	-0.000006	0.000001
0.000270	-0.001170	0.002111	-0.002872	0.002038	0.007852	-0.001011	0.000120	-0.000006	0.000001
0.000238	-0.001034	0.001833	-0.002589	0.001470	0.008195	-0.000946	0.000112	-0.000005	0.000001

Appendix D : $N = (B^T B + h K_z^T K_z)^{-1} B^T$

1 mo	3 mo	6 mo	1 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
0.000206	-0.000895	0.001552	-0.002301	0.000921	0.008493	-0.000854	0.000101	-0.000005	0.000001
0.000175	-0.000758	0.001272	-0.002014	0.000397	0.008743	-0.000734	0.000087	-0.000004	0.000001
0.000144	-0.000623	0.000999	-0.001734	-0.000098	0.008939	-0.000584	0.000069	-0.000003	0.000000
0.000114	-0.000495	0.000739	-0.001467	-0.000557	0.009077	-0.000401	0.000048	-0.000002	0.000000
0.000087	-0.000376	0.000497	-0.001218	-0.000976	0.009152	-0.000184	0.000022	-0.000001	0.000000
0.000062	-0.000269	0.000278	-0.000993	-0.001349	0.009159	0.000069	-0.000008	0.000000	0.000000
0.000040	-0.000175	0.000089	-0.000798	-0.001670	0.009094	0.000360	-0.000043	0.000002	0.000000
0.000022	-0.000096	-0.000072	-0.000631	-0.001944	0.008962	0.000685	-0.000081	0.000004	-0.000001
0.000007	-0.000029	-0.000207	-0.000491	-0.002171	0.008767	0.001041	-0.000121	0.000006	-0.000001
-0.000006	0.000026	-0.000318	-0.000376	-0.002355	0.008514	0.001424	-0.000164	0.000008	-0.000001
-0.000016	0.000069	-0.000405	-0.000284	-0.002500	0.008209	0.001829	-0.000207	0.000009	-0.000001
-0.000023	0.000102	-0.000471	-0.000214	-0.002606	0.007856	0.002255	-0.000250	0.000011	-0.000002
-0.000029	0.000125	-0.000517	-0.000164	-0.002678	0.007459	0.002696	-0.000292	0.000013	-0.000002
-0.000032	0.000139	-0.000544	-0.000134	-0.002717	0.007024	0.003149	-0.000331	0.000015	-0.000002
-0.000033	0.000145	-0.000554	-0.000120	-0.002727	0.006556	0.003611	-0.000368	0.000017	-0.000003
-0.000033	0.000143	-0.000549	-0.000122	-0.002710	0.006058	0.004078	-0.000401	0.000018	-0.000003
-0.000031	0.000134	-0.000530	-0.000138	-0.002669	0.005537	0.004546	-0.000430	0.000020	-0.000003
-0.000027	0.000119	-0.000499	-0.000167	-0.002606	0.004995	0.005012	-0.000452	0.000021	-0.000003
-0.000023	0.000099	-0.000456	-0.000207	-0.002524	0.004439	0.005472	-0.000469	0.000022	-0.000003
-0.000017	0.000074	-0.000404	-0.000256	-0.002425	0.003873	0.005923	-0.000477	0.000022	-0.000003
-0.000010	0.000045	-0.000344	-0.000314	-0.002313	0.003301	0.006362	-0.000478	0.000022	-0.000003
0.000003	0.000013	-0.000278	-0.000378	-0.002189	0.002728	0.006784	-0.000469	0.000022	-0.000003
0.000005	-0.000021	-0.000206	-0.000448	-0.002056	0.002158	0.007187	-0.000451	0.000021	-0.000003
0.000013	-0.000057	-0.000131	-0.000521	-0.001917	0.001596	0.007566	-0.000421	0.000019	-0.000003
0.000022	-0.000095	-0.000054	-0.000596	-0.001774	0.001047	0.007919	-0.000380	0.000017	-0.000003
0.000031	-0.000133	0.000025	-0.000673	-0.001629	0.000514	0.008243	-0.000326	0.000015	-0.000002
0.000039	-0.000170	0.000102	-0.000748	-0.001486	0.000003	0.008533	-0.000258	0.000012	-0.000002
0.000048	-0.000206	0.000178	-0.000822	-0.001346	-0.000483	0.008786	-0.000177	0.000008	-0.000001
0.000056	-0.000241	0.000250	-0.000892	-0.001212	-0.000939	0.009000	-0.000080	0.000004	-0.000001
0.000063	-0.000274	0.000317	-0.000958	-0.001087	-0.001360	0.009171	0.000032	-0.000001	0.000000
0.000070	-0.000303	0.000378	-0.001017	-0.000973	-0.001743	0.009295	0.000162	-0.000007	0.000001
0.000076	-0.000329	0.000434	-0.001070	-0.000869	-0.002089	0.009375	0.000307	-0.000014	0.000002
0.000081	-0.000353	0.000483	-0.001117	-0.000775	-0.002398	0.009412	0.000468	-0.000021	0.000003
0.000086	-0.000374	0.000527	-0.001158	-0.000691	-0.002673	0.009408	0.000642	-0.000029	0.000004
0.000090	-0.000392	0.000565	-0.001194	-0.000617	-0.002914	0.009364	0.000830	-0.000037	0.000006
0.000094	-0.000408	0.000598	-0.001225	-0.000551	-0.003123	0.009284	0.001031	-0.000045	0.000007
0.000097	-0.000421	0.000626	-0.001250	-0.000494	-0.003301	0.009168	0.001243	-0.000054	0.000008
0.000100	-0.000432	0.000650	-0.001271	-0.000446	-0.003451	0.009018	0.001466	-0.000063	0.000009
0.000102	-0.000441	0.000669	-0.001288	-0.000405	-0.003572	0.008836	0.001699	-0.000071	0.000011
0.000103	-0.000448	0.000684	-0.001300	-0.000372	-0.003666	0.008624	0.001941	-0.000080	0.000012
0.000104	-0.000452	0.000694	-0.001308	-0.000346	-0.003734	0.008384	0.002192	-0.000088	0.000013
0.000105	-0.000455	0.000701	-0.001312	-0.000327	-0.003779	0.008117	0.002450	-0.000096	0.000015
0.000105	-0.000456	0.000704	-0.001312	-0.000314	-0.003801	0.007826	0.002714	-0.000104	0.000016
0.000105	-0.000455	0.000704	-0.001308	-0.000307	-0.003800	0.007511	0.002985	-0.000111	0.000017
0.000104	-0.000453	0.000700	-0.001302	-0.000307	-0.003780	0.007174	0.003261	-0.000118	0.000018
0.000103	-0.000449	0.000693	-0.001292	-0.000311	-0.003740	0.006818	0.003541	-0.000124	0.000019
0.000102	-0.000444	0.000683	-0.001279	-0.000321	-0.003683	0.006444	0.003824	-0.000130	0.000020
0.000101	-0.000437	0.000671	-0.001264	-0.000335	-0.003609	0.006054	0.004111	-0.000134	0.000020
0.000099	-0.000429	0.000656	-0.001246	-0.000354	-0.003519	0.005649	0.004399	-0.000138	0.000021
0.000097	-0.000420	0.000639	-0.001226	-0.000376	-0.003415	0.005231	0.004688	-0.000141	0.000021
0.000095	-0.000410	0.000620	-0.001204	-0.000402	-0.003299	0.004802	0.004978	-0.000142	0.000022
0.000092	-0.000400	0.000599	-0.001179	-0.000432	-0.003170	0.004363	0.005267	-0.000143	0.000022
0.000089	-0.000388	0.000576	-0.001153	-0.000465	-0.003031	0.003916	0.005556	-0.000142	0.000022
0.000087	-0.000375	0.000552	-0.001126	-0.000500	-0.002883	0.003463	0.005842	-0.000140	0.000021
0.000083	-0.000362	0.000526	-0.001097	-0.000537	-0.002726	0.003005	0.006125	-0.000137	0.000021

Appendix D : $N = (B^T B + h K_z^T K_z)^{-1} B^T$

1 mo	3 mo	6 mo	1 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
0.000080	-0.000349	0.000499	-0.001067	-0.000577	-0.002563	0.002544	0.006406	-0.000132	0.000020
0.000077	-0.000335	0.000471	-0.001036	-0.000618	-0.002393	0.002082	0.006682	-0.000125	0.000019
0.000074	-0.000320	0.000443	-0.001005	0.000660	0.002219	0.001619	0.006953	-0.000117	0.000018
0.000070	-0.000306	0.000414	-0.000973	-0.000704	-0.002042	0.001159	0.007218	-0.000107	0.000016
0.000067	-0.000291	0.000385	-0.000940	-0.000748	-0.001862	0.000701	0.007478	-0.000095	0.000014
0.000064	-0.000276	0.000355	-0.000907	-0.000792	-0.001681	0.000248	0.007730	-0.000082	0.000012
0.000060	-0.000261	0.000326	-0.000875	-0.000837	-0.001500	-0.000198	0.007974	-0.000066	0.000010
0.000057	-0.000246	0.000296	-0.000842	-0.000881	-0.001320	-0.000637	0.008209	-0.000048	0.000007
0.000053	-0.000231	0.000267	-0.000810	-0.000924	-0.001143	-0.001066	0.008436	-0.000028	0.000004
0.000050	-0.000217	0.000239	-0.000778	-0.000967	-0.000969	-0.001484	0.008652	-0.000006	0.000001
0.000047	-0.000203	0.000211	-0.000747	-0.001008	-0.000799	-0.001890	0.008858	0.000018	-0.000003
0.000044	-0.000189	0.000184	-0.000717	-0.001048	-0.000635	-0.002281	0.009053	0.000045	-0.000007
0.000040	-0.000176	0.000158	-0.000688	-0.001086	-0.000476	-0.002659	0.009236	0.000074	-0.000011
0.000038	-0.000163	0.000133	-0.000660	-0.001123	-0.000323	-0.003023	0.009408	0.000105	-0.000016
0.000035	-0.000151	0.000109	-0.000633	-0.001158	-0.000175	-0.003374	0.009570	0.000138	-0.000021
0.000032	-0.000139	0.000086	-0.000607	-0.001192	-0.000032	-0.003712	0.009721	0.000174	-0.000026
0.000029	-0.000127	0.000063	-0.000581	-0.001224	0.000105	-0.004036	0.009861	0.000212	-0.000032
0.000027	-0.000116	0.000041	-0.000556	-0.001255	0.000237	-0.004348	0.009991	0.000252	-0.000038
0.000024	-0.000105	0.000021	-0.000532	-0.001284	0.000364	-0.004647	0.010111	0.000294	-0.000044
0.000022	-0.000095	0.000000	-0.000509	-0.001312	0.000487	-0.004934	0.010221	0.000337	-0.000051
0.000020	-0.000085	-0.000019	-0.000487	-0.001339	0.000604	-0.005208	0.010321	0.000383	-0.000057
0.000017	-0.000075	-0.000037	-0.000465	-0.001364	0.000716	-0.005471	0.010411	0.000430	-0.000064
0.000015	-0.000066	-0.000055	-0.000445	-0.001388	0.000824	-0.005721	0.010492	0.000480	-0.000071
0.000013	-0.000057	-0.000072	-0.000425	-0.001411	0.000927	-0.005960	0.010563	0.000531	-0.000079
0.000011	-0.000049	-0.000088	-0.000406	-0.001432	0.001026	-0.006187	0.010626	0.000584	-0.000086
0.000009	-0.000041	-0.000104	-0.000387	-0.001453	0.001119	-0.006403	0.010679	0.000638	-0.000094
0.000008	-0.000033	-0.000119	-0.000370	-0.001471	0.001209	-0.006607	0.010724	0.000694	-0.000102
0.000006	-0.000026	-0.000133	-0.000353	-0.001489	0.001294	-0.006801	0.010759	0.000751	-0.000110
0.000004	-0.000019	-0.000146	-0.000336	-0.001506	0.001374	-0.006983	0.010787	0.000810	-0.000118
0.000003	-0.000012	-0.000159	-0.000321	-0.001521	0.001451	-0.007156	0.010806	0.000871	-0.000126
0.000001	-0.000005	-0.000171	-0.000306	-0.001535	0.001523	-0.007317	0.010817	0.000932	-0.000135
0.000000	0.000001	-0.000182	-0.000292	-0.001548	0.001591	-0.007469	0.010819	0.000996	-0.000143
-0.000001	0.000006	-0.000193	-0.000278	-0.001560	0.001655	-0.007610	0.010814	0.001060	-0.000152
-0.000003	0.000012	-0.000203	-0.000265	-0.001571	0.001715	-0.007742	0.010802	0.001126	-0.000161
-0.000004	0.000017	-0.000213	-0.000253	-0.001580	0.001772	-0.007863	0.010781	0.001192	-0.000170
-0.000005	0.000022	-0.000221	-0.000241	-0.001589	0.001824	-0.007975	0.010754	0.001260	-0.000178
-0.000006	0.000026	-0.000230	-0.000230	-0.001597	0.001873	-0.008078	0.010719	0.001329	-0.000187
-0.000007	0.000031	-0.000237	-0.000220	-0.001603	0.001918	-0.008172	0.010677	0.001400	-0.000196
-0.000008	0.000034	-0.000244	-0.000210	-0.001609	0.001959	-0.008256	0.010628	0.001471	-0.000205
-0.000009	0.000038	-0.000251	-0.000201	-0.001613	0.001997	-0.008332	0.010572	0.001543	-0.000214
-0.000010	0.000041	-0.000257	-0.000193	-0.001617	0.002031	-0.008399	0.010510	0.001616	-0.000223
-0.000010	0.000045	-0.000262	-0.000185	-0.001619	0.002062	-0.008457	0.010441	0.001689	-0.000232
-0.000011	0.000047	-0.000267	-0.000177	-0.001621	0.002089	-0.008507	0.010366	0.001764	-0.000241
-0.000012	0.000050	-0.000271	-0.000170	-0.001622	0.002114	-0.008549	0.010285	0.001839	-0.000250
-0.000012	0.000052	-0.000275	-0.000164	-0.001622	0.002135	-0.008583	0.010198	0.001915	-0.000259
-0.000013	0.000054	-0.000278	-0.000158	-0.001621	0.002153	-0.008609	0.010104	0.001992	-0.000268
-0.000013	0.000056	-0.000281	-0.000152	-0.001619	0.002167	-0.008628	0.010006	0.002070	-0.000276
-0.000013	0.000058	-0.000283	-0.000148	-0.001617	0.002179	-0.008638	0.009901	0.002148	-0.000285
-0.000014	0.000059	-0.000285	-0.000143	-0.001613	0.002188	-0.008642	0.009791	0.002226	-0.000294
-0.000014	0.000060	-0.000286	-0.000139	-0.001609	0.002194	-0.008638	0.009676	0.002305	-0.000302
-0.000014	0.000061	-0.000287	-0.000136	-0.001604	0.002197	-0.008627	0.009556	0.002384	-0.000311
-0.000014	0.000062	-0.000288	-0.000133	-0.001598	0.002197	-0.008610	0.009431	0.002464	-0.000319
-0.000014	0.000062	-0.000288	-0.000130	-0.001592	0.002195	-0.008586	0.009301	0.002544	-0.000327
-0.000014	0.000063	-0.000287	-0.000128	-0.001585	0.002190	-0.008555	0.009166	0.002625	-0.000335
-0.000014	0.000063	-0.000287	-0.000126	-0.001577	0.002182	-0.008517	0.009027	0.002706	-0.000343

Appendix D : $N = (B^T B + h K_z^T K_z)^{-1} B^T$

1 mo	3 mo	6 mo	1 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
-0.000014	0.000063	-0.000285	-0.000125	-0.001568	0.002172	-0.008474	0.008883	0.002786	-0.000350
-0.000014	0.000062	-0.000284	-0.000124	-0.001559	0.002160	-0.008425	0.008735	0.002868	-0.000358
-0.000014	0.000062	-0.000282	-0.000124	-0.001549	0.002145	-0.008369	0.008583	0.002949	-0.000365
-0.000014	0.000061	-0.000279	-0.000124	-0.001538	0.002128	-0.008308	0.008427	0.003030	-0.000372
-0.000014	0.000060	-0.000277	-0.000124	-0.001527	0.002108	-0.008241	0.008268	0.003112	-0.000379
-0.000014	0.000059	-0.000273	-0.000125	-0.001515	0.002087	-0.008169	0.008104	0.003193	-0.000386
-0.000013	0.000058	-0.000270	-0.000126	-0.001503	0.002063	-0.008092	0.007937	0.003274	-0.000392
-0.000013	0.000057	-0.000266	-0.000127	-0.001490	0.002037	-0.008009	0.007766	0.003356	-0.000399
-0.000013	0.000055	-0.000262	-0.000129	-0.001477	0.002009	-0.007922	0.007592	0.003437	-0.000405
-0.000012	0.000053	-0.000258	-0.000131	-0.001463	0.001980	-0.007829	0.007415	0.003518	-0.000410
-0.000012	0.000051	-0.000253	-0.000133	-0.001448	0.001948	-0.007732	0.007235	0.003599	-0.000416
-0.000011	0.000049	-0.000248	-0.000136	-0.001433	0.001914	-0.007631	0.007053	0.003679	-0.000421
-0.000011	0.000047	-0.000243	-0.000139	-0.001418	0.001879	-0.007525	0.006867	0.003760	-0.000426
-0.000010	0.000045	-0.000237	-0.000142	-0.001402	0.001842	-0.007415	0.006679	0.003840	-0.000431
-0.000010	0.000043	-0.000231	-0.000146	-0.001386	0.001804	-0.007300	0.006488	0.003919	-0.000435
-0.000009	0.000040	-0.000225	-0.000150	-0.001369	0.001763	-0.007182	0.006295	0.003998	-0.000439
-0.000009	0.000037	-0.000219	-0.000154	-0.001352	0.001722	-0.007060	0.006099	0.004077	-0.000443
-0.000008	0.000035	-0.000212	-0.000158	-0.001334	0.001679	-0.006935	0.005902	0.004156	-0.000446
-0.000007	0.000032	-0.000205	-0.000163	-0.001316	0.001634	-0.006806	0.005702	0.004233	-0.000449
-0.000007	0.000029	-0.000198	-0.000168	-0.001298	0.001588	-0.006673	0.005501	0.004311	-0.000452
-0.000006	0.000025	-0.000191	-0.000173	-0.001279	0.001541	-0.006538	0.005298	0.004387	-0.000454
-0.000005	0.000022	-0.000184	-0.000178	-0.001260	0.001492	-0.006399	0.005093	0.004463	-0.000456
-0.000004	0.000019	-0.000176	-0.000183	-0.001241	0.001442	-0.006258	0.004887	0.004539	-0.000458
-0.000004	0.000016	-0.000168	-0.000189	-0.001221	0.001391	-0.006113	0.004680	0.004613	-0.000459
-0.000003	0.000012	-0.000160	-0.000195	-0.001202	0.001340	-0.005966	0.004471	0.004687	-0.000460
-0.000002	0.000008	-0.000152	-0.000201	-0.001181	0.001287	-0.005817	0.004261	0.004760	-0.000460
-0.000001	0.000005	-0.000144	-0.000207	-0.001161	0.001233	-0.005665	0.004051	0.004832	-0.000460
0.000000	0.000001	-0.000135	-0.000213	-0.001140	0.001178	-0.005511	0.003839	0.004904	-0.000460
0.000001	-0.000003	-0.000127	-0.000220	-0.001120	0.001122	-0.005355	0.003627	0.004974	-0.000459
0.000002	-0.000007	-0.000118	-0.000226	-0.001099	0.001066	-0.005197	0.003414	0.005044	-0.000458
0.000002	-0.000011	-0.000109	-0.000233	-0.001077	0.001008	-0.005037	0.003201	0.005112	-0.000456
0.000003	-0.000015	-0.000100	-0.000240	-0.001056	0.000950	-0.004876	0.002987	0.005180	-0.000454
0.000004	-0.000019	-0.000091	-0.000247	-0.001034	0.000892	-0.004713	0.002773	0.005247	-0.000451
0.000005	-0.000023	-0.000082	-0.000254	-0.001013	0.000833	-0.004548	0.002559	0.005312	-0.000448
0.000006	-0.000027	-0.000072	-0.000261	-0.000991	0.000773	-0.004382	0.002345	0.005376	-0.000444
0.000007	-0.000031	-0.000063	-0.000269	-0.000969	0.000713	-0.004216	0.002131	0.005440	-0.000440
0.000008	-0.000035	-0.000054	-0.000276	-0.000947	0.000652	-0.004048	0.001917	0.005502	-0.000436
0.000009	-0.000039	-0.000044	-0.000284	-0.000925	0.000591	-0.003879	0.001704	0.005562	-0.000431
0.000010	-0.000044	-0.000035	-0.000291	-0.000903	0.000530	-0.003709	0.001491	0.005622	-0.000425
0.000011	-0.000048	-0.000025	-0.000299	-0.000880	0.000468	-0.003539	0.001278	0.005680	-0.000419
0.000012	-0.000052	-0.000015	-0.000306	-0.000858	0.000406	-0.003369	0.001067	0.005737	-0.000412
0.000013	-0.000057	-0.000006	-0.000314	-0.000836	0.000344	-0.003197	0.000856	0.005793	-0.000405
0.000014	-0.000061	0.000004	-0.000322	-0.000813	0.000282	-0.003026	0.000646	0.005847	-0.000397
0.000015	-0.000065	0.000014	-0.000329	-0.000791	0.000219	-0.002855	0.000436	0.005900	-0.000389
0.000016	-0.000070	0.000023	-0.000337	-0.000769	0.000157	-0.002683	0.000229	0.005952	-0.000380
0.000017	-0.000074	0.000033	-0.000345	-0.000747	0.000095	-0.002512	0.000022	0.006001	-0.000371
0.000018	-0.000078	0.000043	-0.000353	-0.000724	0.000033	-0.002341	-0.000184	0.006050	-0.000361
0.000019	-0.000083	0.000052	-0.000361	-0.000702	-0.000030	-0.002170	-0.000387	0.006097	-0.000350
0.000020	-0.000087	0.000062	-0.000368	-0.000680	-0.000091	-0.002000	-0.000590	0.006142	-0.000339
0.000021	-0.000092	0.000072	-0.000376	-0.000658	-0.000153	-0.001831	-0.000791	0.006186	-0.000328
0.000022	-0.000096	0.000081	-0.000384	-0.000636	-0.000215	-0.001662	-0.000990	0.006228	-0.000315
0.000023	-0.000100	0.000091	-0.000391	-0.000614	-0.000276	-0.001494	-0.001187	0.006268	-0.000302
0.000024	-0.000104	0.000100	-0.000399	-0.000593	-0.000337	-0.001327	-0.001382	0.006307	-0.000289
0.000025	-0.000109	0.000110	-0.000407	-0.000571	-0.000397	-0.001161	-0.001575	0.006344	-0.000275
0.000026	-0.000113	0.000119	-0.000414	-0.000550	-0.000457	-0.000996	-0.001766	0.006379	-0.000260

Appendix D : $N = (B^T B + h K_z^T K_z)^{-1} B^T$

1 mo	3 mo	6 mo	1 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
0.000027	-0.000117	0.000128	-0.000422	-0.000529	-0.000517	-0.000833	-0.001954	0.006413	-0.000244
0.000028	-0.000121	0.000137	-0.000429	-0.000508	-0.000576	-0.000671	-0.002141	0.006444	-0.000228
0.000029	-0.000125	0.000147	-0.000436	-0.000487	-0.000634	-0.000510	-0.002324	0.006474	-0.000211
0.000030	-0.000129	0.000156	-0.000444	-0.000466	-0.000692	-0.000351	-0.002506	0.006502	-0.000194
0.000031	-0.000133	0.000164	-0.000451	-0.000446	-0.000749	-0.000194	-0.002684	0.006528	-0.000176
0.000032	-0.000137	0.000173	-0.000458	-0.000426	-0.000805	-0.000039	-0.002860	0.006552	-0.000157
0.000033	-0.000141	0.000182	-0.000465	-0.000406	-0.000861	0.000114	-0.003033	0.006574	-0.000138
0.000033	-0.000145	0.000190	-0.000471	-0.000386	-0.000916	0.000265	-0.003203	0.006594	-0.000118
0.000034	-0.000149	0.000199	-0.000478	-0.000366	-0.000970	0.000414	-0.003370	0.006612	-0.000097
0.000035	-0.000152	0.000207	-0.000485	-0.000347	-0.001023	0.000561	-0.003533	0.006628	-0.000075
0.000036	-0.000156	0.000215	-0.000491	-0.000328	-0.001075	0.000705	-0.003694	0.006642	-0.000053
0.000037	-0.000160	0.000223	-0.000497	-0.000310	-0.001126	0.000846	-0.003851	0.006654	-0.000030
0.000038	-0.000163	0.000231	-0.000503	-0.000292	-0.001176	0.000985	-0.004005	0.006664	-0.000007
0.000038	-0.000167	0.000239	-0.000509	-0.000274	-0.001226	0.001122	-0.004155	0.006672	0.000018
0.000039	-0.000170	0.000246	-0.000515	-0.000256	-0.001274	0.001255	-0.004301	0.006677	0.000043
0.000040	-0.000173	0.000253	-0.000521	-0.000239	-0.001320	0.001386	-0.004444	0.006680	0.000069
0.000041	-0.000176	0.000260	-0.000526	-0.000222	-0.001366	0.001513	-0.004583	0.006681	0.000095
0.000041	-0.000179	0.000267	-0.000531	-0.000205	-0.001411	0.001637	-0.004718	0.006680	0.000122
0.000042	-0.000182	0.000274	-0.000536	-0.000189	-0.001454	0.001758	-0.004849	0.006676	0.000150
0.000043	-0.000185	0.000280	-0.000541	-0.000174	-0.001496	0.001876	-0.004977	0.006671	0.000179
0.000043	-0.000188	0.000287	-0.000546	-0.000158	-0.001536	0.001990	-0.005100	0.006662	0.000209
0.000044	-0.000191	0.000293	-0.000550	-0.000143	-0.001576	0.002101	-0.005218	0.006652	0.000239
0.000044	-0.000193	0.000299	-0.000555	-0.000129	-0.001614	0.002208	-0.005333	0.006639	0.000270
0.000045	-0.000196	0.000304	-0.000559	-0.000115	-0.001650	0.002312	-0.005443	0.006624	0.000302
0.000046	-0.000198	0.000310	-0.000563	-0.000101	-0.001685	0.002412	-0.005550	0.006607	0.000335
0.000046	-0.000200	0.000315	-0.000566	-0.000088	-0.001719	0.002509	-0.005652	0.006587	0.000368
0.000047	-0.000202	0.000320	-0.000570	-0.000075	-0.001752	0.002602	-0.005750	0.006565	0.000402
0.000047	-0.000204	0.000325	-0.000573	-0.000062	-0.001783	0.002693	-0.005844	0.006541	0.000437
0.000048	-0.000206	0.000329	-0.000576	-0.000050	-0.001813	0.002779	-0.005935	0.006515	0.000472
0.000048	-0.000208	0.000334	-0.000579	-0.000039	-0.001842	0.002863	-0.006021	0.006486	0.000508
0.000048	-0.000210	0.000338	-0.000581	-0.000027	-0.001869	0.002944	-0.006104	0.006455	0.000545
0.000049	-0.000211	0.000342	-0.000584	-0.000016	-0.001895	0.003021	-0.006183	0.006423	0.000583
0.000049	-0.000213	0.000345	-0.000586	-0.000006	-0.001920	0.003095	-0.006258	0.006388	0.000621
0.000049	-0.000215	0.000349	-0.000588	0.000005	-0.001944	0.003166	-0.006329	0.006351	0.000660
0.000050	-0.000216	0.000352	-0.000590	0.000015	-0.001967	0.003234	-0.006397	0.006312	0.000699
0.000050	-0.000217	0.000356	-0.000592	0.000024	-0.001988	0.003299	-0.006461	0.006272	0.000740
0.000050	-0.000218	0.000359	-0.000594	0.000033	-0.002009	0.003361	-0.006522	0.006229	0.000781
0.000051	-0.000220	0.000362	-0.000595	0.000042	-0.002028	0.003420	-0.006579	0.006184	0.000822
0.000051	-0.000221	0.000364	-0.000596	0.000051	-0.002046	0.003476	-0.006632	0.006137	0.000864
0.000051	-0.000222	0.000367	-0.000597	0.000059	-0.002063	0.003529	-0.006683	0.006088	0.000907
0.000051	-0.000222	0.000369	-0.000598	0.000067	-0.002079	0.003580	-0.006729	0.006038	0.000950
0.000051	-0.000223	0.000371	-0.000599	0.000074	-0.002093	0.003627	-0.006773	0.005986	0.000994
0.000052	-0.000224	0.000373	-0.000599	0.000081	-0.002107	0.003672	-0.006813	0.005931	0.001039
0.000052	-0.000224	0.000375	-0.000600	0.000088	-0.002120	0.003714	-0.006850	0.005875	0.001084
0.000052	-0.000225	0.000377	-0.000600	0.000095	-0.002131	0.003753	-0.006883	0.005817	0.001130
0.000052	-0.000225	0.000378	-0.000600	0.000101	-0.002142	0.003790	-0.006913	0.005758	0.001176
0.000052	-0.000226	0.000379	-0.000600	0.000107	-0.002151	0.003824	-0.006941	0.005696	0.001223
0.000052	-0.000226	0.000381	-0.000600	0.000113	-0.002159	0.003855	-0.006965	0.005633	0.001271
0.000052	-0.000226	0.000382	-0.000599	0.000118	-0.002167	0.003884	-0.006986	0.005568	0.001319
0.000052	-0.000226	0.000382	-0.000599	0.000123	-0.002173	0.003910	-0.007004	0.005502	0.001367
0.000052	-0.000227	0.000383	-0.000598	0.000128	-0.002179	0.003934	-0.007019	0.005434	0.001416
0.000052	-0.000227	0.000384	-0.000597	0.000132	-0.002183	0.003956	-0.007031	0.005364	0.001466
0.000052	-0.000226	0.000384	-0.000596	0.000136	-0.002187	0.003974	-0.007040	0.005293	0.001516
0.000052	-0.000226	0.000384	-0.000595	0.000140	-0.002190	0.003991	-0.007047	0.005220	0.001567
0.000052	-0.000226	0.000384	-0.000594	0.000144	-0.002191	0.004005	-0.007050	0.005145	0.001618

Appendix D : $N = (B^T B + h K_z^T K_z)^{-1} B^T$

1 mo	3 mo	6 mo	1 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
0.000052	-0.000226	0.000384	-0.000592	0.000147	-0.002192	0.004017	-0.007051	0.005069	0.001670
0.000052	-0.000226	0.000384	-0.000591	0.000150	-0.002192	0.004027	-0.007049	0.004991	0.001722
0.000052	-0.000225	0.000384	-0.000589	0.000153	-0.002191	0.004034	-0.007044	0.004912	0.001775
0.000052	-0.000225	0.000383	-0.000587	0.000155	-0.002189	0.004039	-0.007037	0.004832	0.001828
0.000052	-0.000224	0.000382	-0.000585	0.000157	-0.002187	0.004042	-0.007027	0.004750	0.001882
0.000051	-0.000223	0.000382	-0.000583	0.000159	-0.002183	0.004042	-0.007014	0.004666	0.001936
0.000051	-0.000223	0.000381	-0.000581	0.000161	-0.002179	0.004041	-0.006999	0.004581	0.001990
0.000051	-0.000222	0.000380	-0.000579	0.000163	-0.002174	0.004037	-0.006982	0.004495	0.002045
0.000051	-0.000221	0.000379	-0.000576	0.000164	-0.002168	0.004032	-0.006962	0.004407	0.002101
0.000051	-0.000220	0.000377	-0.000574	0.000165	-0.002161	0.004024	-0.006939	0.004318	0.002157
0.000051	-0.000219	0.000376	-0.000571	0.000166	-0.002154	0.004014	-0.006914	0.004228	0.002213
0.000050	-0.000218	0.000374	-0.000568	0.000166	-0.002146	0.004003	-0.006887	0.004136	0.002270
0.000050	-0.000217	0.000373	-0.000566	0.000167	-0.002136	0.003989	-0.006857	0.004043	0.002327
0.000050	-0.000216	0.000371	-0.000563	0.000167	-0.002127	0.003974	-0.006826	0.003949	0.002385
0.000050	-0.000215	0.000369	-0.000559	0.000167	-0.002116	0.003956	-0.006791	0.003853	0.002443
0.000049	-0.000214	0.000367	-0.000556	0.000166	-0.002105	0.003937	-0.006755	0.003757	0.002501
0.000049	-0.000213	0.000365	-0.000553	0.000166	-0.002093	0.003916	-0.006717	0.003659	0.002560
0.000049	-0.000211	0.000363	-0.000549	0.000165	-0.002081	0.003893	-0.006676	0.003560	0.002619
0.000048	-0.000210	0.000361	-0.000546	0.000164	-0.002067	0.003869	-0.006633	0.003460	0.002679
0.000048	-0.000209	0.000358	-0.000542	0.000163	-0.002053	0.003842	-0.006589	0.003358	0.002739
0.000048	-0.000207	0.000356	-0.000538	0.000162	-0.002039	0.003814	-0.006542	0.003256	0.002799
0.000047	-0.000206	0.000353	-0.000535	0.000160	-0.002023	0.003785	-0.006493	0.003152	0.002860
0.000047	-0.000204	0.000350	-0.000531	0.000158	-0.002008	0.003753	-0.006443	0.003047	0.002921
0.000047	-0.000203	0.000347	-0.000527	0.000156	-0.001991	0.003721	-0.006390	0.002942	0.002982
0.000046	-0.000201	0.000344	-0.000522	0.000154	-0.001974	0.003686	-0.006335	0.002835	0.003044
0.000046	-0.000199	0.000341	-0.000518	0.000152	-0.001956	0.003650	-0.006279	0.002727	0.003106
0.000045	-0.000197	0.000338	-0.000514	0.000149	-0.001938	0.003613	-0.006221	0.002618	0.003169
0.000045	-0.000196	0.000335	-0.000509	0.000147	-0.001919	0.003574	-0.006161	0.002508	0.003231
0.000045	-0.000194	0.000332	-0.000505	0.000144	-0.001900	0.003533	-0.006100	0.002398	0.003294
0.000044	-0.000192	0.000328	-0.000500	0.000141	-0.001880	0.003491	-0.006036	0.002286	0.003358
0.000044	-0.000190	0.000325	-0.000496	0.000138	-0.001859	0.003448	-0.005971	0.002173	0.003421
0.000043	-0.000188	0.000321	-0.000491	0.000134	-0.001838	0.003403	-0.005905	0.002060	0.003485
0.000043	-0.000186	0.000318	-0.000486	0.000131	-0.001816	0.003358	-0.005836	0.001945	0.003549
0.000042	-0.000184	0.000314	-0.000481	0.000127	-0.001794	0.003310	-0.005767	0.001830	0.003614
0.000042	-0.000182	0.000310	-0.000476	0.000124	-0.001772	0.003262	-0.005695	0.001714	0.003679
0.000041	-0.000180	0.000306	-0.000471	0.000120	-0.001749	0.003212	-0.005623	0.001597	0.003744
0.000041	-0.000178	0.000302	-0.000466	0.000116	-0.001725	0.003161	-0.005548	0.001479	0.003809
0.000040	-0.000176	0.000298	-0.000461	0.000111	-0.001701	0.003109	-0.005473	0.001360	0.003875
0.000040	-0.000173	0.000294	-0.000456	0.000107	-0.001677	0.003055	-0.005396	0.001241	0.003940
0.000039	-0.000171	0.000290	-0.000450	0.000103	-0.001652	0.003001	-0.005317	0.001121	0.004007
0.000039	-0.000169	0.000286	-0.000445	0.000098	-0.001626	0.002945	-0.005237	0.001000	0.004073
0.000038	-0.000166	0.000281	-0.000440	0.000093	-0.001601	0.002888	-0.005156	0.000879	0.004139
0.000038	-0.000164	0.000277	-0.000434	0.000088	-0.001574	0.002831	-0.005074	0.000756	0.004206
0.000037	-0.000162	0.000273	-0.000428	0.000083	-0.001548	0.002772	-0.004990	0.000633	0.004273
0.000037	-0.000159	0.000268	-0.000423	0.000078	-0.001521	0.002712	-0.004906	0.000510	0.004340
0.000036	-0.000157	0.000263	-0.000417	0.000073	-0.001493	0.002651	-0.004820	0.000385	0.004408
0.000036	-0.000154	0.000259	-0.000411	0.000068	-0.001466	0.002590	-0.004733	0.000261	0.004475
0.000035	-0.000152	0.000254	-0.000406	0.000062	-0.001438	0.002527	-0.004645	0.000135	0.004543
0.000034	-0.000149	0.000249	-0.000400	0.000057	-0.001409	0.002463	-0.004555	0.000009	0.004611
0.000034	-0.000147	0.000245	-0.000394	0.000051	-0.001381	0.002399	-0.004465	-0.000118	0.004680
0.000033	-0.000144	0.000240	-0.000388	0.000045	-0.001351	0.002334	-0.004374	-0.000245	0.004748
0.000033	-0.000142	0.000235	-0.000382	0.000040	-0.001322	0.002268	-0.004282	-0.000373	0.004817
0.000032	-0.000139	0.000230	-0.000376	0.000034	-0.001292	0.002201	-0.004188	-0.000501	0.004885
0.000031	-0.000137	0.000225	-0.000370	0.000028	-0.001262	0.002133	-0.004094	-0.000630	0.004954
0.000031	-0.000134	0.000220	-0.000363	0.000021	-0.001232	0.002065	-0.003999	-0.000759	0.005023

Appendix D : $N = (B^T B + h K_z^T K_z)^{-1} B^T$

1 mo	3 mo	6 mo	1 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
0.000030	-0.000131	0.000215	-0.000357	0.000015	-0.001201	0.001995	-0.003904	-0.000889	0.005092
0.000030	-0.000128	0.000210	-0.000351	0.000009	-0.001171	0.001926	-0.003807	-0.001019	0.005162
0.000029	-0.000126	0.000205	-0.000345	0.000003	-0.001139	0.001855	-0.003709	-0.001150	0.005231
0.000028	-0.000123	0.000199	-0.000339	-0.000004	-0.001108	0.001784	-0.003611	-0.001281	0.005301
0.000028	-0.000120	0.000194	-0.000332	-0.000010	-0.001077	0.001712	-0.003512	-0.001413	0.005371
0.000027	-0.000117	0.000189	-0.000326	-0.000017	-0.001045	0.001640	-0.003413	-0.001545	0.005441
0.000026	-0.000115	0.000184	-0.000319	-0.000023	-0.001013	0.001567	-0.003312	-0.001677	0.005511
0.000026	-0.000112	0.000178	-0.000313	-0.000030	-0.000980	0.001494	-0.003211	-0.001809	0.005581
0.000025	-0.000109	0.000173	-0.000306	-0.000037	-0.000948	0.001420	-0.003110	-0.001942	0.005651
0.000024	-0.000106	0.000167	-0.000300	-0.000044	-0.000915	0.001345	-0.003007	-0.002076	0.005722
0.000024	-0.000103	0.000162	-0.000293	-0.000051	-0.000883	0.001271	-0.002905	-0.002209	0.005792
0.000023	-0.000100	0.000157	-0.000287	-0.000058	-0.000850	0.001195	-0.002801	-0.002343	0.005863
0.000022	-0.000098	0.000151	-0.000280	-0.000064	-0.000816	0.001119	-0.002698	-0.002478	0.005933
0.000022	-0.000095	0.000146	-0.000274	-0.000071	-0.000783	0.001043	-0.002593	-0.002612	0.006004
0.000021	-0.000092	0.000140	-0.000267	-0.000079	-0.000750	0.000967	-0.002489	-0.002747	0.006075
0.000020	-0.000089	0.000134	-0.000260	-0.000086	-0.000716	0.000890	-0.002384	-0.002882	0.006146
0.000020	-0.000086	0.000129	-0.000254	-0.000093	-0.000682	0.000813	-0.002278	-0.003017	0.006217
0.000019	-0.000083	0.000123	-0.000247	-0.000100	-0.000648	0.000735	-0.002172	-0.003152	0.006288
0.000018	-0.000080	0.000118	-0.000240	-0.000107	-0.000614	0.000657	-0.002066	-0.003288	0.006359
0.000018	-0.000077	0.000112	-0.000234	-0.000114	-0.000580	0.000579	-0.001959	-0.003424	0.006430
0.000017	-0.000074	0.000106	-0.000227	-0.000122	-0.000546	0.000501	-0.001853	-0.003559	0.006501
0.000016	-0.000071	0.000101	-0.000220	-0.000129	-0.000512	0.000422	-0.001745	-0.003696	0.006573
0.000016	-0.000068	0.000095	-0.000213	-0.000136	-0.000478	0.000344	-0.001638	-0.003832	0.006644
0.000015	-0.000065	0.000089	-0.000207	-0.000144	-0.000443	0.000265	-0.001531	-0.003968	0.006715
0.000014	-0.000062	0.000084	-0.000200	-0.000151	-0.000409	0.000186	-0.001423	-0.004104	0.006787
0.000014	-0.000059	0.000078	-0.000193	-0.000158	-0.000374	0.000107	-0.001315	-0.004241	0.006858
0.000013	-0.000056	0.000072	-0.000186	-0.000166	-0.000340	0.000027	-0.001207	-0.004377	0.006930
0.000012	-0.000053	0.000066	-0.000179	-0.000173	-0.000305	-0.000052	-0.001099	-0.004514	0.007001
0.000012	-0.000050	0.000061	-0.000173	-0.000181	-0.000271	-0.000131	-0.000991	-0.004651	0.007072
0.000011	-0.000047	0.000055	-0.000166	-0.000188	-0.000236	-0.000211	-0.000883	-0.004787	0.007144
0.000010	-0.000044	0.000049	-0.000159	-0.000195	-0.000202	-0.000290	-0.000775	-0.004924	0.007215

Appendix E : B N = Benchmark Weights of Benchmark Bonds

Benchmark	Weights									
	1 mo	3 mo	6 mo	1 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
1 month	0.999993	0.000012	(0.000006)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000
3 months	0.000012	0.999976	0.000013	(0.000002)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)
6 months	(0.000006)	0.000013	0.999991	0.000002	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000
1 year	0.000000	(0.000002)	0.000002	0.999999	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)
3 years	(0.000000)	0.000000	(0.000000)	0.000000	1.000000	0.000000	(0.000000)	0.000000	(0.000000)	0.000000
5 years	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	1.000000	0.000000	(0.000000)	0.000000	(0.000000)
7 years	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	1.000000	0.000000	(0.000000)	0.000000
10 years	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	1.000000	0.000000	(0.000000)
20 years	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	1.000000	(0.000000)
30 years	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	(0.000000)	0.000000	0.000000	(0.000000)	1.000000

Appendix F : Interest Rate Scenarios

Benchmark	Yield Rates							
	Base Scenario	Flat Decrease	Flat Increase	Slope Decrease	Slope Increase	Trough	Bump	Twist
1 month	4.00%	3.00%	5.00%	4.50%	3.50%	4.00%	4.00%	4.00%
3 months	4.50%	3.50%	5.50%	4.98%	4.02%	4.47%	4.53%	4.05%
6 months	5.00%	4.00%	6.00%	5.46%	4.54%	4.92%	5.08%	4.25%
1 year	5.50%	4.50%	6.50%	5.91%	5.09%	5.31%	5.69%	4.50%
3 years	6.00%	5.00%	7.00%	6.20%	5.80%	5.41%	6.59%	5.00%
5 years	7.25%	6.25%	8.25%	7.25%	7.25%	6.25%	8.25%	6.50%
7 years	7.50%	6.50%	8.50%	7.46%	7.54%	6.58%	8.42%	7.00%
10 years	7.90%	6.90%	8.90%	7.80%	8.00%	7.10%	8.70%	7.65%
20 years	8.55%	7.55%	9.55%	8.25%	8.85%	8.15%	8.95%	8.45%
30 years	8.60%	7.60%	9.60%	8.10%	9.10%	8.60%	8.60%	8.60%

Benchmark	Prices							
	Base Scenario	Flat Decrease	Flat Increase	Slope Decrease	Slope Increase	Trough	Bump	Twist
1 month	100.053	100.135	99.971	100.012	100.094	100.053	100.053	100.053
3 months	100.068	100.314	99.824	99.950	100.186	100.076	100.060	100.178
6 months	100.000	100.490	99.515	99.777	100.224	100.041	99.959	100.367
1 year	99.904	100.871	98.951	99.515	100.295	100.083	99.725	100.871
3 years	100.542	103.305	97.869	99.991	101.096	102.170	98.945	103.305
5 years	100.000	104.238	95.970	100.000	100.000	104.238	95.970	103.158
7 years	100.000	105.553	94.805	100.215	99.785	105.095	95.208	102.730
10 years	100.000	107.138	93.468	100.686	99.320	105.659	94.729	101.725
20 years	100.000	110.237	91.149	102.914	97.210	103.915	96.307	100.957
30 years	100.000	111.754	90.209	105.603	94.886	100.000	100.000	100.000