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Large Deviation Estimate in Ruin Theory

Jinhua Tao

Central Missouri State University

Warrensburg, MO 64093

Abstract

The main purpose of this paper is to study the ruin probabilities of insurance surplus process over an extended period of time. The traditional large deviation techniques are used to obtain asymptotic exponential bound for the probability of ruin occurring at time t when the aggregate claim process is compound Poisson. The exponential bound is naturally expressed in terms of the large deviation rate function.

1. Introduction

Let $X_1,\ X_2,\ \cdots$ be a sequence of i.i.d. random variables with common distribution function $F(x)$ and mean μ. Let $N(t)$ be a Poison process with intensity λ and be independent of all X_i's. For $t \geq 0$, let S_t denote the aggregate claims up to time t and let $U(t)$ denote the insurer's surplus at time t, then we have the following two processes:

$$S_t = X_1 + X_2 + \cdots + X_{N(t)},$$

the insurance risk process and

$$U(t) = u + ct - S_t,$$

the insurance surplus process. Here u is the initial surplus of the insurer and we assume that premium is collected continuously at rate c. Ruin occurs when $S_t > ct + u$ for the first time. The probability of eventual ruin is defined to be

$$\psi(u) = P(T(u) < \infty)$$

with $T(u) = min\{t : S_t > ct + u\}$.

It is well known that when $c \leq \lambda\mu$, $\psi(u) = 1$ for any $u \geq 0$ and when $c > \lambda\mu$, $0 < \psi(u) < 1$ for $u \geq 0$. In this paper we are mainly concerned with the probability that ruin occurs at the end of a particular planning year, i.e.

$$P\{S_t > u + ct\} \tag{1}$$

when $t =$ positive integers and $c > \lambda\mu$. It will be shown that when t is large this probability is approximately equal to $e^{r(c)t}$ where $r(c) < 0$ and depends only on c and the original process.

2. Main results

We start with a known fact about the mean and moment generating function of S_t.

Lemma 2.1 Let $m(\theta)$ be the moment generating function of X_1, then

(i). $E[S_t] = \lambda\mu t$

(ii). Moment generating function of S_t

$$E[e^{\theta S_t}] = e^{\lambda(m(\theta)-1)t}$$

Next we introduce two important functions which will be used frequently in proving our main results.

Let $h(\theta)$ be a function which is equal to

$$h(\theta) = \lambda[m(\theta) - 1]$$

and let function $r(a)$ be the convex conjugate of $h(\theta)$, i.e.

$$r(a) = \inf_{\theta}\{h(\theta) - a\theta\}. \tag{2}$$

We call function $r(a)$ the rate function of our compound Poisson process for the reason which will become clear in the main results. This function has the following properties.

Lemma 2.2 Assume that $m(\theta) < \infty$ for θ in some open interval D_m containing origin and assume that for each a, the solution θ_a to the equation $a = h'(\theta)$ exists and lies in the interior of D_m. Then

i) $r(a)$ is strictly concave down and infinitely differentiable with maximum 0 attained at $a = \lambda\mu$;

ii) $r(a) = \lambda(m(\theta_a) - 1) - a\theta_a$, where θ_a is the unique solution to the equation

$$a = \lambda m'(\theta).$$

iii) For any $a \geq \lambda\mu$,

$$\inf_{\theta}\{h(\theta) - a\theta\} = \inf_{\theta \geq 0}\{h(\theta) - a\theta\}.$$

Proof: Since $m(\theta)$ is finite in an open interval around origin, $m(\theta)$ is infinitely differentiable in D_m. Furthermore,

$$h''(\theta) = \lambda m''(\theta) > 0,$$

This implies that $h(\theta)$ is strictly concave up and that $a = h'(\theta)$ defines a 1-1 strictly increasing and infinitely differentiable mapping. The corresponding pair a and θ_a satisfies:

$$r(a) = h(\theta_a) - \theta_a a, \ when \ \ a = h'(\theta_a).$$

The inverse mapping is actually given by

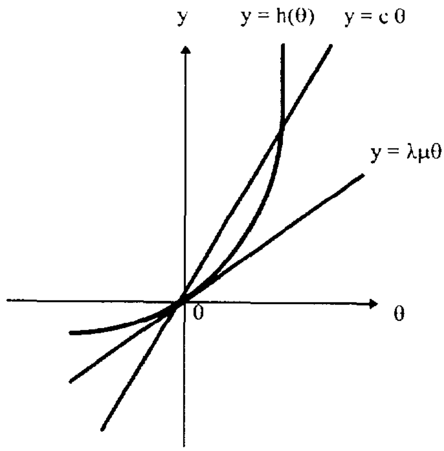
$$\theta = -r'(a).$$

Consequently

$$r''(a) = -\frac{d\theta}{da} = -\frac{1}{h''(\theta)} < 0,$$

which implies that $r(a)$ is a concave down function.

The proof of iii) could be easily obtained from the following picture about the relation between functions $y = h(\theta)$ and $y = a\theta$ with $a > \lambda\mu$.



Our first result is an upper bound for the probability in (1).

Theorem 2.1 Assume that $m(\theta) < \infty$ for θ in some open interval D_m containing origin, then for any $t \geq 0$

$$P\left[S_t > ct + u\right] \leq e^{r(c)t}$$

Proof: For any $\theta > 0$,

$$\begin{aligned} P\left[S_t > ct + u\right] &\leq P\left[S_t > ct\right] \\ &= P\left[e^{\theta S_t} > e^{\theta ct}\right] \\ &\leq e^{-\theta ct} E(e^{\theta S_t}) \\ &= e^{[-\theta c + \lambda(m(\theta)-1]t)}. \end{aligned} \tag{3}$$

(The inequality is Chebycheff's.) Take infimum for all $\theta > 0$ on the right hand side, by Lemma 2.2 iii) we recognize that the exponent is simply $r(c)t$. This proves the upper bound. QED

As to the lower bound we consider the special case when t is a positive integer. Note that by Lemma 2.1, S_t is equivalent to the sum of i.i.d. random variables

$Y_1, \cdots, Y_t$ with $E(Y_1) = \lambda\mu$ and $E[e^{\theta Y_1}] = e^{\lambda(m(\theta)-1)}$. By the Law of large numbers,

$$\lim_{t\to\infty} P\left(\frac{S_t}{t} \to \lambda\mu\right) = 1.$$

The event $\{S_t > u + ct\}$ with $c > \lambda\mu$ is a event for S_t to be away from its central mean $\lambda\mu t$ on a large scale (scale of n). Consequently the probability is very small. Further since u is a fixed constant, the change in u is relatively small (scale of constant) compared with the change in c. Therefore the asymptotic expression for the probability $P(S_t > ct + u)$ in the case of $u = 0$ is almost the same as in the case of $u > 0$. We will provide the proof of the lower bound only for the case that $u = 0$.

Theorem 2.2 Assume that $m(\theta) < \infty$ for θ in some open interval D_m containing origin and assume that the solution θ_c to the equation $c = \lambda m'(\theta)$ exists and lies in the interior of D_m. Then for every $0 < \epsilon < e^{r(c)}$ there exists a number $n_0 > 0$ such that for every positive integer $t \geq n_0$,

$$P\left[S_t > ct\right] \geq (e^{r(c)} - \epsilon)^t.$$

Proof The main idea to prove the lower bound is first to shift the center of the process to ct using Esscher transform, then to use the Central Limit Theorem to estimate the transformed probability.

First we let $Y_1, Y_2, \cdots$ be a sequence of i.i.d.'s whose common moment generating function is equal to $e^{\lambda(m(\theta)-1)}$ and whose distribution function is denoted by $F^*(y)$. Then

$$\begin{aligned} P(S_t > ct) &= P(Y_1 + \cdots + Y_t > ct) \\ &= \int \cdots \int_{y_1+\cdots+y_t > ct} dF^*(y_1) \cdots dF^*(y_t) \\ &= \int \cdots \int_{y_1+\cdots+y_t > ct} e^{-\theta_c(y_1+\cdots+y_t)} e^{\theta_c y_1} dF^*(y_1) \cdots e^{\theta_c y_t} dF^*(y_t). \end{aligned}$$

Define the Esscher transform of $F^*(y)$ by

$$G(y) = \frac{\int_{-\infty}^{y} e^{\theta_c y} dF^*(y)}{e^{\lambda(m(\theta_c)-1)}}.$$

Then

$$P(S_t > ct) = e^{\lambda(m(\theta_c)-1)t} \int \cdots \int_{y_1+\cdots+y_t>ct} e^{-\theta_c(y_1+\cdots+y_t)} dG(y_1)\cdots dG(y_t).$$

Now we claim that $E_G(Y_1) = c$. This is because

$$\inf_{\theta}\{\lambda(m(\theta)-1) - c\theta\} = \lambda(m(\theta_c)-1) - c\theta_c$$

so θ_c satisfies

$$\begin{aligned} c &= \lambda m'(\theta) \\ &= \frac{\int y e^{\theta y} dF^*(y)}{m_Y(\theta)} \\ &= \int y dG(y), \end{aligned}$$

where $m_Y(\theta) = e^{\lambda(m(\theta)-1)}$. Now, by the Central Limit Theorem, we see that if the Y_i were distributed as G, then for any $\epsilon > 0$,

$$P_G\left[c < \frac{Y_1+\cdots+Y_t}{t} < c+\epsilon\right] \to \frac{1}{2} \ as\ n \to \infty.$$

Hence, for each $\epsilon > 0$ there exists an $n_0 > 0$ such that for $t \geq n_0$ we have

$$\int \cdots \int_{c<\frac{y_1+\cdots+y_t}{t}<c+\epsilon} dG(y_1)\cdots dG(y_t) \geq \frac{1}{4}.$$

Now

$$P(S_t > ct) \quad = \quad e^{\lambda(m(\theta_c)-1)t} \int \cdots \int_{y_1+\cdots+y_t>ct} e^{-\theta_c(y_1+\cdots+y_t)} dG(y_1)\cdots dG(y_t).$$

$$\geq e^{\lambda(m(\theta_c)-1)t}\int\cdots\int_{c<\frac{y_1+\cdots+y_t}{t}<c+\epsilon} e^{-\theta_c(y_1+\cdots+y_t)}dG(y_1)\cdots dG(y_t)$$

$$\geq e^{\lambda(m(\theta_c)-1)t}e^{-\theta_c(c+\epsilon)t}\int\cdots\int_{c<\frac{y_1+\cdots+y_t}{t}<c+\epsilon} dG(y_1)\cdots dG(y_t)$$

$$\geq \frac{1}{4}e^{r(c)t}e^{-\epsilon\theta_c t}$$

which implies the lower bound. QED

3. Examples

We look at two specific moment generating functions for the claim size random variables and calculate their rate functions.

Example 1. Let $X_1 = 1$ with probability 1, i.e. all the claim sizes are the same, equal to one unit. Then $m_X(\theta) = e^\theta$ and

$$E(e^{\theta S_t}) = e^{\lambda t(e^\theta - 1)}.$$

This implies that S_t is a Poisson Process with parameter λ. The rate function is calculated by equations:

$$\lambda e^{\theta_a} = a,$$

and

$$r(a) = \lambda(e^{\theta_a} - 1) - \theta_a a.$$

Hence

$$r(c) = (c - \lambda) - c\ln(\frac{c}{\lambda}).$$

Let $c = (1+\alpha)\lambda$, where $\alpha > 0$ is the security loading, then

$$r(c) = \alpha\lambda - (1+\alpha)\lambda\ln(1+\alpha).$$

By Taylor expansion, the right hand side is approximately equal to $-\frac{\alpha^2}{2}\lambda$ (assuming $\alpha < 1$).

Example 2. Let $\lambda = 1$ and $m(\theta) = \frac{1}{2}\theta^2 + 1$. Then

$$E(e^{\theta S_t}) = e^{\frac{1}{2}\theta^2 t},$$

i.e. S_t is a normal random variable with mean 0 and standard deviation $\sqrt{t}$. A simple calculation gives

$$r(c) = \sup_\theta(\frac{\theta^2}{2} - \theta c) = -\frac{c^2}{2}.$$

Theorems 2.1 and 2.2 state that

$$P(S_t > ct) \approx e^{-\frac{c^2}{2}t}. \tag{4}$$

In this special case we can actually estimate the probability directly, since $\frac{S_t}{\sqrt{t}}$ is a standard normal random variable. Hence

$$\begin{aligned} P[S_t > ct] &= P[Z > c\sqrt{t}] \\ &= \frac{1}{\sqrt{2\pi}}\int_{c\sqrt{t}}^{\infty} e^{-z^2/2}dz. \end{aligned}$$

But

$$\begin{aligned} \int_x^\infty e^{-z^2/2}dz &\leq \int_x^\infty \frac{z}{x}e^{-z^2/2}dz \\ &\leq \frac{1}{x}e^{-x^2/2} \end{aligned}$$

and

$$\int_x^\infty e^{-z^2/2}dz \geq \int_x^{x+1/x} \frac{z}{x+1/x}e^{-z^2/2}dz$$

$$\geq \frac{1}{x+1/x}e^{-x^2/2},$$

so

$$P[S_t > ct] \approx \frac{1}{\sqrt{2\pi t}c} e^{-\frac{c^2}{2}t},$$

which agrees with the exponential order in (3.1).

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