Testing Deterministic versus Stochastic Trends in the Lee-Carter Mortality Indexes and Its Implications for Projecting Mortality Improvements at Advanced Ages

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Abstract

In recent years, mortality has improved considerably faster than had been predicted, resulting in unforeseen mortality losses for annuity and pension liabilities. As a result, projection of mortality improvements has become an increasingly important issue among actuaries. Among all projection methods, the Lee-Carter approach has been widely accepted by the actuarial community. In this paper, we investigate the dynamics of the Lee-Carter mortality index (parameter kt). Specifically, we perform statistical hypothesis tests to examine whether the mortality indexes for Canada, England and Wales and the United States are best described by stochastic trends (difference stationary models) or deterministic trends (trend stationary models). Such a distinction is important because mortality forecasts generated from these two classes of time-series models could be highly different. The empirical results favor broken-trend stationary models over difference stationary models, which are used in most previous applications of the Lee-Carter method. The results also give strong statistical evidence that the rates of mortality decline for the three populations have significantly accelerated in mid-1970s. We further analyze the impact of the acceleration of mortality decline on the probability of survival to an advanced age, and provide several recommendations to users of the Lee-Carter approach.

1. Introduction

Mortality assumptions are crucial to many areas of actuarial practice such as life insurance financial reporting and maintenance of private and public pension programs. In recent years, the actuaries' problem regarding mortality assumptions is that people are living longer than they were expected to; for instance, in the United Kingdom, mortality rates for both male assured lives and male pensioners have reduced far more rapidly than the speed of reduction implied by the mortality reduction factors derived in about 15 years ago (see Continuous Mortality Investigation Bureau, 2002). Projection of mortality improvements has therefore become an increasingly important issue among actuaries.

Among all theoretical models of future mortality, the Lee-Carter model (Lee and Carter, 1992) has been widely discussed in the actuarial literature; for example, Li and Chan (2007) analyzed the impact of non-repetitive exogenous interventions (outliers) on Lee-Carter mortality forecasts; Ozeki (2005) fitted the Lee-Carter model to historical Japanese period life tables; Buettner (2002) applied the Lee-Carter methodology to project mortality patterns of the oldest-old; Friedland (1998) summarized how the projected deficit of U.S. Social Security would change if the Lee-Carter approach were used; Tuljapurkar (1998) and Tuljapurkar and Boe (1998) provided a review of the Lee-Carter model and some recommendations for forecasters.

For various reasons, the Lee-Carter model is particularly suitable for actuarial applications. First, the model has a relatively small number of parameters, and the parameters are fairly easy to interpret. Second, it attaches probabilistic confidence intervals to central mortality forecasts so that actuaries can access how light (and how heavy) future mortality improvements may turn out to be. Third, sample paths of future mortality can be generated via the stochastic components of the model, allowing actuaries to quantify the risk of unanticipated mortality improvement by using prevalent risk measures such as value at risk (VaR) and conditional tail expectation (CTE). Although there are many other ways to produce stochastic mortality forecasts, other models, for example, the P-splines regression (Currie et al., 2004), tend to smooth the progression of death rates over time. This prohibits us to investigate the possibility of structural changes in the dynamics of mortality.

However, unlike process-oriented methods, which take account of expert opinions and changes in the pattern of deaths by different causes, the Lee-Carter approach is based entirely on extrapolation, presuming that the forces of change that were effective in the experience period are going to be in effect in the future. Because of such an extrapolative nature, forecasters must be careful when specifying a time-series process for the time-varying component (parameter k_t, often known as mortality index) in the Lee-Carter model. In Li and Chan (2005), forecasters are reminded to be wary of outliers in the mortality index, since they may possibly lead to an erroneous mortality forecast, particularly if they are located near the forecast origin.

In this paper, we further investigate the dynamics of the Lee-Carter mortality index. Our first objective is to perform statistical hypothesis tests to examine whether the mortality indexes for various developed countries are best described by difference stationary models (stochastic trends), which are used in most previous applications of the Lee-Carter method (see, e.g., Lee and Carter, 1992; Tuljapurkar et al., 2000) or trend stationary (deterministic trends) models, which are rather uncommon in the context of mortality modeling. Such a distinction is crucially important because mortality forecasts generated from these two classes of time-series models could be highly different.

In recent years, several demographers (e.g., Kannisto et al., 1994; Vaupel, 1997) observed that, for many developed countries, the reduction of mortality rates has significantly accelerated in the 1970s. While their observation has important implications for social, health and research policy, the demographers made no attempt to verify the statistical significance of the sudden change in the pace of mortality decline. The second objective of this study is to statistically detect and model any structure changes in the dynamics of the Lee-Carter mortality index during the past 50 years, and to evaluate the impact of such structural changes on the resulting mortality projections.

The rest of this paper is organized as follows: in Section 2, we define the notation used throughout this paper and state all sources of data; in Section 3, we briefly review the Lee-Carter model and describe in detail the time-series models under consideration: difference stationary models, simple trend stationary models and broken-trend stationary models; then we discuss the appropriateness of these models for the Lee-Carter mortality indexes derived from the historical mortality data of different developed countries; in Section 4, we present the empirical results with emphasis on the probability of survival to advanced ages; finally, in Section 5, we conclude the paper with some recommendations for practitioners.

2. Notation and Data

Let us define the following notation:

$D_{x,t}$: the number of deaths between ages x and x+1 in year t;

$E_{x,t}$: the number of exposures-to-risk ages x and x+1 in year t;

$m_{x,t}=D_{x,t} / E_{x,t}$: the central rate of death at age x in year t.

For detailed interpretations of the above notation, we refer readers to Chapter 3 of Bowers et al. (1997).

We apply the theoretical mortality models to the populations of Canada, England and Wales and the United States. The required historical data—death counts ($D_{x,t}$) and mid-year population estimates (proxy for $E_{x,t}$) for x = 0,1, ..., 99 and t = 1950, 1951, ..., 2002—are obtained from the Human Mortality Database (2007).

3. Methodology

3.1 The Lee-Carter Model

The Lee-Carter model assumes that central death rates for all ages are driven by a single time-varying component, denoted by k_t, which is also referred to as mortality index. Mathematically,

$$\ln(m_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t}, \qquad (1)$$

where a_x is an age-specific parameter that indicates the average of level of $\ln(m_{x,t})$ over time, b_x is another age-specific parameter that characterizes the sensitivity of $\ln(m_{x,t})$ to changes in the mortality index k_t; and $\varepsilon_{x,t}$ is the error term that captures all remaining variations and shows no long-term trend.

Note that all parameters on the right-hand side of equation (1) are unobservable. Hence, we are not able to fit the model by simple methods like the ordinary least squares. To solve this problem, researchers have proposed a few alternative approaches including the method of singular value decomposition (SVD) considered by Lee and Carter (1992), the method of maximum likelihood estimation (MLE) implemented by Wilmoth (1993) and Brouhns et al. (2002) and the method of generalized linear models (GLM) employed by Renshaw and Haberman (2006).

In this paper, we adopt the method of SVD, which is easier to implement then the other methods. The SVD procedure can be implemented by using various standard mathematical/statistical packages such as GENSTAT, MATLAB and IMSL MATH/ LIBRARY. Using this method of SVD, we first take a_x as the arithmetic average of $\ln(m_{x,t})$ over time; then we apply SVD the matrix of $\{\ln(m_{x,t}) - a_x\}$. The first left and right singular vectors give initial estimates of b_x and k_t, respectively. To satisfy the constraints for parameter uniqueness, the estimates of b_x and k_t are normalized so that they sum to one and zero, respectively. Note that the fitted number of deaths derived from the initial estimates of b_x and k_t may not be the same as the observed number of deaths. To reconcile the fitted and observed number of deaths, we re-estimate parameter k_t so that for all t, the following condition is satisfied:

$$\sum_x D_{x,t} = \sum_x E_{x,t} \exp(\hat{a}_x + \hat{b}_x \hat{k}_t) . \qquad (2)$$

where $\hat{a}_x$, $\hat{b}_x$ and $\hat{k}_t$ are the estimates of a_x and b_x, and k_t, respectively.

As a matter of empirical fact, the mortality indexes demonstrate a long-term stability (see Figure 1) and explain a large proportion of variability in the historical central death rates. As a matter of empirical fact, the mortality indexes demonstrate a long-term stability (see Figure 1) and explain a large proportion of variability in the historical central death[4] (see Table 1). As a result, we may view the time-varying component k_t as a dominant temporal "signal" in the historical data and model k_t by an appropriate univariate time-series process.

The time-series process is of crucial importance because the entire mortality forecast is based on an extrapolation of k_t via this process. Let T be the forecast origin and $\hat{k}_{T+s}$ be the s-period ahead forecast of k_t; then, the s-period ahead forecast of $m_{x,t}$ is given by

$$\hat{m}_{x,T+s} = \exp(\hat{a}_x + \hat{k}_{T+s}\hat{b}_x) . \qquad (3)$$

Using the Lee-Carter approach, probabilistic confidence intervals for many demographic quantities can be computed easily without the need of generating sample paths: for future death rates and life expectancies, we may use the semi-analytic

[4] The ratio $s_1^2 / \sum_i s_i^2$, where s_i is the i^{th} singular value in the SVD, measures the proportion of total temporal variance in the logarithmically transformed central death rates explained by the mortality index.

expressions provided in Appendix B of Lee and Carter (1992); for future annuity rates, we may use the semi-analytic expression given in the Appendix of Li and Chan (2007). Alternatively, forecasters may compute confidence intervals by using a parametric or residual bootstrap. The method of bootstrapping is detailed in Brouhns et al. (2005), Koissi et al. (2005) and Li et al. (2006).

While it is possible to consider more principal components when we apply SVD, the mortality indexes in the extra principal components are highly non-linear. Since the entire Lee-Carter mortality forecast is based on an extrapolation of the time-varying component(s), the non-linearity makes forecasting more complicated.

FIGURE 1

Estimates of k_t for the General Populations of Canada, England and Wales and the United States

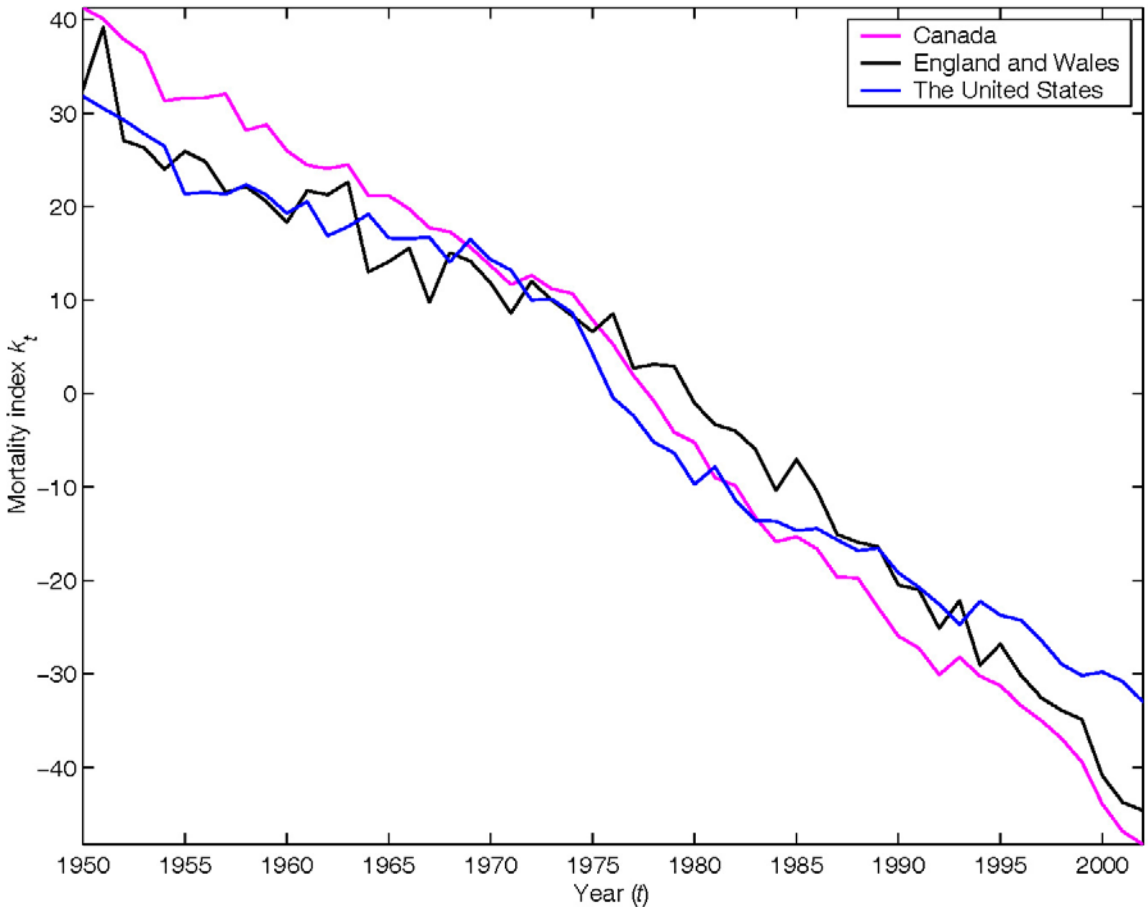


TABLE 1

Proportion of Variance of $\ln(m_{x,t})$ Explained by the Mortality Index k_t.

Population	Proportion of variance explained by k_t
Canada	0.9687
England and Wales	0.9439
The United States	0.9705

3.2 Difference Stationary versus Trend Stationary

In most previous applications of the Lee-Carter method, the dynamics of k_t are modeled by autoregressive integrated moving average (ARIMA) models. An ARIMA (p,d,q) model in its general form can be written as

$$\Phi(B)(1-B)^d k_t = \Theta(B)e_t, \qquad (4)$$

where B is the backshift operator such that $B_s k_t = k_{t-s}$, $\Phi(B) = 1-\varphi_1 B - \ldots - \varphi_p B^p$, $\Theta(B) = 1-\theta_1 B - \ldots - \theta_q B^q$, and $\{e_t\}$ is a sequence of white noise random variables, iid with zero mean and constant variance. The class of ARIMA models is called *difference stationary* models since the time-series $\{k_t\}$ is stationary[5] after differencing d times.

[5] A time-series $\{y_t\}$ is said to be strictly stationary if the joint distribution of $(y_{t_1},\ldots,y_{t_k})$ is invariant under time drift. In practice, a weaker version of stationarity is often assumed. A time-series $\{y_t\}$ is weakly stationary if both $E(y_t)$ and $\mathrm{cov}(y_t, y_{t-l})$, where l is an arbitrary integer, are time invariant. In this paper, we consider the weaker version of stationarity. We refer readers to Tsay (2002) and Wei (2006) for further

The ARIMA order (p,d,q) can be determined by the Box and Jenkins (1976) approach. In most cases, the dynamics of k_t can be effectively captured by an ARIMA(0,1,0) model. Although a similar model with an additional AR or MA term may be marginally superior, the order of (0,1,0) is usually preferred on the ground of parsimony. For instance, in the original work of Lee and Carter for the U.S. population, ARIMA(0,1,0) was found to be the optimal ARIMA model for $\{k_t\}$; Tuljapurkar et al. (2000) noted also that ARIMA(0,1,0) seems to be a "universal model" for the mortality indexes of the G7 countries. ARIMA(0,1,0), which is sometimes known as random walk with drift, can be expressed as follows:

$$k_t = c + k_{t-1} + e_t, \qquad (5)$$

where c is drift of the random walk process.

Although not often applied to mortality modeling, *trend stationary* models, which achieve stationarity by de-trending (removal of deterministic trends), may be equally suitable for modeling the dynamics of k_t. In particular, the apparent linearity in the mortality indexes (see Figure 1) may be well described by a simple trend stationary model, which can be expressed as follows:

$$k_t = \alpha + \beta t + e_t, \qquad (6)$$

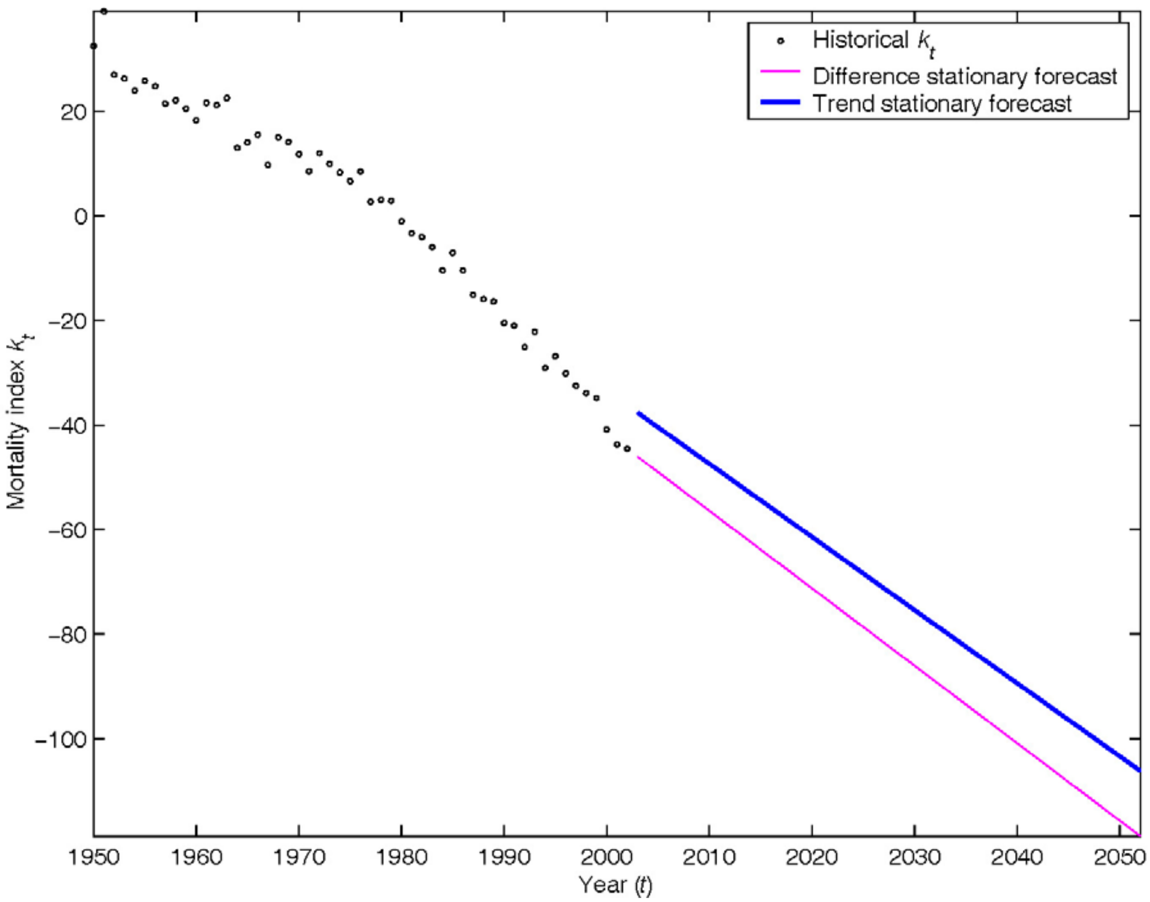
where α and β are the intercept and slope parameters, respectively. The distinction

information regarding the concept of stationarity.

between trend and difference stationarity is potentially important in mortality forecasting, since medium- and long-run mortality projections implied by trend and difference stationary models could be highly different (Diebold and Senhadji, 1996; Rudebusch, 1993). This can be seen by comparing the forecasts of k_t from both models (see Figure 2). The forecast from the trend stationary model (equation (6)) revert to the trend quickly, in sharp contrast to that from the difference stationary model (equation (5)), which remains permanently lower.

FIGURE 2

The Mortality Index k_t for the English and Welsh Population, Followed by the Optimal Forecasts of k_t from the Best-Fitting Difference and Trend Stationary Models for the Period 2003-2053



3.3 The Broken-Trend Stationary Model

In Figure 1, we observe that the slopes (and y-intercepts) of the mortality indexes have somewhat changed in the middle of the experience period: it seems that the pattern of k_t can be better described by a piecewise straight line than a straight line with a constant slope. This observation motivates us to consider the broken-trend stationary model, which is defined as follows:

$$k_t = \alpha_1 + \beta_1 t + (\alpha_2 - \alpha_1)\Delta_t(T^*) + (\beta_2 - \beta_1)\Psi_t(T^*) + e_t, \qquad (7)$$

where $\Delta_t(T^*) = \begin{cases} 1 & \text{if } t > T^* \\ 0 & \text{otherwise} \end{cases}$ and $\Psi_t(T^*) = \begin{cases} t - T^* & \text{if } t > T^* \\ 0 & \text{otherwise} \end{cases}$. The model allows a structural break point at time T^* and permits exogenous changes in both the level (intercept) and the growth rate (slope) of the time-series after the break.

The broken-trend stationary model was proposed originally by Perron (1989) to examine the effects of the Great Crash in 1929 and the oil-price shock in 1972 on macroeconomic data series. Since its introduction, the broken-trend stationary model has been widely applied in the field of econometrics; for example, Lee et al. (2005) used the model to study the convergence of income disparity between Japan and ASEAN-5 economies; Narayan and Smyth (2004) employed the model to investigate the efficiency of the stock market in South Korea; Yan and Felmingham (2006) applied the model to an analysis of the Shanghai and Shenzhen share price indexes.

We may be able to distinguish between trend and difference stationarity by a Dickey-Fuller test (Dickey and Fuller, 1979) or an Augmented Dickey-Fuller test (Said and Dickey, 1984). Unfortunately, most tests using the Dickey-Fuller and Augmented Dickey-Fuller techniques are considered to have low power; that is, there is a high probability that the null hypothesis of difference stationarity is not rejected even if the time-series is in fact trend stationary. Furthermore, Dickey-Fuller tests are not appropriate for testing broken-trend stationary models.

Zivot and Andrews (1992) derived a method for testing difference stationarity (equation (5)) against broken-trend stationarity (equation (7)). We apply Zivot and Andrews' statistical test to the mortality indexes of the three populations. The results, which we summarize in Table 2, point to the following two important conclusions:

1. for each of the three countries, the *t*-statistic favors a broken-trend stationary model over an ARIMA(0,1,0) model for modeling the Lee-Carter mortality index;
2. for each of the three countries, the break-year T^*, which is detected statistically from the historical mortality index data, is located in mid-1970s; the timing of the structural breaks and the *t*-values indicate that the mortality patterns of the three developed countries underwent major structural changes at approximately the same time. The structural changes can be visualized in the plots of trended mortality indexes in Figure 3.

Our results are consistent with Renshaw and Haberman's (2003) work on estimating mortality reduction factors for the English and Welsh population. Renshaw and Haberman found that the reduction factors (in log scale) for males in England and Wales are better modeled by a time-covariate model with a hinge sited in year 1975 that one without a hinge.

A problem of the original Lee-Carter model is that the model does not fit the age-specific mortality data exactly at the forecast origin; that is, $m_{x,T} \neq \exp(\hat{a}_x + \hat{b}_x \hat{k}_T)$. This situation would inevitably lead to error, which would be especially important in early years of forecast. Bell (1997) and Lee (2000) noted that the error at the forecast origin caused significant bias in the forecasts for the first decade. This problem can be resolved by setting $a_x = m_{x,T}$ and $k_T = 0$, thereby fitting the age-specific mortality data at the forecast origin exactly. We apply the extended model (with the forecast origin correction) to the mortality data, and perform Zivot and Andrews' test using the re-estimated mortality indexes. The conclusions remain the same. In Figure 4, we observe that the patterns of the re-estimated mortality indices are similar to that of the original ones. The results suggest that our methodology is robust with respect to the choice of parameter constraints.

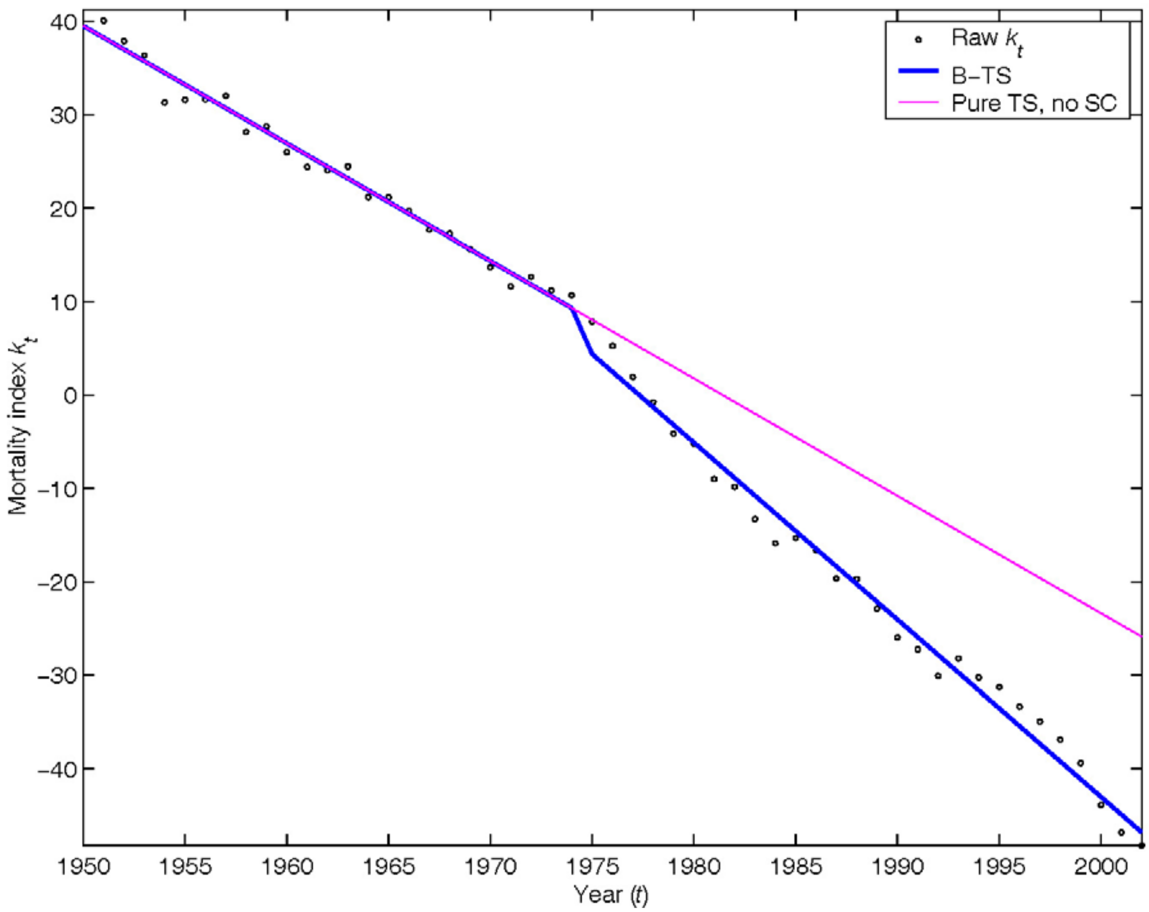
TABLE 2

Results of Zivot and Andrews' Test

Population	*t*-statistic	Conclusion	Detected break-year
Canada	–4.447	Broken-trend stationary	1975
England and Wales	–7.008	Broken-trend stationary	1974
The United States	–5.341	Broken-trend stationary	1974

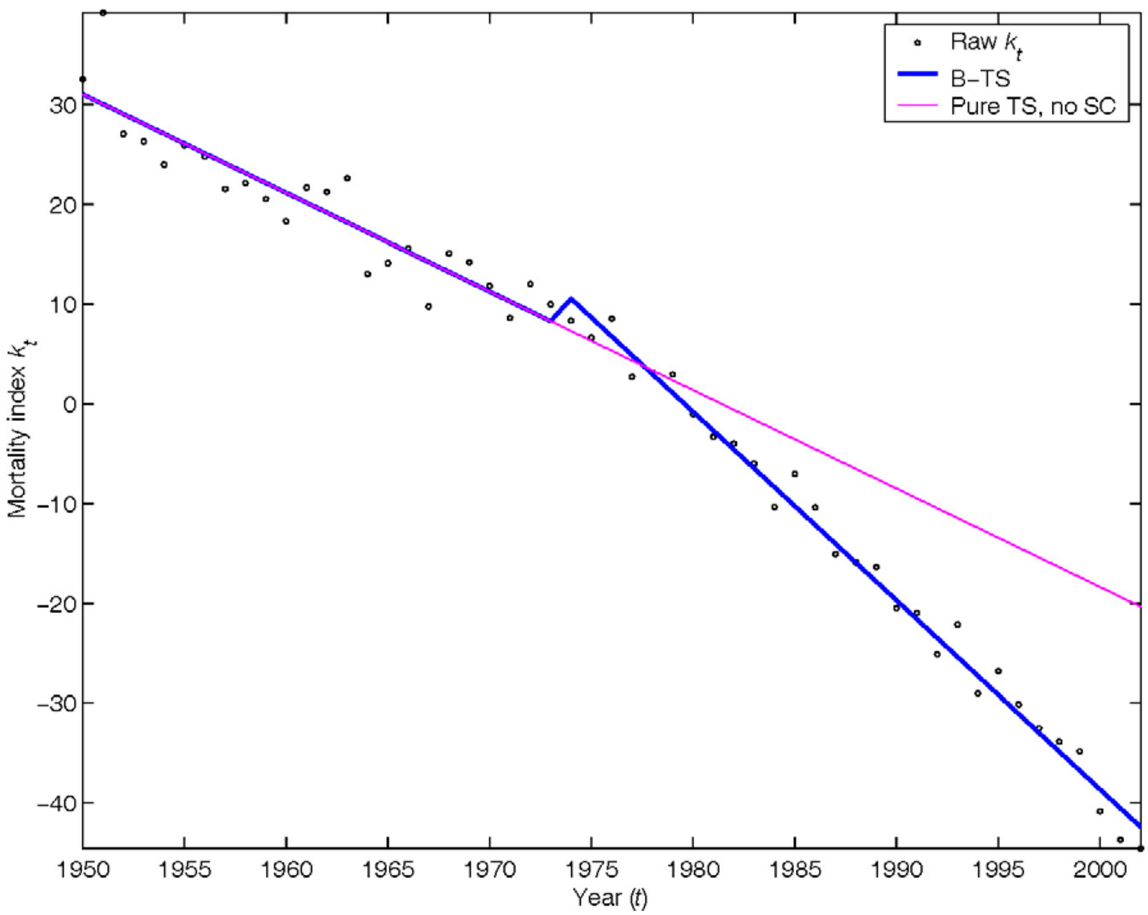
FIGURE 3

Trended k_t Based on the Best-Fitting Broken-Trend Stationary Model (B-TS) and a Hypothetical Trend of k_t under the Assumption that No Structural Change in the 1970s (pure TS, no SC)



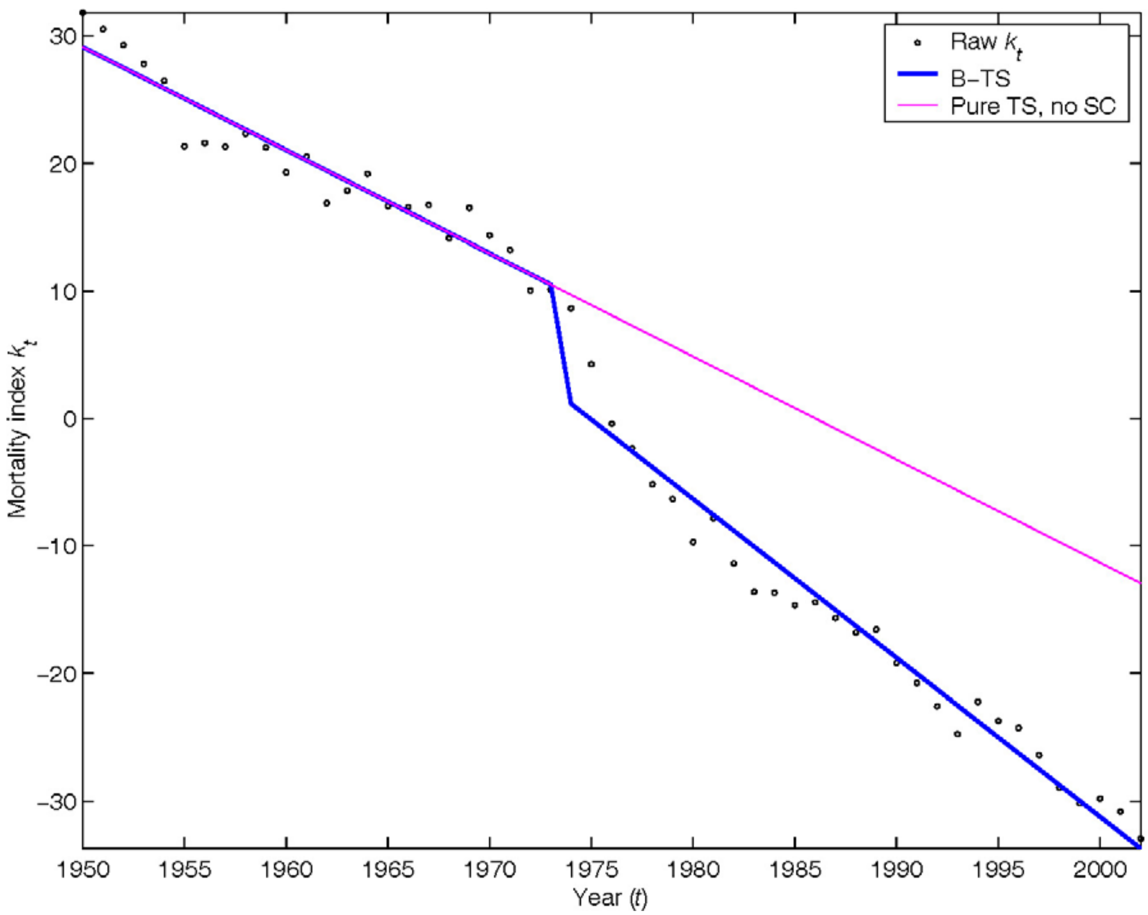
Canada

FIGURE 3—Continued



England and Wales

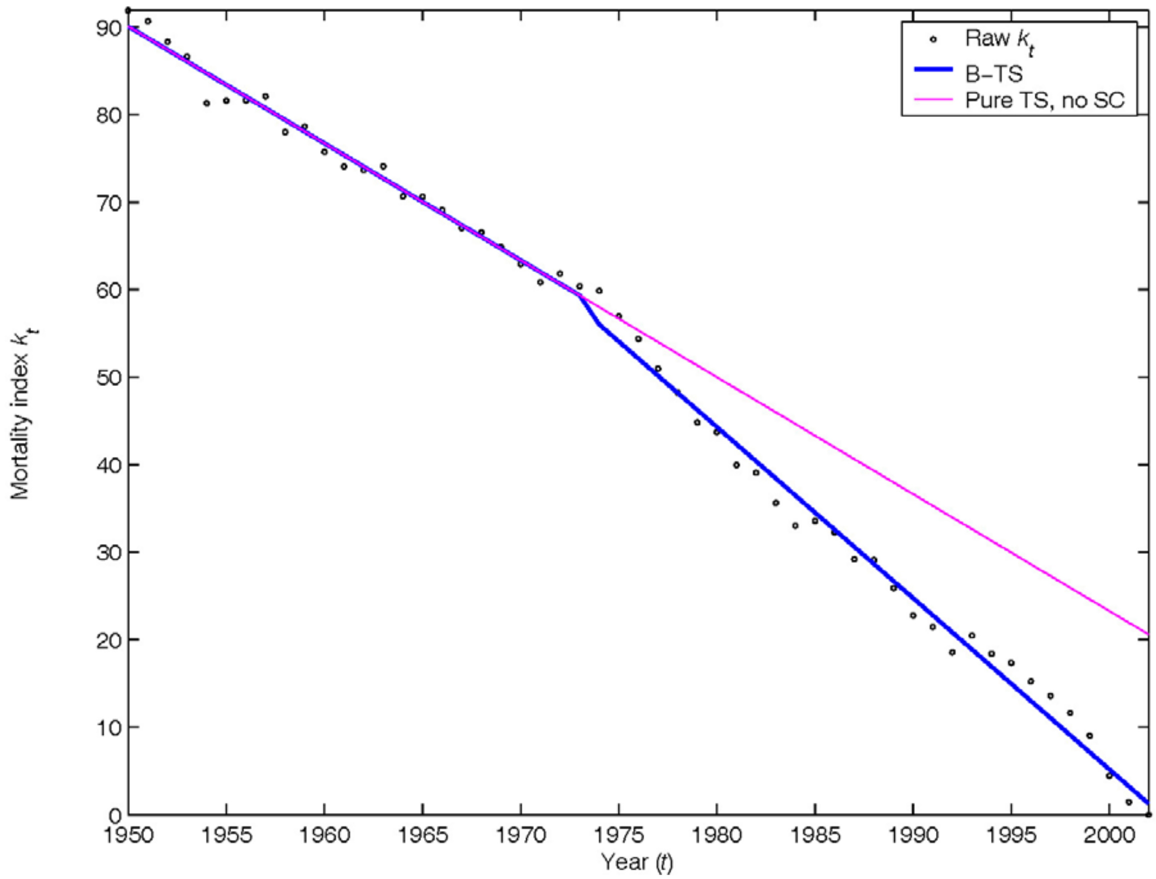
FIGURE 3—Continued



The United States

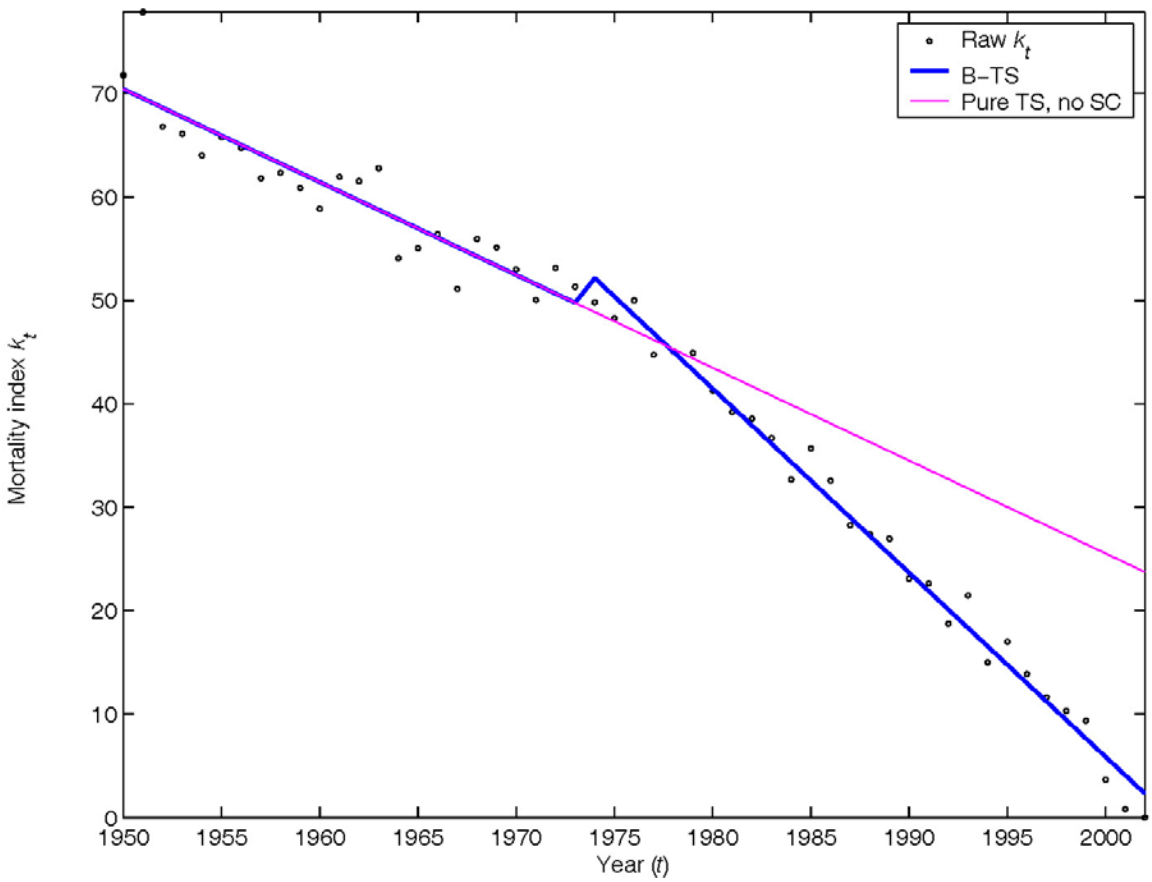
FIGURE 4

Trended k_t (Taken Account of the Forecast Origin Correction) Based on the Best-Fitting Broken-Trend Stationary Model (B-TS) and a Hypothetical Trend of k_t under the Assumption that No Structural Change in the 1970s (pure TS, no SC)



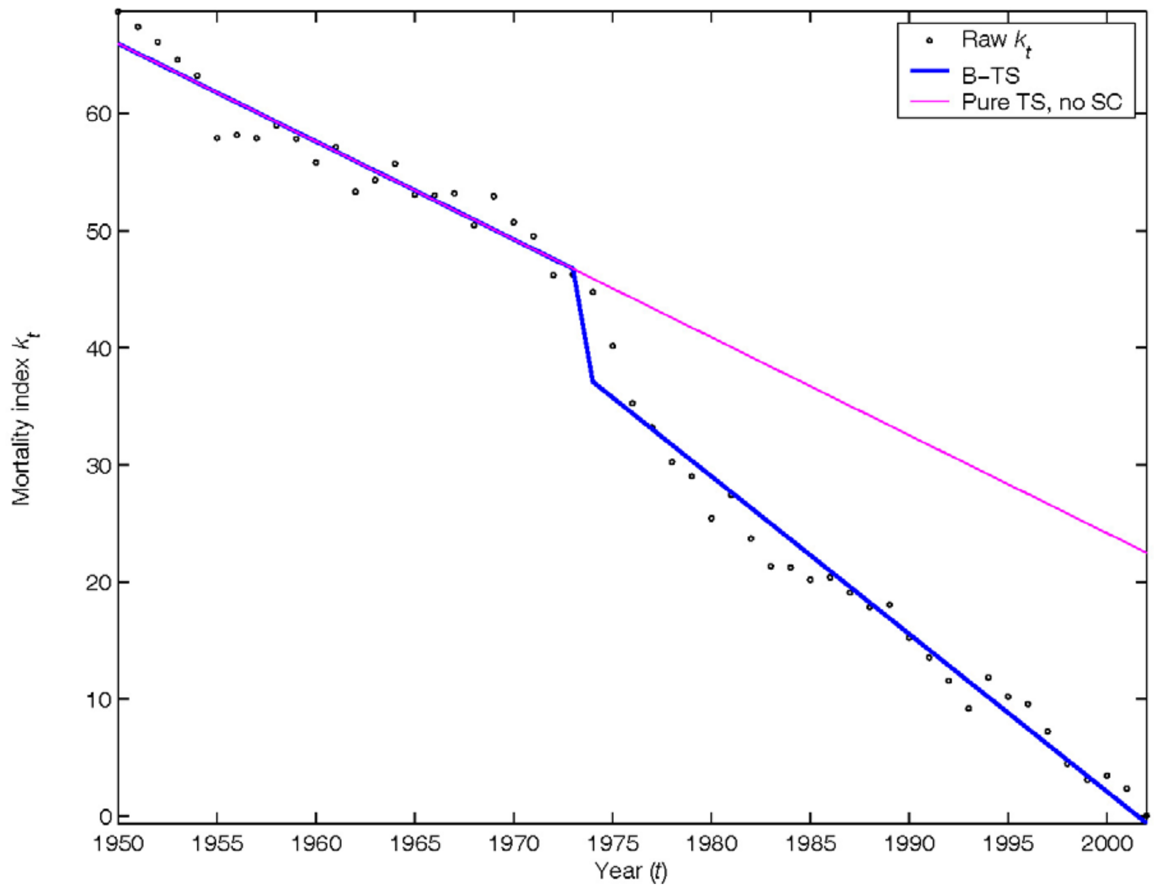
Canada

FIGURE 4—Continued



England and Wales

FIGURE 4—Continued



The United States

4. Empirical Results

4.1 Point Forecasts of k_t

Model parameters in both difference and broken-trend stationary models can be estimated by either the method of maximum likelihood or the principle of least squares. Given the parameter estimates, we can compute the *s*-step ahead forecasts of the mortality indexes by using the following equations:

1. for an ARIMA(0,1,0) model:

$$\hat{k}_{T+s} = \hat{c}s\,; \qquad (8)$$

2. for a broken-trend stationary model:

$$\hat{k}_{T+s} = \hat{\alpha}_2 + \hat{\beta}_2 s\,; \qquad (9)$$

where $\hat{c}$, $\hat{\alpha}_2$ and $\hat{\beta}_2$ are the estimates of c, α_2, and, β_2, respectively; T is the forecast origin; and $\hat{k}_{T+s}$ is the s-step ahead forecast of k_t.

In Figure 5, we show the point forecasts of k_t from the best fitting ARIMA(0,1,0) and broken-trend stationary models. We observe that, for Canada and England and Wales, the forecasts of k_t diverge, leading to materially different mortality forecasts in medium and long run; for instance, the broken-trend stationary forecast of k_t for the English and Welsh population in 2075 is −178.71; and this level of mortality will not be attained by the corresponding ARIMA(0,1,0) forecast until 2093. The observed patterns suggest that the distinction between difference and broken-trend stationarity is particularly important to analyses which require long horizon mortality forecasts; for example, should the structural change in mid-1970s be permanent, the infinite horizon social security imbalance could be significantly more severe than the estimate made by Lee and Anderson (2005).

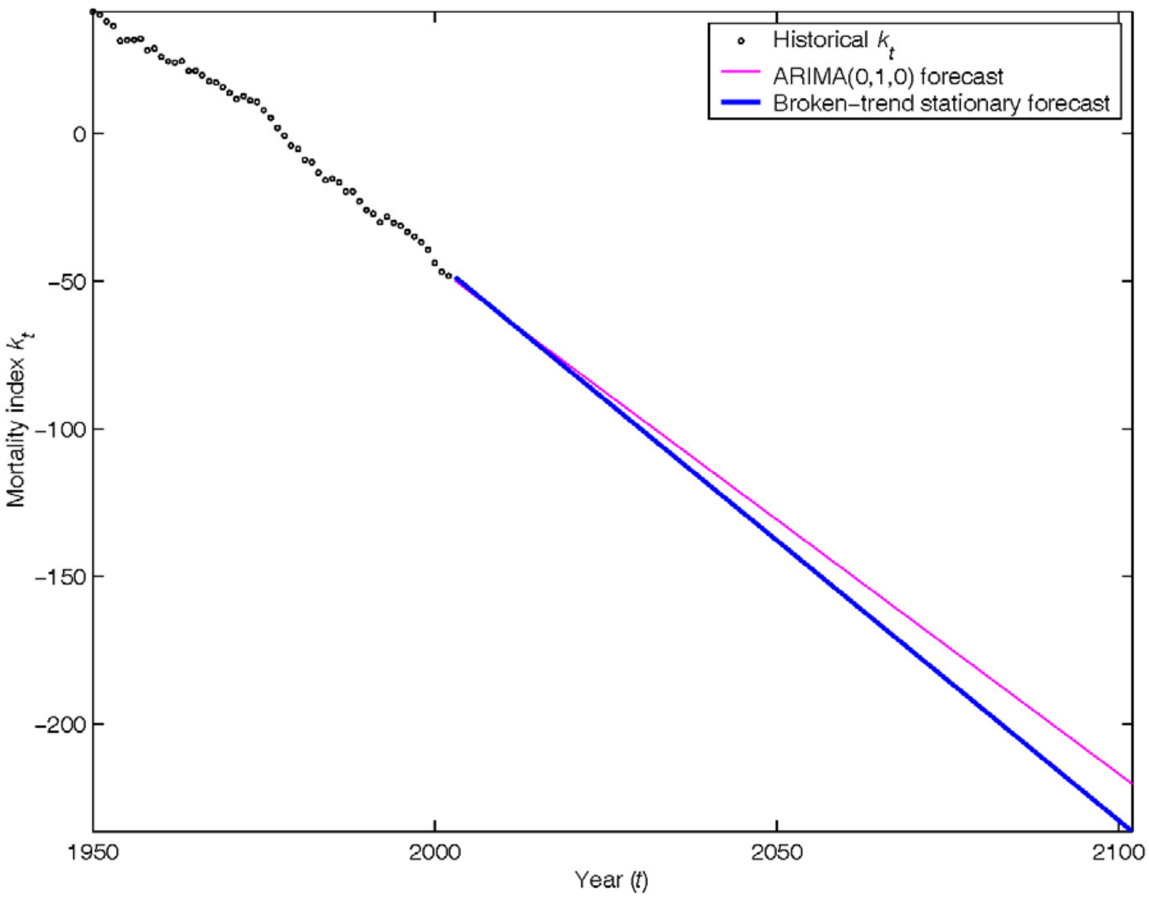
One may question why the two types of models yield similar forecasts of k_t for the U.S. population in Figure 5. This may be explained by the properties of ARIMA(0,1,0) forecasts. The slope of an ARIMA(0,1,0) forecast is $\hat{c}$; and in most parameter estimation methods, for example, the conditional least squares method, the value of $\hat{c}$ for a time-series $\{X_t;\ t = 1, ..., n\}$ is computed using the following expression:

$$\hat{c} = \frac{X_n - X_1}{n-1}, \qquad (10)$$

which depends on X_1 and X_n only. As a result, an ARIMA(0,1,0) forecast is highly subject to the idiosyncratic features of the first and last observation in the experience period. For the U.S. population, at $\hat{k}_{1950}$ and $\hat{k}_{2002}$ gives a value of $\hat{c}$ that is coincidentally close to the slope ($\hat{\beta}_2$) of the trend stationary forecast, resulting in the two forecasts being similar. However, if we begin the experience period in a later year, say 1955, instead of 1950, the divergence between the two forecasts would become highly significant (see Figure 6).

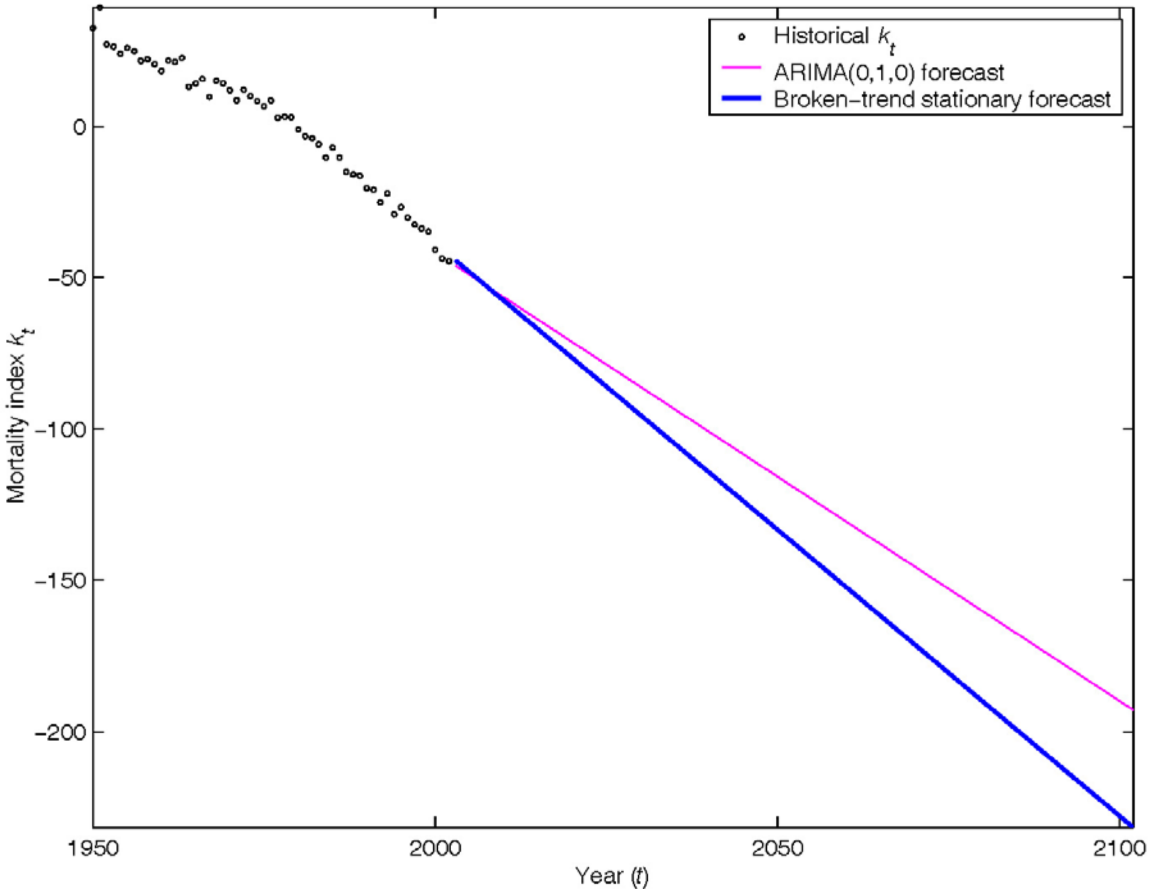
FIGURE 5

Optimal Forecasts of k_t from the Best-Fitting ARIMA(0,1,0) and Broken-Trend Stationary Models for the Period 2003-2103; Experience Period is 1950-2002



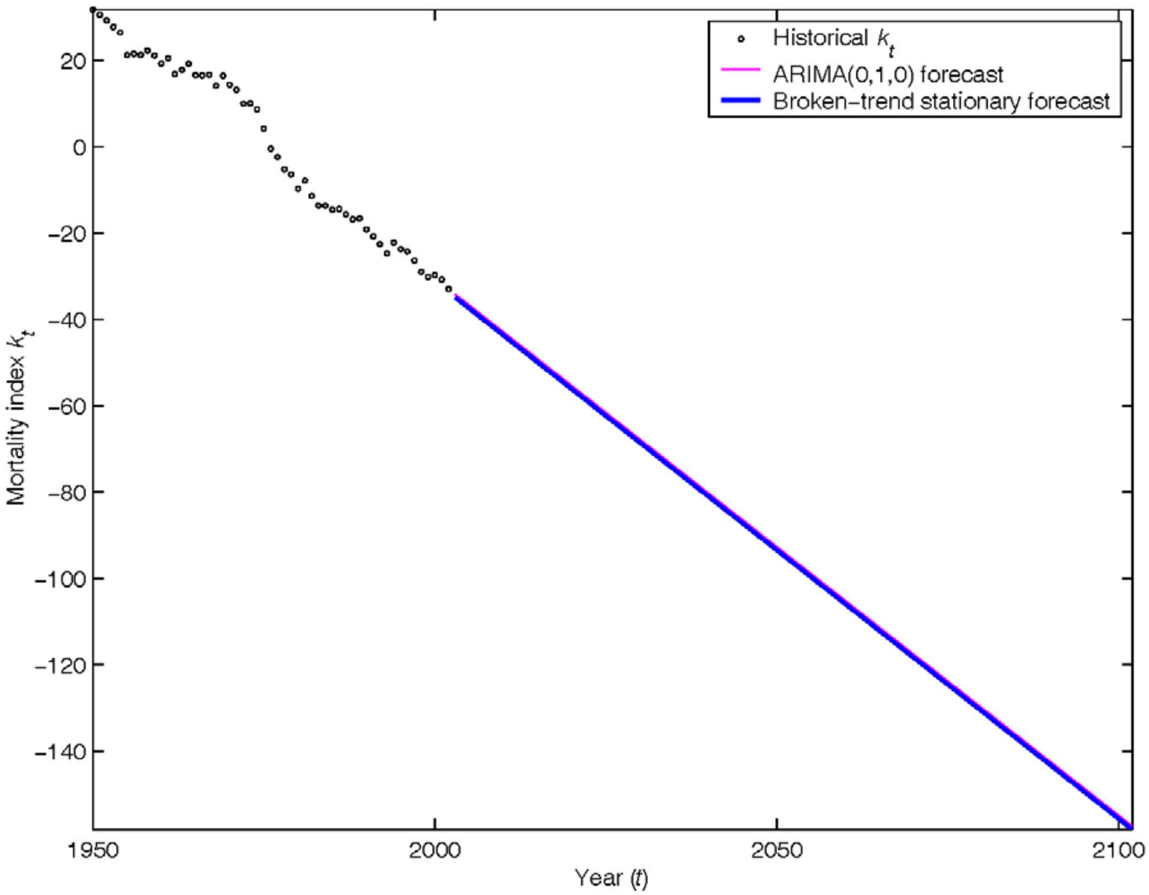
Canada

FIGURE 5—Continued



England and Wales

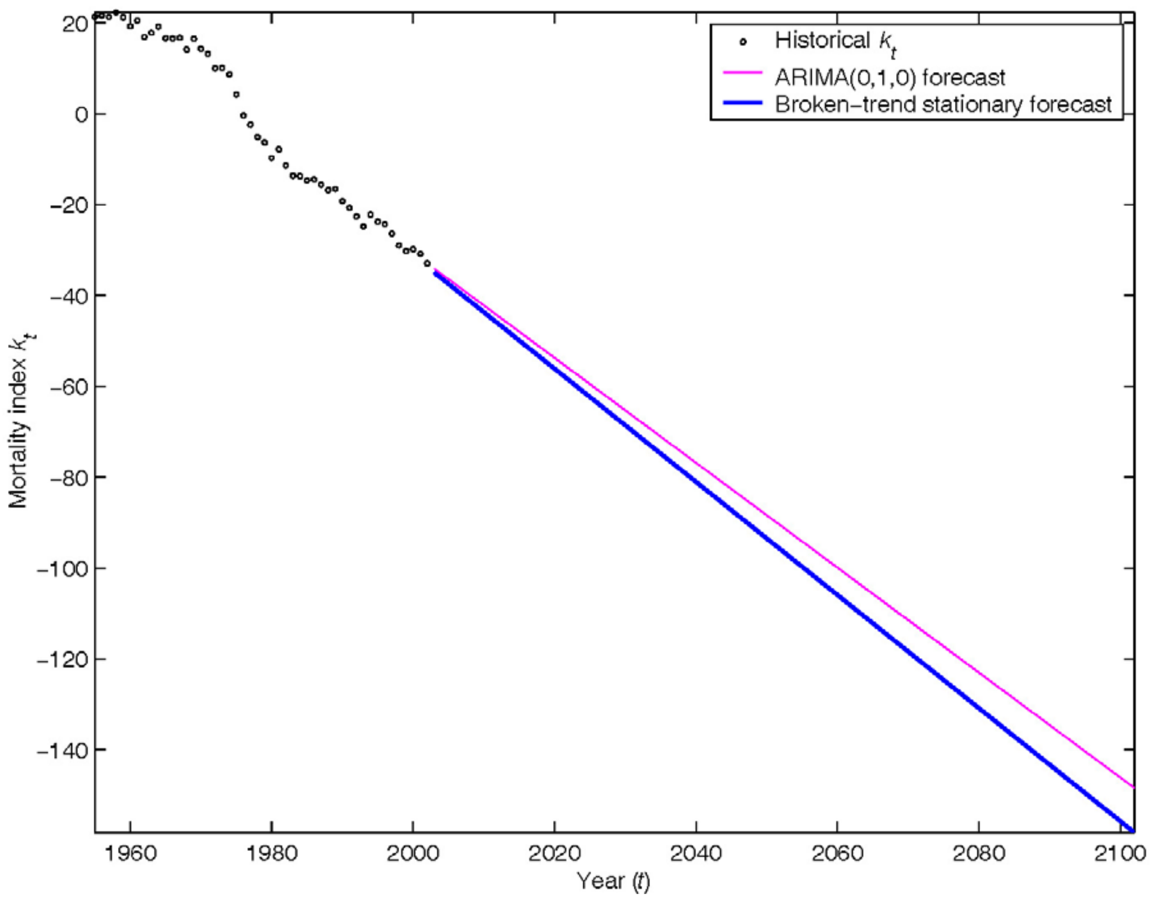
FIGURE 5—Continued



The United States

FIGURE 6

Optimal Forecasts of k_t for the U.S. Population from the Best-Fitting ARIMA(0,1,0) and Broken-Trend Stationary Models for the Period 2003-2103; Experience Period is 1955-2002



4.2 Impact on Age-specific Death Rates

Given an extrapolation of k_t, we can easily compute the implied projection of future age-specific death rates. Furthermore, if the decline in k_t is linear, then each age-specific central death rate declines at its own constant exponential rate.

Let $R(x,t) = 1 - \frac{m_{x,t+1}}{m_{x,t}}$ be the annual reduction of central death rate at age x and in year t. Retrospectively, the annual rates of reduction implied by a broken-trend stationary model can be summarized as follows:

$$\hat{R}(x,t) = \begin{cases} 1 - \exp(\hat{b}_x \hat{\beta}_1), & t \le T \\ 1 - \exp(\hat{b}_x \hat{\beta}_2), & t > T^* \end{cases}. \qquad (11)$$

In Table 3 we compare the annual rates of reduction implied by the best fitting broken-trend stationary model before and after the structural break point T^*. In using Zivot and Andrews' method, the structural break point T^* is determined statistically from the mortality index data series; and for all three populations under consideration, the detected structural break points are located in either 1974 or 1975. The comparison in Table 3 thus indicates a common (and abrupt) acceleration of mortality decline has occurred in mid-1970s.

We are not the first to identify the mutual acceleration of mortality decline in the 1970s. Kannisto et al. (1994) observed in the historical mortality data of 27 countries that, except for a few Eastern European countries, rates of mortality improvement for both sexes have significantly accelerated in the 1970s. Vaupel (1997) further observed in the Kannisto-Thatcher oldest-old mortality database that remarkable progress has been made since the 1970s in improving survival at older ages, even at the most advanced ages. However, both Kannisto et al. and Vaupel did not support their empirical observations with any mathematical/statistical models. This study fills in this gap by offering a rigorous statistical justification for the observed universal acceleration of mortality decline in previous studies.

Prospectively, the future rate of reduction per annum can be summarized by the following equations:

1. for an ARIMA(0,1,0) model:

$$\hat{R}(x, T+s+1) = 1 - \exp(\hat{b}_x \hat{c})\,; \qquad (12)$$

2. for a broken-trend stationary model:

$$\hat{R}(x, T+s+1) = 1 - \exp(\hat{b}_x \hat{\beta}_2)\,; \qquad (13)$$

where T is the forecast origin and s = 1,2, ... In Table 4 we show the annual percentage reduction implied by both models at various ages. We observe that, for Canada and England and Wales, the reduction factors derived from a broken-trend stationary model and an ARIMA(0,1,0) model are highly different: the average difference is approximately 10 percent for Canada and 20 percent for England and Wales. For the United States, the two models yield almost identical reduction factors because of the coincidental similarly between $\hat{c}$ and $\hat{\beta}_2$ we mentioned in Section 4.1.

TABLE 3

Annual Reduction in Age-Specific Death Rates Implied by the Best-Fitting Broken-Trend Stationary Models Before and After the Structural Break in Year T^*.

	Canada		England and Wales		The United States	
Age	Before T^*	After T^*	Before T^*	After T^*	Before T^*	After T^*
45	1.31%	1.97%	0.98%	1.86%	0.79%	1.22%
55	1.19%	1.78%	0.97%	1.85%	0.84%	1.30%
65	1.14%	1.71%	0.89%	1.70%	0.75%	1.15%
75	0.99%	1.49%	0.80%	1.53%	0.72%	1.10%
85	0.76%	1.15%	0.74%	1.41%	0.74%	1.13%
95	0.40%	0.61%	0.44%	0.84%	0.61%	0.94%

TABLE 4. Annual Reduction in Future Age-Specific Death Rates Implied by the Best-Fitting ARIMA(0,1,0) and Broken-Trend Stationary Models

	Canada		England and Wales		The United States	
Age	Before T^*	After T^*	Before T^*	After T^*	Before T^*	After T^*
45	1.79%	1.97%	1.46%	1.86%	1.22%	1.22%
55	1.62%	1.78%	1.45%	1.85%	1.30%	1.30%
65	1.55%	1.71%	1.33%	1.70%	1.15%	1.15%
75	1.35%	1.49%	1.20%	1.53%	1.10%	1.10%
85	1.04%	1.15%	1.10%	1.41%	1.13%	1.13%
95	0.55%	0.61%	0.66%	0.84%	0.94%	0.94%

4.3 Impact on Survival Probabilities

From an actuarial viewpoint, an increased probability of survival to extreme ages can lead to more low-frequency-high-severity losses in businesses that provide some kinds of "living benefits." Prime examples include life annuities, reverse mortgages (i.e., Equity Release Mechanisms in the United Kingdom) and defined-benefit pension plans. More specifically, changes in the probability of survival to an extreme age usually have little effect on the expected loss of an insurer; however, this is obviously not the case if we consider risk measures such as VaR and CTE, since the odds of an extreme loss are largely determined by the tail of the survival distribution.

In Section 4.2, we show that, for Canada and England and Wales, the annual rates of reduction in future age-specific death rates implied by the broken-trend stationary models are systematically higher than that implied by the ARIMA(0,1,0) models by 10-20 percent. However, the difference between the survival functions implied by the two types of models can be even more significant, since each survival probability is dependent on multiple age-specific death rates.

Here we consider the probability of survival to age 100 for the cohort who was born in 2007. Under the assumption of constant force of mortality for fractional ages, this survival probability, which is actuarially denoted by $s(100)$, can be computed by using the following expression:

$$\begin{aligned}\hat{s}(100) &= \prod_{t=0}^{99} \exp(-\hat{m}_{i,2007+i}) \\ &= \prod_{t=0}^{99} \exp(-\hat{m}_{0,2002} \exp(\hat{b}_x \hat{\gamma}(5+i)),\end{aligned} \tag{14}$$

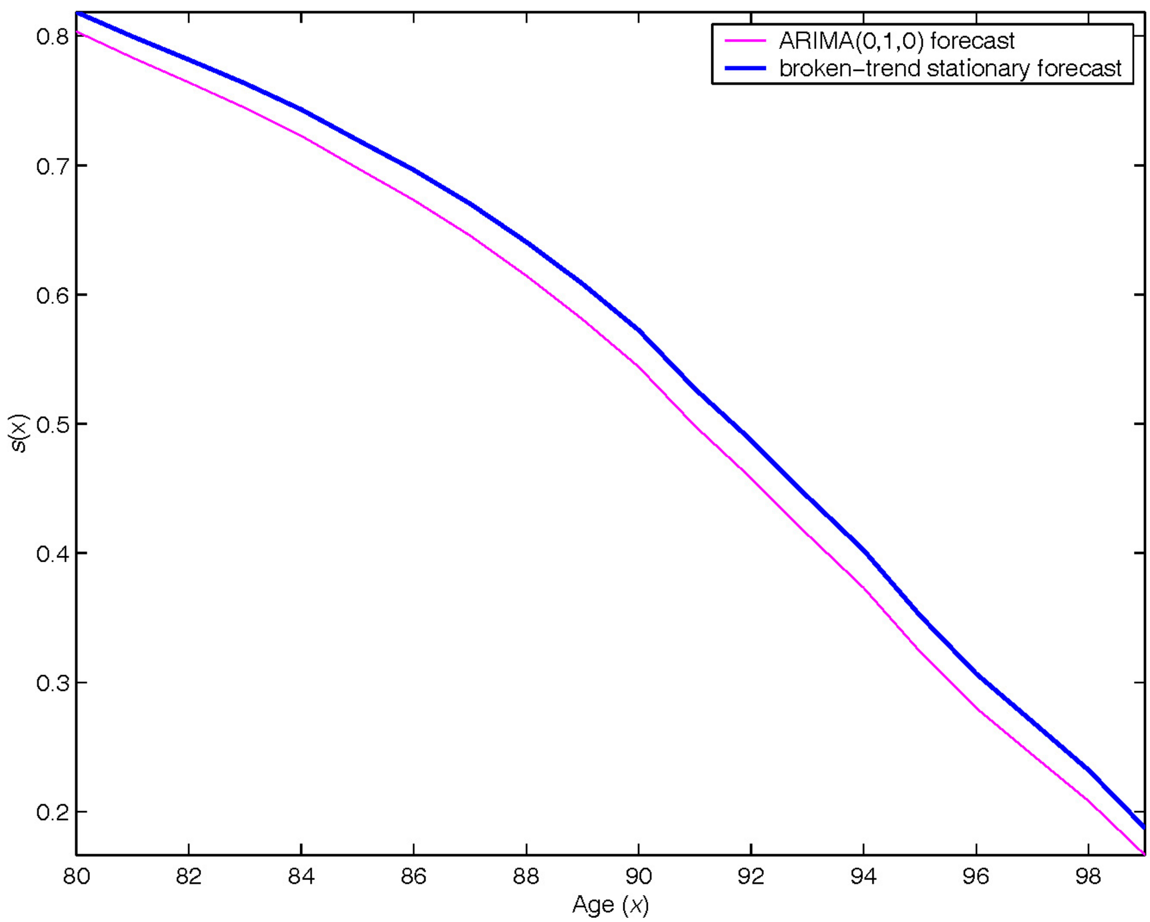
where $\hat{\gamma} = \hat{c}$ if an ARIMA(0,1,0) model is used and $\hat{\gamma} = \hat{\beta}_2$ if a broken-trend stationary model is used. The value of $m_{0,2002}$ is obtained from the available data. The results, which we show in Table 5, imply that the use of an ARIMA(0,1,0) model in projecting the Lee-Carter mortality index may potentially lead to a serious underestimation of the probability of survival to an extreme age, say 100. For Canada, the value of $s(100)$ implied by the broken-trend stationary model is 12.5 percent higher than that implied by the ARIMA(0,1,0) model; for England and Wales, the difference is more than 40 percent.

The plausibility of the projected mortality schedules is an important issue in mortality forecasting. In Figure 7, we illustrate of the shape of the projected mortality schedules from age 80 to 99 for the cohort who was born in 2007. We observe no anti-intuitive behaviors in all the projected mortality schedules.

TABLE 4. Estimates of $s(100)$ Based on the Best-Fitting ARIMA(0,1,0) and Broken-Trend Stationary Models

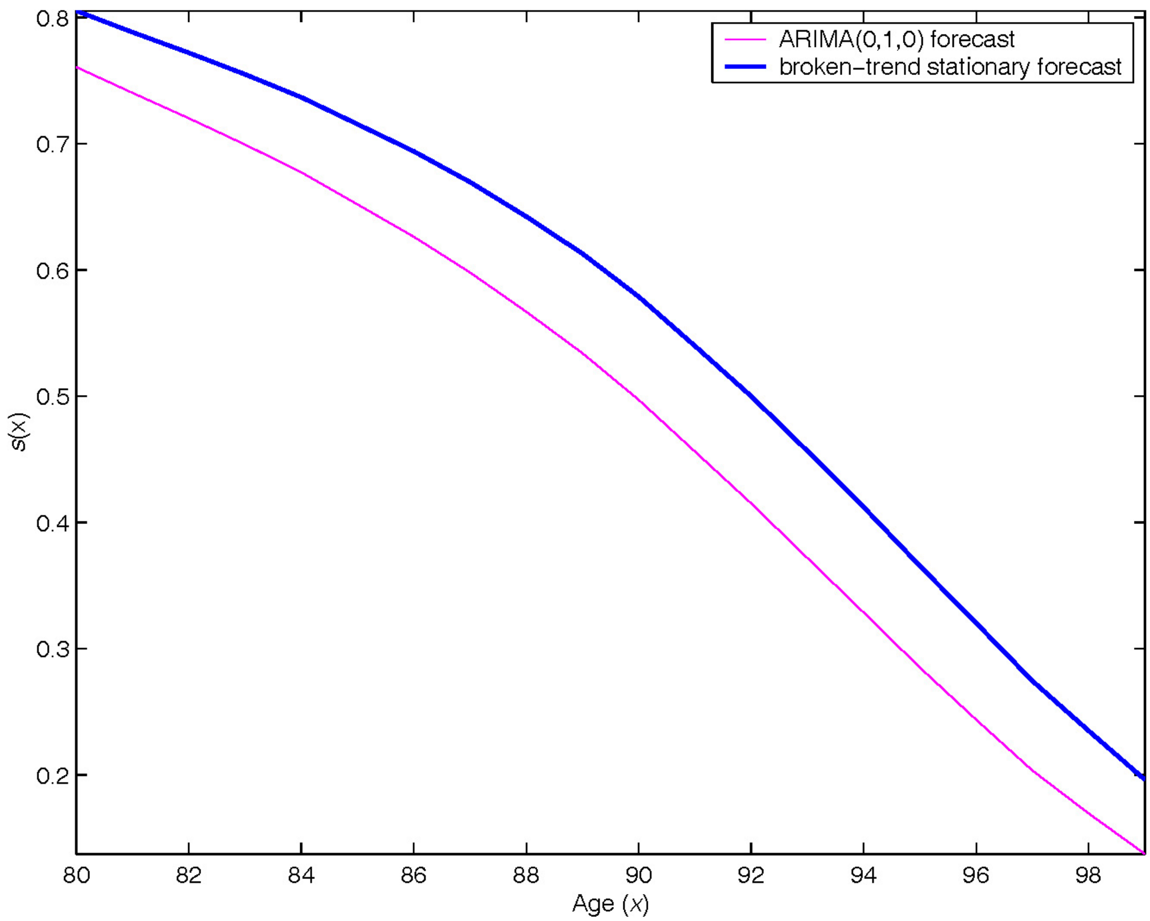
Population	Difference stationary	Broken-trend stationary
Canada	0.1664	0.1872
England and Wales	0.1373	0.1960
The United States	0.2402	0.2423

FIGURE 7. Projected Mortality Schedules at Old Ages for the Cohort Who Was Born in 2007, Using the Best-Fitting ARIMA(0,1,0) and Broken-Trend Stationary Models; Experience Period is 1955-2002



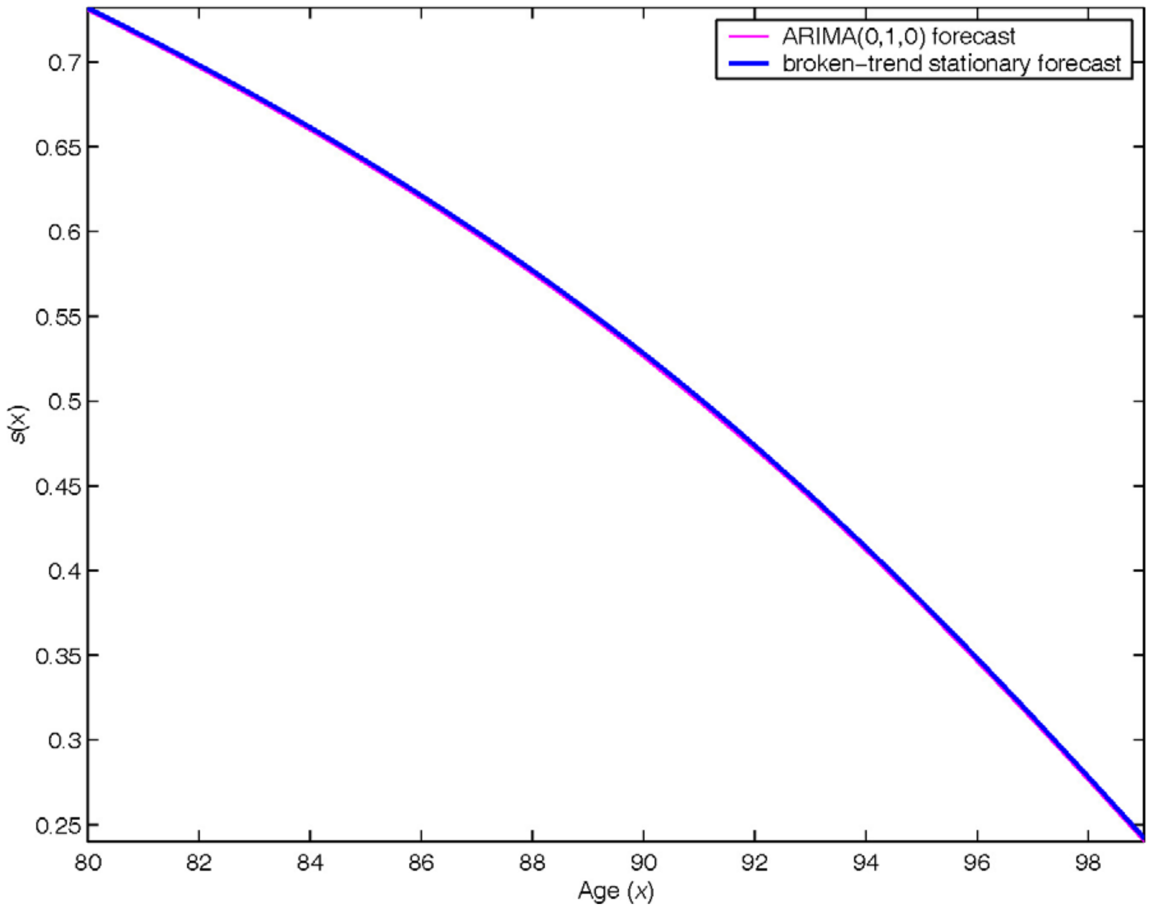
Canada

FIGURE 7—Continued



England and Wales

FIGURE 7—Continued



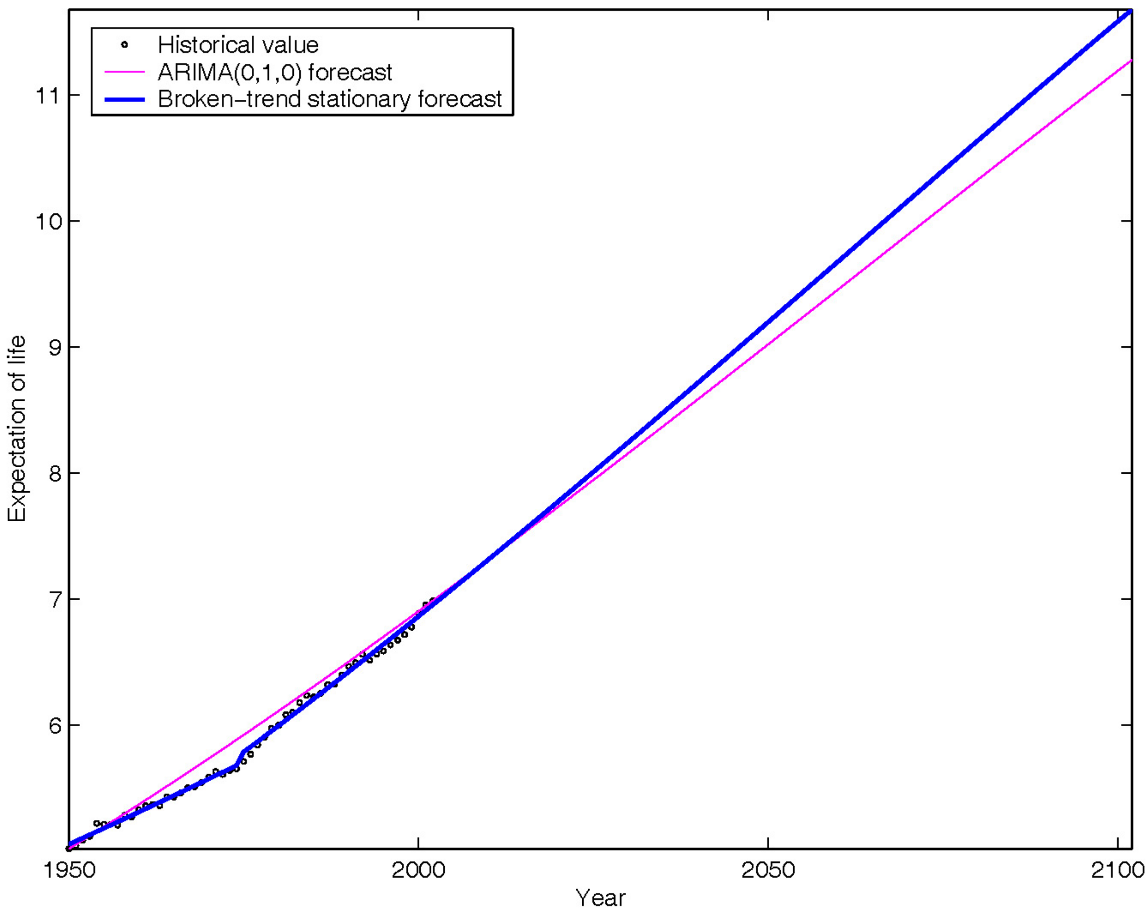
The United States

4.4 Impact on Life Expectancies and Annuity Premiums

Here we investigate the impact on two actuarial measures: (1) the expectation of life at age 85 ($\mathring{e}_{85}$); and (2) the actuarial present value of a whole life annuity of $1 sold to a life-aged-65 ($\ddot{a}_{65}$). Measure (1) can be considered as a convenient summary of old-age mortality, while measure (2) has a wide range of actuarial applications, such as pension plan valuation. Figures 8 and 9 show, for each population, the projected values of $\mathring{e}_{85}$ and $\ddot{a}_{65}$, respectively.

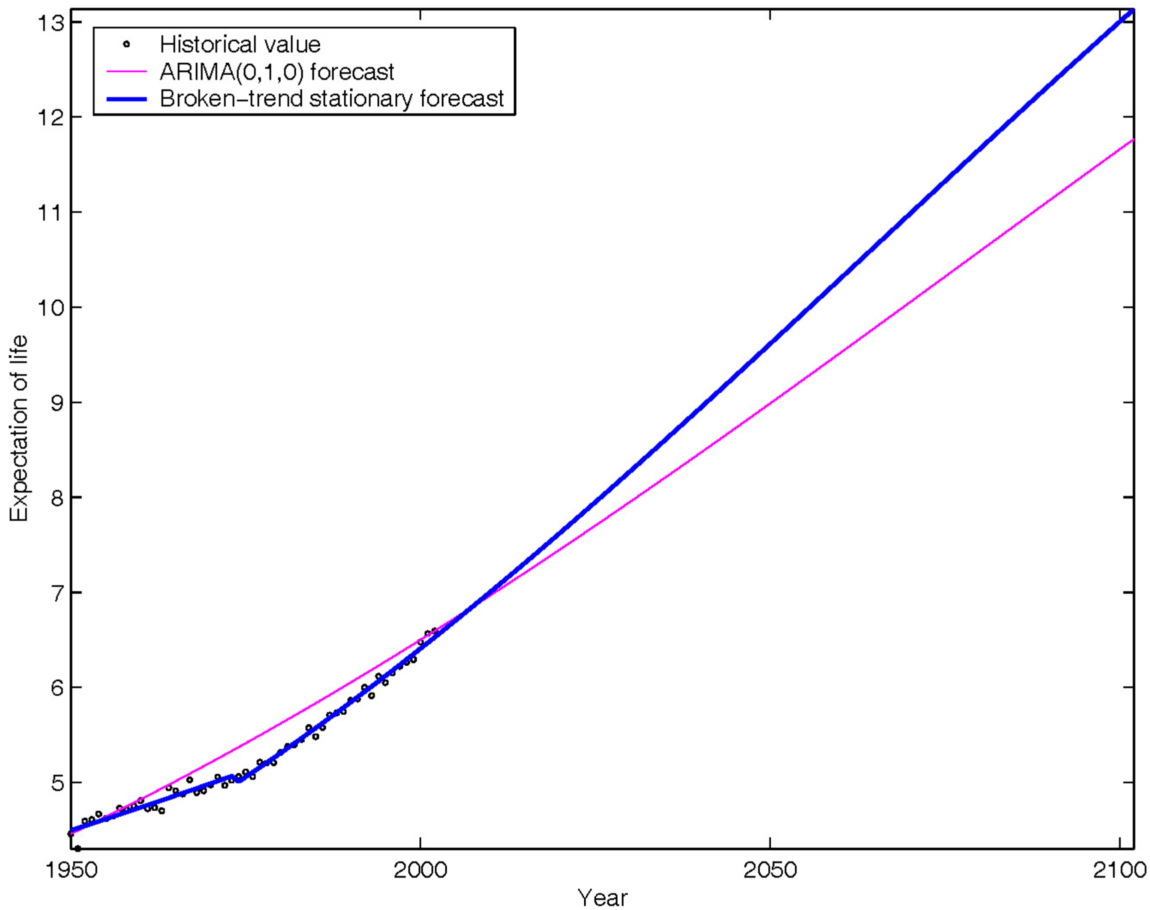
FIGURE 8

Optimal Forecasts of $\mathring{e}_{85}$ from the Best-Fitting ARIMA(0,1,0) and Broken-Trend Stationary Models for the Period 2003-2103; Experience Period is 1950-2002



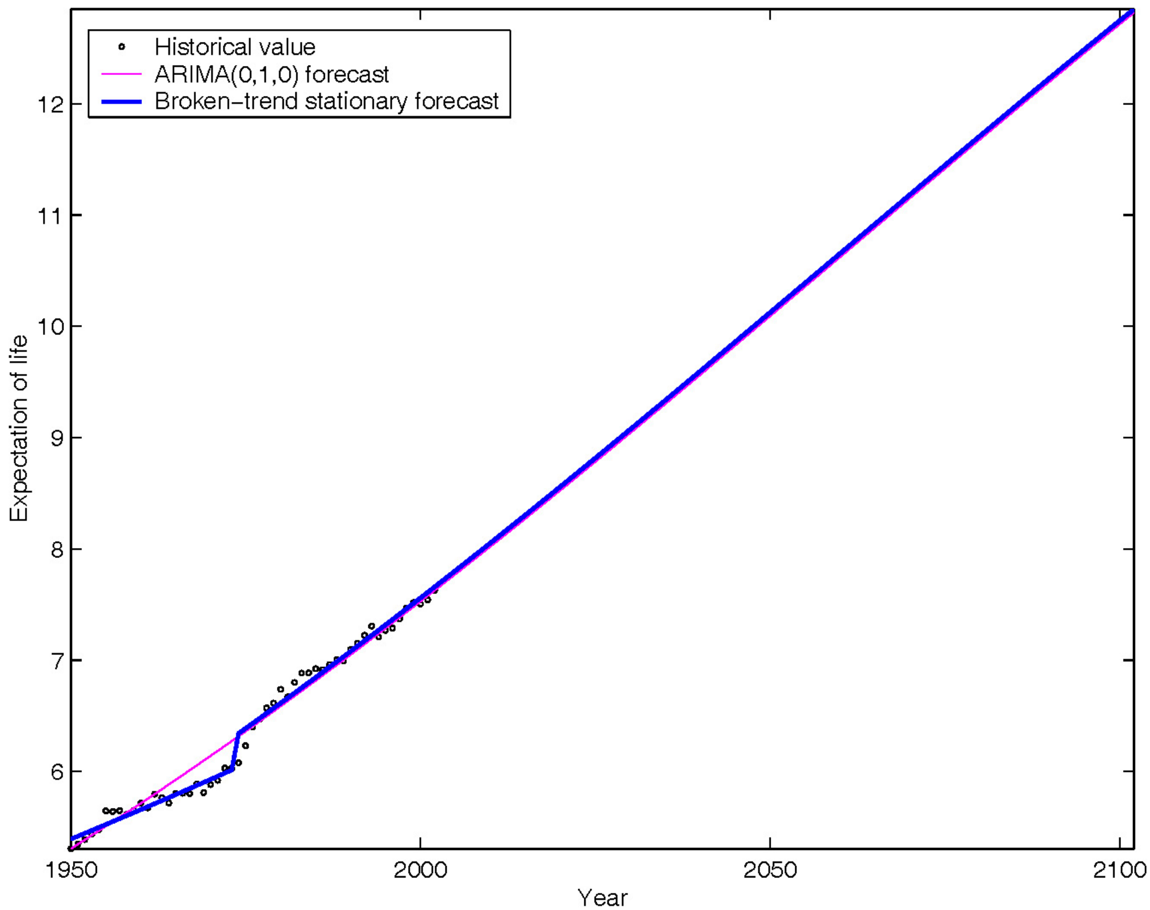
Canada

FIGURE 8—Continued



England and Wales

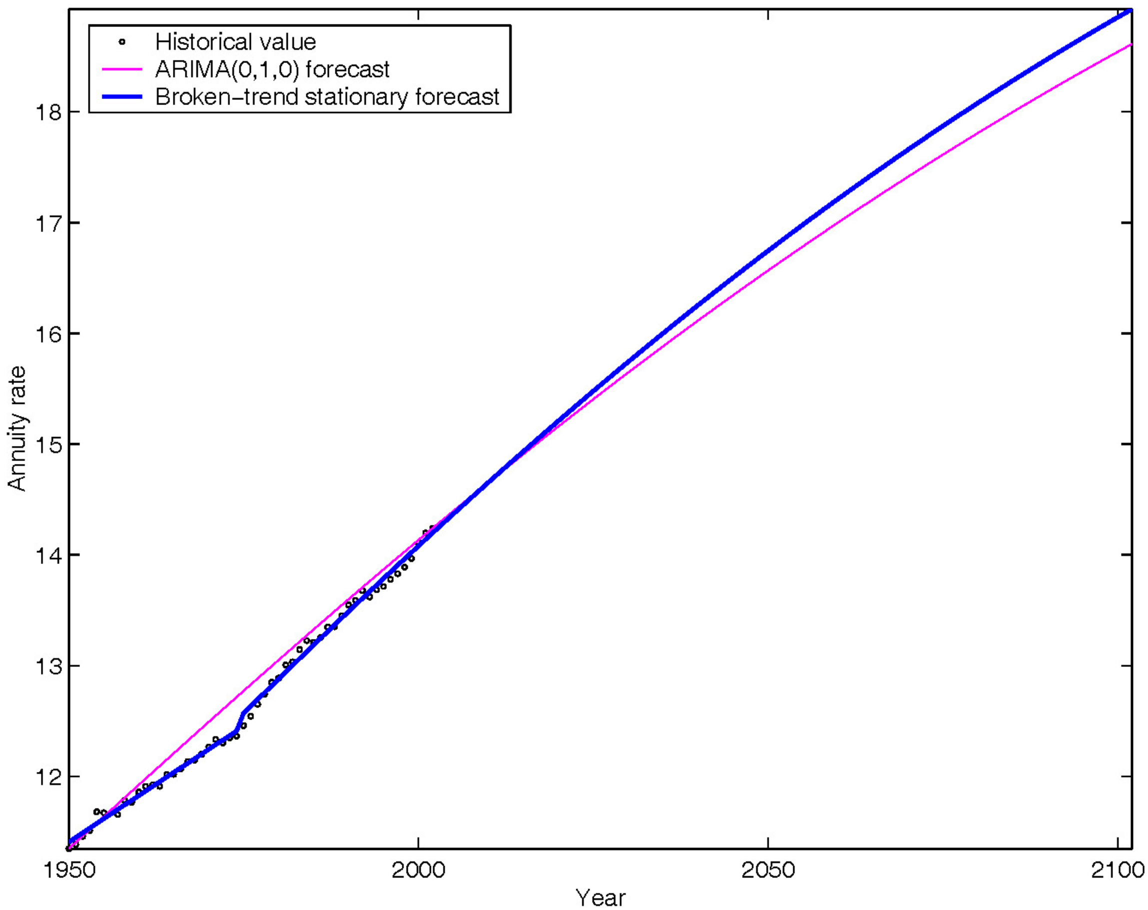
FIGURE 8—Continued



The United States

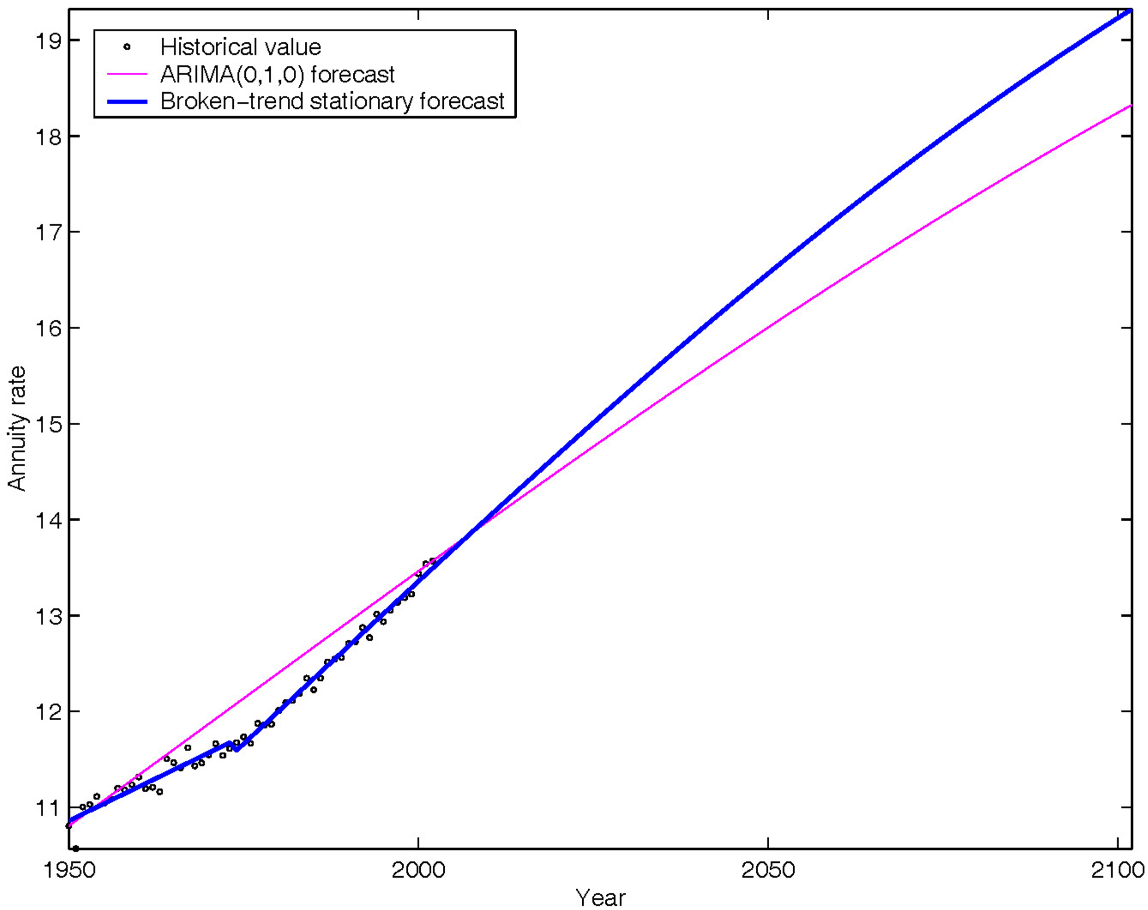
FIGURE 9

Optimal Forecasts of $\ddot{a}_{65}$ from the Best-Fitting ARIMA(0,1,0) and Broken-Trend Stationary Models for the Period 2003-2103; Experience Period is 1950-2002



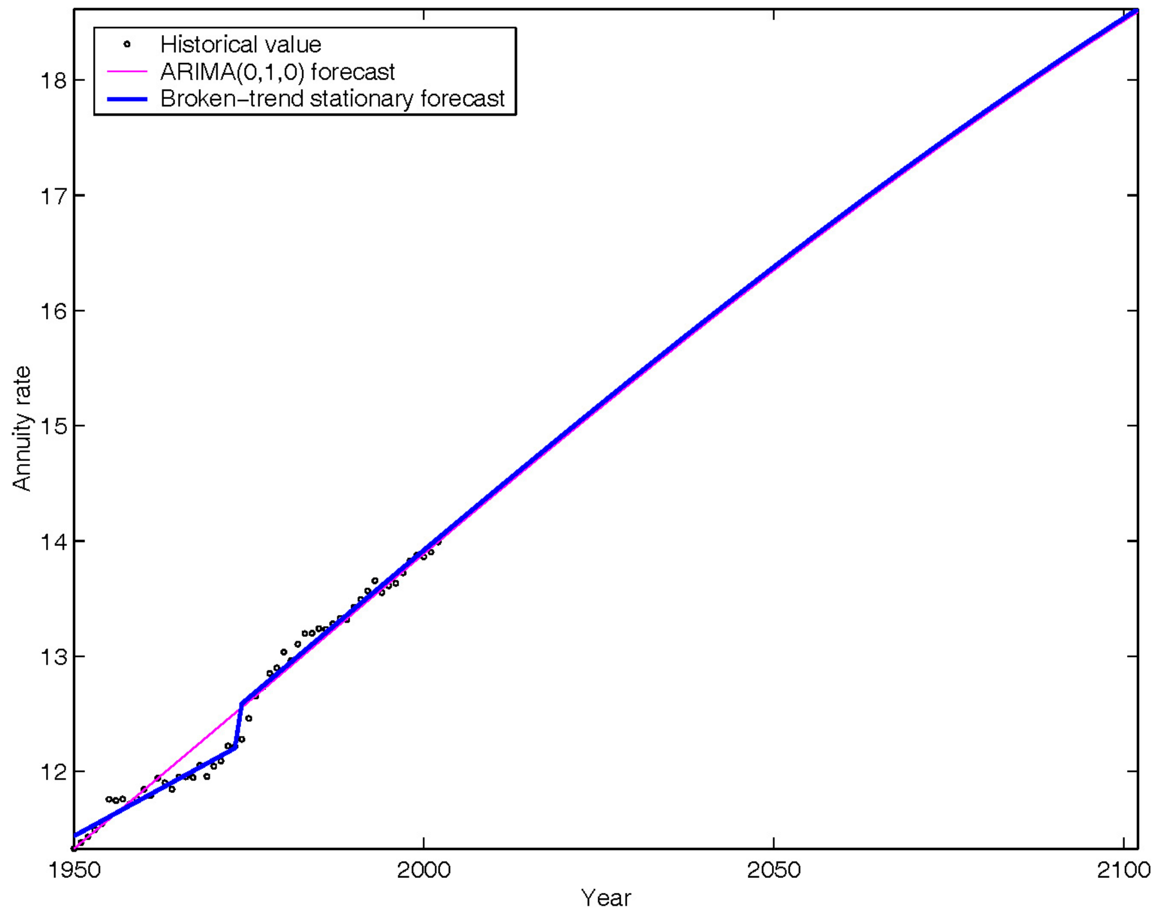
Canada

FIGURE 9—Continued



England and Wales

FIGURE 9—Continued



The United States

5. Discussion and Conclusion

In using the Lee-Carter approach, the time-series process for the mortality index (parameter k_t) is of critical importance because the entire mortality forecast is determined by an extrapolation of k_t via this process. In this study, we examined whether the time-series of k_t for the populations of Canada, England and Wales and the United States are stochastic trends (difference stationary) or deterministic trends (trend stationary). Specifically, we used Zivot and Andrews' method to identify the type of stationarity to which the mortality indexes belong. The *t*-values favor broken-trend stationary models over difference stationary (ARIMA) models, which are used in most previous applications of the Lee-Carter methodology.

Zivot and Andrews' method also tests for any possible structural changes in the mortality indexes. The results indicate that, for each of the populations we considered, the gradient of the mortality index has significantly increased after the structural break point T^*. It is interesting that the structural break points, which are detected statistically from the historical data, are all located in mid-1970s. Our findings therefore provide strong statistical evidence for the acceleration of mortality decline which Kannisto et al. (1994) and Vaupel (1997) observed.

There are three lessons we can learn from the results of this study. First, while the patterns of mortality reduction become increasingly complex, ARIMA models cannot capture any structural changes in the indexes. Should the structural changes in the 1970s be permanent, the use of ARIMA models in projecting the mortality indexes will lead to a significant overestimation of future mortality rates. Actuaries should therefore bear in mind this potential consequence when they employ the Lee-Carter method in setting central mortality assumptions.

Second, the broken-trend stationary forecasts are more consistent with the increased pace of mortality decline, which has lasted for more than 30 years. However, no different from other time-series models, broken-trend stationary models yield linear forecasts, presuming that there will be no structural changes in the future. Given that future structure changes are entirely possible, forecasters should closely monitor the rates of mortality decline and perform model recalibration from time to time, no matter which time-series model is used.

Third, forecasters should be cautious when interpreting the probabilistic confidence intervals attached to a Lee-Carter mortality forecast. These confidence intervals, which are based on a linear time-series model, exclude deep structural changes and kinds of shocks and trend breaks that were not observed in the past. The true amount of uncertainty can be substantially higher than that included in the intervals.

For the difference stationary and simple trend stationary versions, we can apply bootstrapping techniques (see, e.g., Brouhns et al., 2005; Koissi et al., 2005) to perform stochastic simulation on the actuarial measures considered. However, for the broken-trend stationary version, the bootstrapping techniques will underestimate the underlying uncertainty since they assume that the model structure is known and fixed (i.e., structure changes will not occur in the future). Further research on how to incorporate the possibility of future structural changes into the prediction intervals is warranted.

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