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INTERVAL ESTIMATES FOR RISK LOADS FOR INSURERS

WILLIAM F. BAILEY

ABSTRACT

In Volume LXXV of the Proceedings there appeared a paper entitled Risk Loads for Insurers by Feldblum. One of the methods described in that paper was based on calculating for property/casualty lines of business pointestimates of the type of β's used in the Capital Asset Pricing Model.

Starting with TABLE 4 (Profit Margins and Their Standard Deviations by Line of Business (1979-1988)) of his paper, I use operational bootstrapping to generate interval estimates for the β's shown in his TABLE 5.

The method of operational bootstrapping has already been described:

(a) for the univariate case, in the APPENDIX of the paper entitled "A Method for Determining Confidence Intervals for Trend in Volume XLIV (pp 1-29) of the Transactions of the Society of Actuaries, and

(b) for the bivariate case, in an APPENDIX of the paper entitled "Six Bridges to Ψ's" in ARCH 1993.1.

The method of operational bootstrapping for the 4-dimensional case, used to generate the interval estimates in the current paper, is a natural generalization of the bivariate case.

INTRODUCTION

Confidence intervals for the betas in TABLE 4 of Feldblum's paper can be generated using the assumption of underlying normal distributions.[1] However, bootstrapping (operational or not) for interval estimates does not require the normality assumption. The method of operational[2] bootstrapping does not use random

[1] See Hogg, R. and Craig, A., *Introduction to Mathematical Statistics*, New York: Macmillan Publishing Co., Inc., 1978, 296-298

[2] The term "operational bootstrapping" first appeared in the paper entitled "Bootstrapping for Order Statistics Sans Random Numbers (Operational Bootstrapping)" in:

numbers.

The betas in TABLE 5 of Feldblum's paper were calculated for each line of business by regressing the ten profit margins for that line of business on the ten profit margins for the total of all lines of business combined, using calendar year from 1979 to 1988, inclusive.

For each line of business I generated at least two distributions of betas, one by bootstrapping using random numbers and the other by operational bootstrapping (which does not involve random numbers). Of course, the bootstrap distributions generated using random numbers tended to become more accurate as more random numbers (trials) were used. Only the operational bootstrap distributions of β are shown in my paper.

METHOD OF BOOTSTRAPPING WITH RANDOM NUMBERS FOR INTERVAL ESTIMATES FOR BETAS

Focusing on one particular line of business, a representation of the distribution of beta for that line of business is obtained by resampling with replacement from the bivariate empirical distribution of bivariate profit margins

$$(x_i, y_i)_{i=1,\ldots,10}$$

where

x_i is the profit margin for that line of business and

y_i is the corresponding profit margin for the total of all lines of business combined

and i refers to the i^{th} year in the observation period (1979 to 1988, inclusive).

Each resample involves selecting (with replacement) a sample (resample) of size 10 from the bivariate empirical distribution

LePage, R. and Billard, L., *Exploring the Limits of Bootstrap*. New York: John Wiley & Sons, 1992, 1-29

$$\begin{bmatrix} x_1 & y_1 & .1 \\ x_2 & y_2 & .1 \\ & \cdots & \\ & \cdots & \\ & \cdots & \\ x_{10} & y_{10} & .1 \end{bmatrix}$$

For each resample we calculate four sums; namely,

$$\sum_{j=1}^{j=10} x_{i_j}$$

$$\sum_{j=1}^{j=10} x_{i_j} \cdot y_{i_j}$$

$$\sum_{j=1}^{j=10} y_{i_j}$$

$$\sum_{j=1}^{j=10} y_{i_j} \cdot y_{i_j}$$

where each i_j is an i (1<=i<=10) obtained using a random number and looking up a number pair (x_{i_j}, y_{i_j}) which corresponds to the first cumulative probability higher than that random number in the above bivariate distribution. If we generate n such resamples (of size 10), we would associate the probability 1/n with each such 4-tuple. The resulting 4-dimensional distribution would be transformed according to the following formula into a univariate distribution of betas:

Formula I

$$\left[\frac{\sum_{j=1}^{j=10} x_{i_j} \cdot y_{i_j} - \sum_{j=1}^{j=10} x_{i_j} \cdot \frac{1}{10} \cdot \sum_{j=1}^{j=10} y_{i_j}}{\sum_{j=1}^{j=10} y_{i_j} \cdot y_{i_j} - 10 \cdot \frac{1}{10} \cdot \sum_{j=1}^{j=10} y_{i_j} \cdot \frac{1}{10} \cdot \sum_{j=1}^{j=10} y_{i_j}} \qquad 1/n \right]$$

METHOD OF OPERATIONAL BOOTSTRAPPING FOR INTERVAL ESTIMATES FOR BETAS

Resampling in the case of operational bootstrapping involves the use of 4-dimensional numerical convolutions to generate a representation of the above 4-dimensional distribution, as accurate as might be achieved if bootstrapping were accomplished using an infinite number of random numbers. The univariate distribution of betas would be obtained from this 4-dimensional distribution using Formula I above.

First we transform the bivariate empirical distribution

$$\left[\begin{array}{ccc} x_1 & y_1 & .1 \\ x_2 & y_2 & .1 \\ & \cdots & \\ & \cdots & \\ & \cdots & \\ x_{10} & y_{10} & .1 \end{array} \right]$$

into the 4-dimensional distribution

$$\left[\begin{array}{ccccc} x_1 & x_1 \cdot y_1 & y_1 & y_1 \cdot y_1 & .1 \\ x_2 & x_2 \cdot y_2 & y_2 & y_2 \cdot y_2 & .1 \\ & & \cdots & & \\ & & \cdots & & \\ & & \cdots & & \\ x_{10} & x_{10} \cdot y_{10} & y_{10} & y_{10} \cdot y_{10} & .1 \end{array} \right].$$

Next we convolute 10 of these 4-dimensional distributions together. This involves either 9 numerical convolutions for sums or, if we use the binary buildup approach (1+1=2,2+2=4,4+4=8,8+2=10), 4 numerical convolutions for sums. It turns out here that the 9 convolutions take less computer time than the 4 convolutions, because in each of the 9 convolutions *one of the distributions has only 10 lines; in the case of the* 4 convolutions the number of lines in each of the distributions being convoluted grows larger.

See an APPENDIX of my paper "Six Bridges to Ψ's" in ARCH 1993.1 for a description of how to perform bivariate numerical convolutions. Four-dimensional numerical convolutions are performed in an analogous fashion.[3]

Finally, Formula I above is applied to transform the resulting 4-dimensional distribution into a univariate distribution of betas.

BOOTSTRAP AND OPERATIONAL BOOTSTRAP DISTRIBUTIONS OF β

Comparing corresponding bootstrap and operational bootstrap distributions you would notice that the operational bootstrap results are available further out in the tails of the distributions. The bootstrap distributions involving random numbers tend to be accurate near the center of the distributions, and become progressively less accurate as you move into either tail of the distributions. On the other hand, the operational bootstrap distributions (which by definition do not involve random numbers) maintain accuracy even in the tails of the distributions.

The mean of the (bootstrap or) operational bootstrap distributions of β for any particular line of business need not be exactly equal to the corresponding point estimate of beta shown in TABLE 5. *The focus should be on the percentiles of the* distributions, rather than giving much weight to the (sensitive) means and variances.

The implications of the following tables are staggering, but I don't know what they are!

[3] The operational bootstrapping distributions shown in this paper were generated using COCONUT™, a proprietary command-and-convolute C-language program available from MathWare.

BOOTSTRAP DISTRIBUTIONS OF BETA

Cumulative	LINE OF BUSINESS: Fire	Allied	Farmowners	Homeowners
1.000E-06	-471.82	-0.34	-1692.75	
0.00001	-380.91	-0.28	-848.80	-6971.25
0.0001	-112.44	-0.22	-548.41	-166.11
0.001	-26.05	-0.11	-147.07	-14.78
0.01	-2.57	-0.01	-11.08	-2.14
0.025	-0.70	0.11	-3.20	-0.64
0.05	-0.22	0.22	0.02	0.13
0.1	-0.01	0.40	0.66	0.35
0.2	0.28	0.64	0.98	0.48
0.3	0.49	0.84	1.17	0.57
0.4	0.64	1.03	1.31	0.64
0.5	0.79	1.22	1.44	0.70
0.6	0.93	1.43	1.57	0.77
0.7	1.10	1.69	1.72	0.86
0.8	1.27	2.02	1.96	0.97
0.9	1.53	2.55	2.58	1.18
0.95	1.86	3.05	3.70	1.51
0.975	2.53	3.55	5.93	2.24
0.99	5.10	4.24	13.63	4.70
0.995	10.43	4.74	24.86	9.32
0.9999	99.50	6.38	551.18	214.52
0.99999	272.71	7.89	1680.90	436.02
BETA	.72*	1.04	1.35	0.65
	FI	AL	FA	HO

*The .72 was erroneously shown as .92
in the original TABLE 5.

BOOTSTRAP DISTRIBUTIONS OF BETA

Cumulative	LINE OF BUSINESS:			
	CMP	Ocean Marin	Inl.Marine	Group A&H
1.000E-06		-2226.82	-9544.60	-1034.65
0.00001		-455.63	-1022.95	-626.18
0.0001	-376.83	-141.00	-235.67	-147.29
0.001	-67.03	-21.40	-43.87	-16.80
0.01	-9.65	-2.27	-4.72	-2.85
0.025	-2.64	-1.12	-1.07	-1.63
0.05	1.65	-0.76	-0.15	-1.23
0.1	1.94	-0.52	0.20	-1.00
0.2	2.18	-0.31	0.49	-0.80
0.3	2.33	-0.17	0.67	-0.68
0.4	2.45	-0.07	0.82	-0.59
0.5	2.57	0.06	0.96	-0.50
0.6	2.70	0.14	1.10	-0.42
0.7	2.86	0.25	1.26	-0.33
0.8	3.10	0.40	1.48	-0.21
0.9	3.80	0.61	1.87	-0.06
0.95	5.07	0.86	2.41	0.19
0.975	7.63	1.25	3.65	0.57
0.99	16.22	2.43	6.72	2.00
0.995	29.80	4.12	11.44	4.15
0.9999	613.99	115.95	791.58	171.27
0.99999	2165.55	481.53	3805.60	349.22
BETA	2.38*	0.04	0.88	-0.46
	CM	**OM**	**IM**	**GA**

*The 2.38 was erroneously shown as 2.78
in the original TABLE 5.

BOOTSTRAP DISTRIBUTIONS OF BETA

Cumulative	LINE OF BUSINESS: Other A&H	Work Comp	Other Liab	Med Mal
1.000E-06	-1143.12	-13275.32	-39653.60	-3479.54
0.00001	-1087.77	-10941.85	-6630.95	-2475.78
0.0001	-220.38	-6271.86	-428.46	-1572.13
0.001	-10.23	-28.69	-124.60	-456.09
0.01	-2.16	-3.64	-13.26	-13.45
0.025	-1.34	-1.44	-3.72	-3.90
0.05	-1.00	-0.57	2.67	1.70
0.1	-0.82	-0.16	2.80	2.07
0.2	-0.70	0.12	2.90	2.34
0.3	-0.64	0.28	3.00	2.53
0.4	-0.59	0.40	3.09	2.69
0.5	-0.55	0.51	3.18	2.86
0.6	-0.51	0.61	3.28	3.04
0.7	-0.46	0.71	3.43	3.28
0.8	-0.40	0.84	3.74	3.60
0.9	-0.27	1.05	4.63	4.42
0.95	0.02	1.32	6.20	5.98
0.975	0.39	1.88	9.26	8.98
0.99	1.32	3.74	19.11	17.45
0.995	3.25	7.23	33.86	30.89
0.9999	228.96	403.90	616.23	438.04
0.99999	730.46	1233.99	3746.64	3158.14
BETA	-0.51	0.46	2.98	2.65
	OA	WC	GL	MM

BOOTSTRAP DISTRIBUTIONS OF BETA

Cumulative	LINE OF BUSINESS: Aircraft	PPA Liab	CA Liab	PPA Ph Dam
1.000E-06	-3737.18	-2551.33		-1229.42
0.00001	-3587.87	-513.02		-676.28
0.0001	-2288.34	-96.77	-2024.05	-354.26
0.001	-32.84	-26.02	-145.38	-29.62
0.01	-3.92	-3.04	-11.89	-1.98
0.025	-1.71	-1.01	-3.27	-0.54
0.05	-1.00	-0.38	1.42	-0.10
0.1	-0.60	-0.10	1.78	0.04
0.2	-0.33	0.12	2.01	0.17
0.3	-0.17	0.26	2.19	0.26
0.4	-0.03	0.37	2.32	0.33
0.5	0.09	0.48	2.46	0.40
0.6	0.23	0.58	2.60	0.47
0.7	0.38	0.69	2.77	0.55
0.8	0.55	0.83	3.03	0.64
0.9	0.83	1.07	3.76	0.80
0.95	1.15	1.42	5.08	1.00
0.975	1.83	2.08	7.86	1.38
0.99	4.74	3.89	16.79	2.67
0.995	11.64	6.46	33.85	4.56
0.9999	19633.47	177.81	945.07	1662.49
0.99999		548.25	1149.50	3666.90
BETA	0.07	0.45	2.27	0.37
	AI	PP	CA	PD

BOOTSTRAP DISTRIBUTIONS OF BETA

Cumulative	LINE OF BUSINESS:			
	CA Phy Dam	Fidelity	Surety	Burglary
1.000E-06	-17493.89	-3437.43	-2563.11	-2698.11
0.00001	-2434.91	-1898.49	-1307.04	-1698.31
0.0001	-196.34	-702.55	-508.75	-1346.27
0.001	-77.44	-349.66	-167.16	-552.16
0.01	-7.64	-15.74	-6.77	-8.53
0.025	-2.20	-4.38	-2.77	-2.71
0.05	0.28	0.38	-1.62	-1.28
0.1	0.74	1.12	-1.03	-0.76
0.2	1.08	1.63	-0.60	-0.35
0.3	1.30	1.95	-0.35	-0.10
0.4	1.48	2.23	-0.14	0.12
0.5	1.65	2.50	0.04	0.31
0.6	1.83	2.77	0.22	0.54
0.7	2.04	3.11	0.42	0.79
0.8	2.32	3.59	0.66	1.11
0.9	2.90	4.65	1.05	1.73
0.95	3.83	6.94	1.47	2.57
0.975	6.08	11.50	2.25	5.16
0.99	12.86	23.91	5.06	12.16
0.995	20.88	38.09	8.70	21.47
0.9999	596.51	971.24	199.05	3527.99
0.99999	603.59	2537.00	1906.36	7333.97
BETA	1.52	2.32	0.04	0.33
	CD	FY	SU	BU

BOOTSTRAP DISTRIBUTIONS OF BETA

Cumulative	LINE OF BUSINESS: Boiler & M	Reinsurance	Other LInes
1.000E-06	-106879.82	-1470.54	-3248.17
0.00001	-1338.93	-1362.33	-3038.72
0.0001	-79.40	-366.25	-619.06
0.001	-35.47	-60.01	-55.93
0.01	-7.78	-6.23	-7.62
0.025	-2.62	-1.64	-4.35
0.05	-1.33	0.71	-3.21
0.1	-0.85	1.10	-2.58
0.2	-0.22	1.40	-2.22
0.3	0.24	1.59	-2.03
0.4	0.60	1.75	-1.89
0.5	0.91	1.89	-1.75
0.6	1.21	2.04	-1.63
0.7	1.75	2.20	-1.50
0.8	2.16	2.41	-1.34
0.9	2.67	2.81	-1.06
0.95	3.53	3.52	-0.47
0.975	5.40	4.87	0.83
0.99	11.33	8.84	3.98
0.995	18.32	14.28	12.12
0.9999	170.70	256.27	184.13
0.99999	402.92	1998.90	7819.25
BETA	0.87	1.74	-1.62
	BM	RE	OL