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Comparing Needs for Initial Surplus in Collective Risk Models

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Abstract. The initial risk reserves in collective risk models vary according to the underlying claim distribution, and a suitable level of "ruin" probability. A thorough analysis of the needed initial surplus for various ruin probability levels is provided in eight meaningful examples. The claim distributions are drawn from various fields of insurance, including property damage and liability insurance.

Keywords: Collective risk models, infinite time ruin function, claim distribution, numerical algorithm.

1. Introduction

In general, the claim distribution for a portfolio of risks is not known as the insurance line develops. However, the actuary may have a set of claim distributions which he/she feels may be appropriate. If one considers the future arrival of claims, what amount of initial surplus is needed to hold the probability of "ruin" for this new insurance portfolio to a suitably small number?

In recent years, a number of papers have presented methods for approximating $\psi(u)$, the infinite time ruin function. Goovaerts and DeVylder (1984) develop a recursive algorithm to obtain upper and lower bounds on $\psi(u)$, and hence on the error in the approximation. They consider a Pareto claim distribution, and present a table of values for upper and lower bounds for $\psi(u)$ for three values of u. Panjer (1986) presents a simple recursive method for calculating $\psi(u)$ values. He also derives expressions for bounds (upper and lower) on $\psi(u)$, and hence on the error in the approximated values. Panjer's method is especially appropriate when the claim distribution is discrete. Gerber and Dufresne (1989) investigate three methods to compute $\psi(u)$ values. Their first method is essentially due to Goovaerts and DeVylder (1984). The second method assumes that the claim distribution is a combination of exponential or translated exponential distributions. The third method relies on the fact that $\psi(u)$ is related to the stationary distribution of an appropriate process, and that that process can be simulated in an efficient manner. Ramsay (1992a) develops an improved version of Goovaerts and DeVylder's (1984) stable recursive algorithm for approximating $\psi(u)$. Ramsay (1992b) presents an algorithm for approximating $\psi(u)$. This approximation uses the first four sample moments of the claim distribution. It is illustrated in fourteen tables, and compares favorably with previously developed approximations to $\psi(u)$.

We will use the Gerber-Dufresne description of the Goovaerts-DeVylder method to obtain our initial surplus values in eight examples. Some of the claim distributions will be drawn from the monograph Hogg-Klugman (1984). A summary table will provide lower and upper bounds on $\psi(u)$ obtained by their method. As a further check on our interpretation of their algorithm, we applied it to the claim distribution used by Gerber-Dufresne (1989) on pages 76-77, and obtained the lower and upper bounds on $\psi(u)$ given in Table 1, loc. cit.

A key idea in applying the first Gerber-Dufresne (Goovaerts-DeVylder) method is the ability to obtain an $H(x)$ function from a claim distribution. This can be difficult. The second method explained by Gerber-Dufresne is not applicable in Examples 3, 4, 6, and 7, since it assumes that the claim distribution is a combination of exponential or translated exponential distributions. It is reasonable to say that to apply Panjer's method, or the third method of Gerber-Dufresne, would be an onerous task in any one of Examples 3, 4, 6, and 7. We could not apply Ramsay's (1992b) algorithm in our Example 8 where $Var(X) = +\infty$. This was one reason why we used the Gerber-Dufresne version of the Goovaerts-DeVylder method. Just as Ramsay (1992a) uses Richardson's extrapolation method, the present authors also used that algorithm, but in a different manner. This will be discussed in Section 3.

We will assume that $\{X_i\}$ is a sequence of independent, identically distributed random variables with a common distribution function $P(x)$. The random variable X_i represents the amount of the ith claim. We assume that $P(0) = 0$, i.e. all the claims are positive, and that $E(X_i) = p_1 < \infty$.

The claim numbers process $\{N(t), t \geq 0\}$ is assumed to be independent of the $\{X_i\}$, and to be a Poisson stochastic process, with $E\{N(t)\} = t$, $t \geq 0$. Thus, we are using operational time (see, e.g. page 36 of Beekman (1974)), and a classical compound Poisson risk model is the model for aggregate claims. It is assumed that premiums are received continuously at a constant rate $c > p_1$. The initial surplus is denoted by u. Then, the ruin function is denoted and defined by

$$(1.1) \qquad \psi(u) = P\left\{max_{0\leq t<\infty}\left[\sum_{i=1}^{N(t)} X_i - ct\right] > u\right\}.$$

Let

$$(1.2) \qquad Q(u) = \begin{cases} 0, & u < 0 \\ 1 - P(u), & u \geq 0 \end{cases}.$$

Now, $c = (1+\theta)p_1$ for some $\theta > 0$. Let us use the notation

$$(1.3) \qquad K(u) = \int_u^\infty Q(v)dv.$$

Since $p_1 < \infty$, and $p_1 = \int_0^\infty [1 - P(v)]dv$, we know that

$$(1.4) \qquad K(u) = \int_u^\infty [1 - P(v)]dv < \infty, u \geq 0.$$

Thus, the integral equation for $\psi(u)$ can be expressed as

$$p_1(1+\theta)\psi(u) = K(u) + \int_0^u \psi(v)Q(u-v)dv. \tag{1.5}$$

We will now use the Gerber-Dufresne (1989) description of the Goovaerts-DeVylder (1984) method to explain how one can obtain lower and upper bounds for $\psi(u)$.

A key ingredient in the method is the function

$$H(x) = \frac{1}{p_1}\int_0^x [1-P(y)]\, dy, x > 0 \tag{1.6}$$

where $P(y)$ is the claim distribution, and p_1 is its expected value.

One now uses equations (13) and (14), page 74 of Gerber-Dufresne (1989):

$$h_k^l = H(k+1) - H(k),\ k = 0,1,2,\ldots \tag{1.7}$$

$$h_k^u = H(k) - H(k-1),\ k = 1,2,3,\ldots \tag{1.8}$$

A parameter $q = (1+\theta)^{-1}$ enters into the recursive formulas:

$$f_0^l = \frac{1-q}{1-qh_0^l} \tag{1.9}$$

$$f_i^l = \frac{q}{1-qh_0^l}\sum_{k=1}^{i} h_k^l f_{i-k}^l,\ i = 1,2,\ldots \tag{1.10}$$

$$f_0^u = 1-q \tag{1.11}$$

$$f_i^u = q\sum_{k=1}^{i} h_k^u f_{i-k}^u,\ i=1,2,\ldots \tag{1.12}$$

These are formulas (18), (19), (20), and (21) on page 74 of Gerber-Dufresne (1989). With these functions, $\psi(u)$ is bounded:

$$1-\sum_{i=0}^{u-1} f_i^l \le \psi(u) \le 1-\sum_{i=0}^{u} f_i^u, u=0,1,\ldots \tag{1.13}$$

This is formula (17), loc. cit.

2. Examples

In this section, we will consider eight claim distributions. For simplicity, we will use $\theta = 0.3$ in each example. We have chosen versions of the distributions so that $p_1 = 1$ in each case. Thus, money is being measured in average claim size units. We will determine values u_1, u_2, and u_3 such that $\psi(u_1) = 0.1$, $\psi(u_2) = 0.05$, and $\psi(u_3) = 0.01$.

Example 1.

$$P(x) = \begin{cases} 0, & x<0 \\ 1-e^{-x}, & x \ge 0. \end{cases} \tag{2.1}$$

This produces

$$Q(u) = \begin{cases} 0, & u < 0 \\ e^{-u}, & u \geq 0 \end{cases} \tag{2.2}$$

and

$$K(u) = \int_u^\infty Q(v)dv = e^{-u}. \tag{2.3}$$

Equation (1.5) now becomes

$$1.3\psi(u) = e^{-u} + \int_0^u \psi(v)e^{-(u-v)}dv. \tag{2.4}$$

This claim distribution has been used in many references, and it is known (see page 45 of Beekman (1974)) that $\psi(u) = \frac{1}{1.3}e^{-\frac{.3}{1.3}u}, u \geq 0$. The reader can check easily that this function does satisfy equation (1.5).

From that functional form, we obtain $u_1 \doteq 8.8410$, $u_2 \doteq 11.8446$, $u_3 \doteq 18.8188$. Our approximate solution of the integral equation gave values of $u_1 \doteq 8.8410$, $u_2 \doteq 11.8446$, $u_3 \doteq 18.8189$. The function $H(x)$ from (1.6) is $H(x) = 1 - e^{-x}, x \geq 0$.

<u>Example 2.</u> Swedish non-industry fire insurance, 1948-51 (Cramér (1955),

pp. 43-45).

$$(2.5) \qquad P'(y) = \begin{cases} Ae^{-\alpha y} + B(y+b)^{-\beta}, & 0 < y < 500 \\ 0, & y > 500 \end{cases}$$

where $A = 4.897954$, $B = 4.503$, $b = 6$, $\alpha = 5.514588$, $\beta = 2.75$.

For this distribution, $p_1 = 1$, and

$$(2.6) \quad P(y) = \frac{A}{\alpha}(1 - e^{-\alpha y}) + \frac{B}{\beta - 1}(b^{-\beta+1} - (y+b)^{-\beta+1}), 0 < y < 500.$$

Table VII, p. 45, loc. cit., provides the following values of $\psi(u)$:

u	20	40	60	80	100
$\psi(u)$	0.5039	0.3985	0.3280	0.2757	0.2346

Our approximate solution of the integral equation gave these values:

u	20	40	60	80	100
$\psi(u)$	0.5039	0.3985	0.3280	0.2756	0.2346

The ten, five, and one percent u values for $\psi(u)$ are $u_1 = 219.5718$, $u_2 = 320.4490$, and $u_3 = 536.4131$.

The function $H(x)$ is

$$\begin{aligned} H(x) &= x - \frac{A}{\alpha}x + \frac{A}{\alpha^2}(1 - e^{-\alpha x}) - \frac{B}{\beta - 1}b^{-\beta+1}x + \\ (2.7) \qquad & \frac{B}{(\beta-1)(\beta-2)}\{b^{-\beta+2} - (x+b)^{-\beta+2}\},\ x > 0, \end{aligned}$$

where $A = 4.897954$, $B = 4.503$, $b = 6$, $\alpha = 5.514588$, and $\beta = 2.75$.

Example 3. Modified lognormal distribution.

Assume that X has the distribution function

$$(2.8) \qquad P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(kx)-\mu}{\sigma}} e^{\frac{-y^2}{2}} dy,\ x > 0,\ k > 0.$$

Thus, $X = W/k$ where W has a lognormal distribution

$$(2.9) \qquad P(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln w-\mu}{\sigma}} e^{\frac{-y^2}{2}} dy,\ w > 0.$$

See Hogg-Klugman (1984), pages 45, 109, 229. This distribution was used to model automobile bodily injury claims on page 162 (loc. cit.). We will choose $\mu = 0.5$, and $\sigma^2 = 1$. Then $E(W) = e$, and with $k = e$, $E(X) = 1$. The function $H(x)$ was evaluated by numerically approximating the integral of (1.6). Details are in Section 3. The ten, five, and one percent u values for $\psi(u)$ are $u_1 = 12.4516$, $u_2 = 17.4628$, and $u_3 = 29.9741$.

Example 4. Modified lognormal distribution.

Here, we choose $\mu = 1$, and $\sigma^2 = 2$ in the preceding example. Thus,

$E(W) = e^2$, and with $k = e^2$, $E(X) = 1$. Now $u_1 = 33.6686$, $u_2 = 51.5323$, and $u_3 = 106.5362$.

Example 5. Weibull distribution.

$$P(x) = 1 - \exp(-cx^{\tau}),\ x > 0 \tag{2.10}$$

for $c > 0$, $\tau > 0$.

Hogg and Klugman state (p. 24, loc. cit.) that "this distribution provides a good model for size of claims in casualty insurance (malpractice, windstorms, etc.), particularly when $0 < \tau < 1$." Also note pages 109, 218, 231, and 232, loc. cit., in particular

$$E(X) = \frac{\Gamma(1 + \frac{1}{\tau})}{c^{1/\tau}}. \tag{2.11}$$

We will let $c = \sqrt{2}$, and $\tau = 0.5$. The $H(x)$ function is

$$H(x) = \left[1 - (1 + \sqrt{2x})e^{-\sqrt{2x}}\right], x > 0. \tag{2.12}$$

The ten, five, and one percent u values for $\psi(u)$ are $u_1 = 27.8867$, $u_2 = 38.6634$, and $u_3 = 64.0883$.

Example 6. Gamma Distribution.

$$P(x) = \int_0^{\lambda x} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy, \; x > 0 \tag{2.13}$$

for $\alpha > 0$, $\lambda > 0$.

Hogg and Klugman state (p. 25, loc. cit.) that "the gamma distribution is a good model for many situations involving the size of loss in casualty insurance." Page 227 of their monograph provides three sample gamma distributions. For our sixth example, we will use $\alpha = 7.5, \lambda = 7.5$. Here, $Var(X) = 1/(7.5)$.

The $H(x)$ function was evaluated by numerically approximating the integral of (1.6). Details are in Section 3.

The appropriate u values are $u_1 = 4.8547$, $u_2 = 6.4468$, and $u_3 = 10.1438$.

Example 7. Gamma distribution.

Here we choose the more dangerous loss distribution with $\alpha = 0.3$, $\lambda = 0.3$, and $Var(X) = 10/3$.

Our approximate solution of the integral equation produces $u_1 = 19.5368$, $u_2 = 26.3572$, $u_3 = 42.1948$. Note that much larger values of u were required than in Example 6.

Example 8. Pareto distribution.

$$(2.14) \qquad P(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}, \quad x > 0$$

for $\alpha > 0$, $\lambda > 0$.

This is a good model for dangerous claim situations, and as pointed out on p. 222, Hogg-Klugman (1984), the nth moment only exists for $\alpha > n$. We will choose $\alpha = 1.5$ and $\lambda = 0.5$. Hence $E(X^2) = +\infty$.

The $H(x)$ function is

$$(2.15) \qquad H(x) = \left[1 - (1 + 2x)^{-0.5}\right], x > 0.$$

This produces $u_1 = 531.7017$, $u_2 = 2,198.3100$, and $u_3 = 55,607.0454$. Thus, large initial risk reserves are needed to hold the ruin probabilities down to values of 0.10, 0.05, and 0.01.

The following table is a summary of the u_1, u_2, and u_3 values for the eight examples which were determined for $\psi(u)$ equalling 0.10, 0.05, and 0.01, respectively. It also provides the lower and upper bounds for the

TABLE

EXAMPLES' VALUES OF u, INITIAL SURPLUS,

RANKED BY DEMAND ON SURPLUS

Example No.	Example $P(x)$	Desired $\psi(u)$	u Value	Lower Bound on $\psi(u)$	Upper Bound on $\psi(u)$	cpu time minutes
6	Gamma $\alpha = 7.5$ $\lambda = 7.5$	0.10	4.8547	0.09927	0.10072	9.40
		0.05	6.4468	0.04935	0.05065	
		0.01	10.1438	0.00967	0.01033	
1	$1 - e^{-x}$	0.10	8.8410	0.09932	0.10070	0.45
		0.05	11.8446	0.04939	0.05061	
		0.01	18.8188	0.00970	0.01031	
3	Modified lognormal distr. 1	0.10	12.4516	0.09942	0.10058	1.62
		0.05	17.4628	0.04950	0.05050	
		0.01	29.9741	0.00978	0.01022	
7	Gamma $\alpha = 0.3$ $\lambda = 0.3$	0.10	19.5368	0.09935	0.10065	13.75
		0.05	26.3572	0.04942	0.05058	
		0.01	42.1948	0.00971	0.01029	
5	Weibull $c = 0.024$ $\tau = 0.5$	0.10	27.8867	0.09942	0.10059	0.58
		0.05	38.6634	0.04949	0.05051	
		0.01	64.0883	0.00915	0.01025	
4	Modified lognormal distr. 2	0.10	33.6686	0.09954	0.10047	1.92
		0.05	51.5323	0.04963	0.05037	
		0.01	106.5362	0.00987	0.01013	
2	Swedish fire insurance See 2.5	0.10	219.5718	0.09956	0.10047	0.92
		0.05	320.4490	0.04961	0.05042	
		0.01	536.4131	0.00979	0.01023	
8	Pareto $\alpha = 1.5$ $\lambda = 0.5$	0.10	531.7017	0.09987	0.10016	1.11
		0.05	2,198.3100	0.04993	0.05009	
		0.01	55,607.0454	0.00999	0.01002	

$\psi(u)$ function, and the computer times expended on a Digital Equipment Corporation VAX 6620.

3. Implementation of the Algorithm

The problems of this paper are not just concerned with the evaluation of $\psi(u)$, but with a solution of $\psi(u) = r$, where r is 0.10, 0.05, or 0.01. Therefore an iterative method is required to approximate u.

For our iteration method we used $10p_1$ as our first estimate for the solution of $\psi(u) = 0.10$. For the first estimate of $\psi(u) = 0.05$ we used the solution of $\psi(u) = 0.10$; and for our first estimate for $\psi(u) = 0.01$, we used the solution of $\psi(u) = 0.05$. Letting the first estimate be $u^{(1)}$, our second estimate was $u^{(2)} = 1.2u^{(1)}$. The secant method (see, e.g., page 78 Conte and de Boor (1980)) was used to obtain our third estimate

$$u^{(3)} = u^{(2)} - \frac{(\psi(u^{(2)}) - r)(u^{(2)} - u^{(1)})}{\psi(u^{(2)}) - \psi(u^{(1)})} \tag{3.1}$$

For subsequent approximations, $u^{(i)}$, Müller's method (see, e.g., page 120, loc.cit.) was used which uses the previous three values of $u^{(i-1)}, u^{(i-2)}$, and $u^{(i-3)}$. These methods seemed to give good results with approximately seven iterations required to reach a stopping condition of

$$\left|\frac{\psi(u^{(i)}) - r}{r}\right| \leq 10^{-7} \tag{3.2}$$

For efficient and accurate evaluation of $\psi(u)$ and for u being nonintegral valued, we followed the subdivision method as given on page 55 of Goovaerts-DeVylder (1984). The interval $[0, u]$ is divided into 2^n subintervals of $\left[0, \frac{u}{2^n}\right]$, $\left[\frac{u}{2^n}, \frac{2u}{2^n}\right]$, ..., $\left[\frac{(k-1)u}{2^n}, \frac{ku}{2^n}\right]$, ..., $\left[\frac{(2^n-1)u}{2^n}, u\right]$ so that $H(k)$ in (1.7) and (1.8) becomes $H(\frac{ku}{2^n})$. While various values of n were used in initial experiments, $n = 12$ was used in all final computations as reported in this paper.

Since the computational times were large for several of the examples, Richardson's extrapolation method (see e.g. page 157 of Burden and Faires (1989)) was applied to the computation of $\psi(u)$ for a given u as was done by Ramsay (1992a) in his computation of lower and upper bounds for $\psi(u)$. However, in our computations we applied the extrapolation method to the average of the lower and upper bound values which we used for the approximation of $\psi(u)$.

The computer programming language of Fortran was used along with subroutines of DNORDF, DGAMI, DQDAGI, and DQDAG from the International Mathematics and Statistics Library (IMSL, (1991)). All computing was done in double precision arithmetic on a Digital Equipment Corporation VAX 6000 model 620.

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