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RELATIVE IMPORTANCE OF RISK SOURCES IN INSURANCE SYSTEMS

by

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ABSTRACT

Relative Importance of Risk Sources in Insurance Systems

Actuaries and other managers of risk identify factors in modeling insurance risks because (i) they feel that these factors may cause the outcome of a risk or (ii) that the factors can be managed, thus allowing analysts a degree of control over the system risk. The purpose of this paper is to propose a framework for quantifying the relative importance of a source of risk. The intent is that, with a quantitative measure of relative importance, risk managers will be able to sharpen their intuition about the relative importance of risks and be better custodians of financial security systems.

This paper proposes a measure of relative importance that has its roots in both the statistics and economics literatures. The measure is intuitively appealing when assessing the effectiveness of basic risk management techniques including risk exchange, pooling and financial risk management. The risk measure is also shown to be useful in multivariate situations where several factors affect a risk simultaneously. The paper illustrates this usefulness by considering a pool of policies that is subject to mortality, the risk of a disaster that is common to all policies and to a common investment environment.

Outline

RELATIVE IMPORTANCE OF RISK SOURCES IN INSURANCE SYSTEMS

Section 1 Introduction

Insurance systems are collections of agreements among parties in which one party agrees to accept another's risk. Initially, we may think of "risk" as the financial consequences of uncertain events. Unlike gambling systems, insurance systems are designed to reduce the total system risk. Insurance systems are organized by private and/or public entities that have agreements to reimburse individuals, corporations or other groups in the event of an unforeseen event (loss).

Models of insurance systems can be used for a variety of purposes. Two important purposes are (i) establishing premiums to price agreements and (ii) establishing liabilities, or reserves, to ensure an orderly matching of annual revenues to cost. Another purpose, that has important public policy implications, is understanding the event of an insurance organization becoming unable to meet its financial obligations, that is, becoming insolvent.

Typically, actuaries managing insurance systems can identify several factors that potentially affect the realization of the risk. To illustrate, basic life insurance modeling decomposes risk into a mortality and an investments component. Basic property and casualty insurance decomposes risk into a frequency and severity component. As a more detailed example, consider projections of obligations by the Social Security Administration. Here, the basic components consist of economic and demographic factors. The economic factors include investment returns, inflation, unemployment and wage rate increases. The demographic components include mortality, fertility, marriage/re-marriage rates, disability, immigration and retirement rates.

Actuaries and other managers of risk identify factors in modeling insurance risks because (i) they feel that these factors may in fact cause the outcome of a risk or (ii) that the factors can be managed, thus allowing analysts a degree of control over the system risk. Actuaries manage insurance risks through (i) classical pooling techniques, (ii) risk transference techniques including reinsurance and (iii) financial risk management techniques such as hedging. These broad categories, and a plethora of special cases and variations, of risk management techniques exist to enable actuaries and other financial analysts to cope with the many sources of risk that exist in the world today. There are many rules of thumb and guidelines for deciding upon the appropriate risk management tool.

In models of insurance risks, relative importance of risk sources have traditionally been assessed via sensitivity analysis. That is, systems of risk are evaluated with a small change in parameter values and the resulting system behavior is observed. To illustrate, projections of obligations by the Social Security Administration are provided under "low, intermediate and high" cost assumptions of the future behavior of the economy. This enables users of the projections to get a sense of the relative impact of the economic assumptions. Expert panels reviewing Social Security projections have acknowledged that a method of "deciding which assumptions are the most important is somewhat subjective." (Social Security Technical Panel Report, 1990, Chapter 3). However, for managing the Social Security system, "the panel concluded that three economic variables ... had the greatest potential impact on the actuarial balance of the OASDI program and therefore deserved the most attention."

Assessing the relative importance of causal factors arises in many other areas of actuarial science. Recently, Parker (1996) has studied the relative importance between investment and insurance risks. More traditionally, actuaries have sought to explain deviations of actual from expected results based on the experience of several different factors. To illustrate, see Anderson (1971) for a description of pension plan analysis of gains and losses and Saunders (1986) for a description of analysis of gains and losses for life insurance company operations.

Assessing the relative importance of causal factors is also an area of concern that arises in many other scientific disciplines. Kruskal and Majors (1989) discuss some of the many approaches used in different disciplines concerned with different structures of stochastic information. These approaches include (i) using simple statistical measures such as averages, ratios, and percentages and (ii) more complex statistical measures of significance such as p-values. Much of the work of this paper is motivated by measures used in regression analysis. This literature is described in Subsection 5.2.

Yet, even if an analyst identifies a factor as a determinant of system risk, how influential is that factor? Moreover, if the factor can be managed, how does this affect the behavior of the system's risk? As a tool for assessing relative importance in models of insurance systems, in this paper we describe an approach for decomposing risks into several sources of uncertainty. Decomposing sources of uncertainty will help us understand the relative importance of each source that in turn will allow us to design better systems for managing risk. By decomposing risks into several sources, managers will be able to quantify the contributions of each source of risk. This will aid financial security organizations in managing their risk portfolios and in designing contracts that meet demands for insurance by the public in a fiscally responsible manner.

The goal of this paper is to provide a framework for decomposing risk. Because the development will be firmly rooted in insurance applications, Section 2 introduces several risk sources of insurance contracts. Section 3 explores the variance decomposition method by focusing on these insurance risks. This section introduces a statistic, the proportion of risk attributable to a source, and illustrates its use in three principal risk management techniques: (i) pooling, (ii) risk exchange, and (iii) investment diversification.

Section 4 discusses the risk due to a common disaster in a pool of policies because of its importance as a threat to an insurer's solvency. Again, the intent is not to introduce potential remedies for this risk such as reinsurance exchanges or contract exclusions. Rather, the intent is to quantify the effect that a single, common source of risk may have on a pool of policies.

Section 5 provides background material on (i) the choice of the variance parameter as a measure of risk and on (ii) assessing relative importance of different risk sources. The former topic is based on the financial economics literature and the latter topic is based on the statistics literature. For practicing financial analysts, this section may be skipped without loss of continuity.

Section 6 extends the variance decomposition method to handle several sources of risks. Although the extensions are straightforward in principle, certain difficulties arise when interpreting proportions of risk attributable to each source. Examples from different areas of insurance practice illustrate how to resolve these difficulties.

Section 2 Models of Insurance

We describe models of insurance systems using three types of stochastic elements: (i) financial risk, (ii) timing of contingency type and (iii) cash flow amounts. We will argue that these elements are hierarchical in the sense that the timing of contingency type may depend on the financial risk element and the cash flow amount may depend on the timing of contingency type and/or the financial risk element. This hierarchical nature will be important for the risk decompositions that will be presented subsequently. We will see that the presence of a hierarchy will enable us to attribute risk to various sources in a meaningful way. We now discuss each element, beginning with the highest order in the hierarchy.

Section 2.1 Severity and Frequency Risks

The cash flow amount, denoted by C, represents the random amount of money that is a net outgo, or payment, of the financial security organization. Henceforth, we refer to the financial security organization as the "insurer" for brevity. In the event that all net inflows, or premiums, are made at contract initiation the cash flow amount may simply represent the payments made by an insurer. In this case, the distribution of C is called the claims or "severity" distribution. The simplest type of a contract is a single period property and casualty contract, for example, a flood-loss policy, where C represents the random amount of claims due to damage of a home from flooding. Of course, the amount of claims need not be equal to the amount of loss to the policyholder. Various contractual clauses, such as the existence of deductibles, coinsurance and excess of loss provisions may serve to transform the policyholder's loss distribution to the insurer's claims distribution.

To describe the frequency of the contingency, we consider a random element F that describes (i) the type of contingency and (ii) the number of contingent losses over each period. To illustrate, for a single period property and casualty contract, F may indicate whether a claim occurs. As another example, consider a whole life insurance with an accidental death provision. In this case, we may use $F = \{F_{1t}, F \}$, where F_{1t} is an indicator variable for death (due to all causes other than accidents) and F_{2t} is an indicator variable for accidental death in the tth period.

Specifically, we will consider c different contingent event types, each of which may induce payments over one of T time periods. Define F_{jt} to be a variable that gives the number of events of type j in the tth period. To define an insurance contract, we will use F_{jt} over $j=1, \ldots, c$ and $t=1, \ldots, T$. The variables $\{F_{jt}\}$ have the flexibility to represent basic building blocks for describing insurance systems, as follows.

In this paper, we represent the risk arising from an insurance contract by a vector of cash flows, **R**. We assume the $T \times 1$ vector of insurance risks **R** is a function of the timing of contingency type and the severity. That is,

$$\mathbf{R} = \mathrm{G}(\{F_{jt}\}, \{C_{jt}\}, t=1, \ldots, T, j=1, \ldots, c) \tag{2.1}$$

where G is a known function specified by the type of insurance policy. Here, C_{jt} represents the amount of cash flow for the jth type of event at the tth time point.

Section 2.2 Incorporating Financial Risks

Up to this point, our model of insurance risks has been a traditional one, see, for example, Bowers et al (1986) or Panjer and Willmot (1992). We now extend the model by including financial risks as a stochastic element. We use financial risks to represent random events based on movements of asset streams associated with contractual agreements. As emphasized by Boyle (1992), an important distinction between financial and insurable risks is that pooling as a risk reduction technique is generally ineffective for controlling financial risks, at least when compared to insurable risks. It is well-known in financial economics that diversification is a limited risk reduction technique which has led to the introduction of other techniques, such as hedging strategies and securitization.

Financial risks enter our model in two ways. First, we will allow for the fact that either cash flow amounts and/or the timing of contingency type may depend on the financial stochastic elements. So that we do not unnecessarily restrict our considerations, we use $\mathcal{F}$ to represent the information generated by the investment risk. Generally, both F_{jt} and C_{jt} may be a function of investment performance before time t. For example, C_{jt} may represent cash flows associated with a universal life policy, a type of life insurance contract where premiums and benefits depend on investment performance. Further, F_{jt} may indicator whether or not someone has withdrawn from an insurance contract. It is well-known, for insurance contracts with investment guarantees, that withdrawal rates depend on investment performance.

Second, financial performance will be used to discount the vector of cash flows, **R**, to a single present value number. We will use

$$S = \mathrm{D}(\mathbf{R}, \mathcal{F}),$$

that is, the random present value of cash flows can be expressed as a known function D of the vector of cash flows **R** and interest environment, $\mathcal{F}$. For example, $\mathcal{F}$ could represent the information contained in a constant force of interest, a force that varies deterministically through time, an *i.i.d.* series, a time series such as an autoregressive moving average process or even a multivariate version. To illustrate, recently Lai and Frees (1995) investigated a nonlinear version of an ARCH (autoregressive conditionally heteroscedastic) process as a model of interest rates. Alternatively, a time series transfer function model, such as discussed by an Institute of Actuaries' Working Party in Geoghegan el al (1992), could be explored.

These models provide mechanisms for summarizing future cash flows that result from a policy using a stochastic discounting. Because some cash flows are revenues and others are disbursements, the sum of discounted cash flows S is called *discounted surplus*. Thus, discounted surplus summarizes, at one point in time, losses or profits that the insurance organization will eventually realize, and is an important measure of risk in insurance and related industries. Other choices of D are of interest in actuarial science, for example, the choice associated with ruin probabilities. This paper focuses on the discounted surplus measure.

Section 2.3 Risk Hierarchy

Sections 3 and 5 will introduce several methods of attributing, or decomposing, risk to more than one source. Each method will turn out to be *hierarchical* in nature in that the risk attributable to a source will depend on the risk attributable to sources that are higher in the hierarchy. The hierarchy will be established in conjunction with the purpose of analysis of the risk system. To illustrate, in financial security systems we would establish the risk in congruence with risk management systems that are generally available. The idea is that if a source of risk can be eliminated, then it is useful to understand it's contribution to the overall riskiness. Providers of financial security are compensated for assuming risks and it is important to understand how much risk is being assumed.

To illustrate a risk hierarchy, in the context of insurance risks, we will generally assume: (i) that the risk of claim amounts (C) depend on claim frequency (F) and the interest risk ($\mathcal{I}$) and (ii) that the risk of F depends on $\mathcal{I}$. In principle, it is possible to calculate the risk of $\mathcal{I}$ as a function of F. However, this is difficult to interpret because it would mean that the uncertainty of the financial capital markets depends on insurable events such as mortality or flooding. Although there may be extreme "insurable" events such as a national epidemic or a major earthquake, it seems far more likely that most insurable risks have no impact on the financial markets and thus on financial risks. Further, in the event that C is deterministic, determining the risk associated with $\mathcal{I}$ given F is equivalent to determining the risk associated with $\mathcal{I}$ for known cash flows. In this case, we will establish that the risk is known from work in financial economics and may be zero, in the event of perfect immunization.

Thus, we assume that financial risks are most fundamental, followed by the timing of contingency type and finally the cash flow amount. Within each category, however, there may be several types of risks where no ordering is possible.

Section 3 Risk Decomposition Using Variances

The variance functional may be the most widely applied summary measure for quantifying the risk of a random variable. It is ubiquitous in the study of financial risks, in part due to the pioneering work of Markowitz (1959) on mean-variance tradeoffs of financial returns. The decomposition of uncertainty through analysis of variance has a central role in the study of data through regression analysis. Further, the variance functional as a measure of risk is an important special case in other, more general, frameworks for analyzing risks that will be introduced in Section 5. As we will see, the examples of this paper primarily concern measuring the risk of large pools of insurance obligations. Because our first order approximation of the distribution of these pools involve central limit theorem arguments, it is natural to focus on the variance as a measure of uncertainty. In this section we measure the risk of a random variable Y through its variance, denoted by Var Y.

Section 3.1 Basic Measures for Risk Decomposition

Even when restricting ourselves to variance measures, there are at least two different measures that can be used for risk attribution. To simplify the discussion, suppose that the risk Y is composed of two sources of risk, X and Z, such that $Y = G(X, Z)$, where G(.) is a bivariate function. Using variance as a measure of risk, we would like to measure the variability of Y that is due to X. For some of our insurance applications, we will use $Y = S$ for discounted surplus and $X = \mathit{i}$ for interest rate risk. In the context of this application, the results of this section are related to the recent work of Parker (1996).

One measure is E (Var $(Y \mid Z)$), where E is the expectations operator and Var $(Y \mid Z)$ is the variance of Y given Z. This measure can be interpreted as follows. Assume that Z takes on a known value, say z_0. Then, $G(X, z_0)$ is a random quantity due only to X and, hence, its riskiness can be summarized by Var $(G(X, z_0))$. An alternative expression for this is Var $(G(X, Z) \mid Z=z_0)$ = Var $(Y \mid Z=z_0)$. Thus, the measure E (Var $(Y \mid Z)$) can be interpreted to be the expected, or "average," riskiness of Y due to X, where the averaging is over values of Z.

A second measure is Var($E(Y \mid X)$). The regression function, $E(Y \mid X) = E(G(X, Z) \mid X)$, averages over all values of Z. Thus, the many potential values of Z in $G(X, Z)$ are replaced by the conditional average. Thus, the riskiness in the quantity $E(G(X, Z) \mid X)$ is due solely to the riskiness of X. The measure Var($E(Y \mid X)$) summarizes the variability of this random quantity.

Both measures are dominated by the overall variability, Var Y, a desirable criterion for a measure of riskiness due to a source. This is due to a basic relationship in mathematical statistics,

$$\text{Var } Y = \text{Var}(\, E(Y \mid X)\,) + E(\, \text{Var}(Y \mid X)\,). \tag{3.1}$$

Thus, Var $Y \geq$ Var(E(Y | X)). By a similar line of argument, Var $Y \geq$ E(Var(Y | Z)).

Both measures reduce to Var X in the case of linear independent risks. That is, if X, Z are independent and Y = G(X, Z) = $X + Z$, then it is easy to check that E(Var (Y | Z)) = Var X = Var(E(Y | X)) .

In this paper, we use Var(E(Y | X)) as our measure of the risk of Y that is attributable to X. This choice is motivated by three reasons. First, in many applications it is not possible to identify a second variable Z in order to compute Var (Y | Z). Computing E (Y | X) will turn out to be a more natural device. Second, the choice of Var(E(Y | X)) will be motivated by appealing to arguments using utility theory that will be discussed in Section 5. Third, this choice is also motivated by a purpose of risk attribution, decomposing risks into components that can be controlled through risk management techniques. This motivation is developed in Subsection 3.2. A rescaled version of our basic measure is

$$\rho^2_{YX} = \frac{\text{Var (E }(Y|X)\text{)}}{\text{Var } Y} \tag{3.2}$$

interpreted to be the proportion of Y's risk that is attributable to X. An advantage of this rescaled measure is that it does not depend on the units that Y is measured in. Thus, ρ^2_{YX} is said to be *dimensionless*.

Section 3.2 Measuring the Effectiveness of Risk Management Techniques

This subsection develops the basic risk attribution measure, defined in equation (3.2), by applying it to each of our three basic risk management techniques: risk exchange, pooling, and financial risk management. Through these applications, we will be able to quantify the effectiveness of the risk management techniques using our risk attribution measure. We consider each technique in turn.

Example 3.1 Exchanging Risks in a Single Period Property and Casualty Risk

Suppose that C represents the amount of a claim that is payable in one year. Let X represent the random present value of \$1 payable in one year. We assume, because the obligation cannot be determined in advance, that it will be funded through a money market instrument and hence is random. Further, we also assume that X is independent of C. Thus, the random present value of the obligation is $Y = X\,C$.

As described in the risk hierarchy section, our interest is in measuring the risk of the random present value of claims, Y, that is due to the interest rate risk, X. This is because management may elect to reduce the riskiness of claims amount C through the use of tools such as deductibles or stop-loss reinsurance. In fact, the entire claims distribution may be exchanged for a nonrandom payment. For example, using the net premium principle, the value of the obligation is E(Y | X) = X E C. Note that the agreement of the exchange can be made without regard to the effects of the interest rate risk. Thus, we may interpret E (Y | X) to be the value of an obligation that can be achieved using risk reduction techniques.

It is instructive to compare the two measures of risk attribution introduced in Section 3.1 in this example. Straight-forward calculations show that

$$\mathrm{Var}(\, \mathrm{E}(Y \mid X)\,) \;=\; \mathrm{Var}(\, X\,)\, (\mathrm{E}\, C)^2$$

and

$$\mathrm{E}(\, \mathrm{Var}(Y \mid C)\,) \;=\; \mathrm{Var}(\, X\,)\, \mathrm{E}\, C^2 \;=\; \mathrm{Var}(\, X\,)\, (\mathrm{Var}(\, C) \;+\; (\mathrm{E}\, C)^2)\, .$$

These calculations establish $\mathrm{E}(\, \mathrm{Var}(Y \mid C)\,) \geq \mathrm{Var}(\, \mathrm{E}(Y \mid X)\,)$, with equality if and only if $(\mathrm{Var}\, X)(\mathrm{Var}\, C) = 0$. Thus, in this example, the measure $\mathrm{Var}(\, \mathrm{E}(Y \mid X)\,)$ is not affected by the variability of C, although $\mathrm{E}(\, \mathrm{Var}(Y \mid C)\,)$ is affected.

How effective is it to replace the original risk Y by the new obligation $\mathrm{E}(Y|X) = X\, \mathrm{E}\, C$? Straightforward calculations show that

$$\rho^2_{YX} = \left(1 + \frac{1 + \mathrm{CV}(X)^{-2}}{1 + \mathrm{CV}(C)^{-2}}\right)^{-1},$$

where $\mathrm{CV}(X) = (\mathrm{Var}\, X)^{(1/2)}/\mathrm{E}X$ is the coefficient of variation of X, and similarly for $\mathrm{CV}(C)$. Thus, for example, we have that ρ^2_{YX} tends to be zero as $\mathrm{CV}(X)$ tends to be zero, and that ρ^2_{YX} tends to be one as $\mathrm{CV}(C)$ tends to be zero. We may interpret this to mean that the proportion of risk attributable to X is small when the coefficient of variation X is small, and is large when the coefficient of variation of C is small. This interpretation is not surprising. However, we also have that ρ^2_{YX} tends to $(\mathrm{CV}(C)^2+1)/(2\mathrm{CV}(C)^2+1)$ as $\mathrm{CV}(X)$ tends to infinity. This ratio is bounded by ½ and 1, indicating that a substantial proportional of risk is not explained by X even when the coefficient of variation of X becomes infinitely large. This is not a deficiency in the definition of ρ^2_{YX}; it is due to the multiplicative nature of the relation $Y = X\, C$. ∎

Thus, the measure $\mathrm{Var}(\, \mathrm{E}(Y \mid X)\,)$ can be motivated based on a conditional exchange of risks, that is, exchanging Y for $\mathrm{E}(Y \mid X)$ where, of course, the value of $\mathrm{E}(Y \mid X)$ depends upon the occurrence of X. This exchange is based on the net premium principle (see, for example, Bowers et al, 1986, Chapter 6). More generally, one could consider the exchange of Y for $\mathrm{H}(Y \mid X)$, where $\mathrm{H}(Y \mid X)$ is a general "premium principle." Premium principles H(.) are methods for determining equitable, nonstochastic premiums H as a function of the distribution of the risk Y. Thus, the premium $\mathrm{H}(Y \mid X)$ is, conditional on X, a nonstochastic quantity that is readily exchangeable for Y. See, for example, Gerber, 1979, for an introduction to premium principles. Introducing more general premium principles implicitly means introducing a bias. Of course, these biases exist in practice; they may be due to risk premia, expense loadings, or market inefficiencies. The approach of this paper is to quantify the variability of risks; it is anticipated that risk managers will analyze these tradeoffs using their own perspectives on variability and bias. It is possible to handle the biases automatically by using a mean square error in lieu of a variance operation. However, we prefer to split the bias from the variance calculation, thus forcing risk managers to explicitly account for the two types of deviation from an anticipated result.

Another important risk management tool is the pooling of risks. Insurance organizations spread risk through a pooling of risks. The basic premise that risks can be reduced through pooling, or sharing, is predicated on the fact that the uncertain events upon which risks are based are not perfectly related. Indeed, the smaller the relationship, the more effective is the pooling mechanism. The pooling mechanism is much less effective when a common disaster strikes many of an organization's clients, such as an earthquake in property and casualty insurance, an AIDS epidemic in health insurance or a series of junk bond defaults in life and pension insurance (see Section 4).

To assess the effects of pooling, let $Y_i = G_i(X)$, $i=1, \ldots, n$, be n individual risks. Here, X is a source of risk that is common to the individual risks and $Y = \Sigma_{i=1}^{n} Y_i$ is the pool of risks. We assume, conditional on X, that the risks are independent. By the conditional independence, we have $\mathrm{Var}\ (Y \mid X) = \Sigma_{i=1}^{n} \mathrm{Var}\ (Y_i \mid X)$. This and equation (3.1) yield

$$\mathrm{Var}(\ \mathrm{E}(Y \mid X)\ ) = \mathrm{Var}\ Y - \mathrm{E}(\ \mathrm{Var}(Y \mid X)\ ) = \mathrm{Var}\ Y - \Sigma_{i=1}^{n} \mathrm{E}(\ \mathrm{Var}(Y_i \mid X)\ ). \tag{3.4}$$

One application of equation (3.4) is to see, for pools with a large number of risks, that the pool risk can be largely attributable to the common effect, X. That is, suppose that $\mathrm{E}(\ \mathrm{Var}(Y_i \mid X)\ ) < C$ for some constant C, for each i. Then, from equation (3.2), we have

$$\left| \mathrm{Var}(\ \mathrm{E}(\frac{Y}{n} \mid X)\ ) - \mathrm{Var}\ \frac{Y}{n} \right| < \frac{C}{n}.$$

That is, the difference between the overall riskiness of the pool average, Y/n, and the riskiness attributable to X, is less than C/n. This difference becomes small as the number of individual risks, n, becomes large.

Equation (3.4) can also be used to calculate the exact proportion of riskiness that is attributable to X. Consider the following illustration, based on Example 4.1 of Frees (1990).

Example 3.2 Block of Whole Life Policies

Consider a block of level premium whole life business, categorized into three groups of size N so that the total block size is $n = 3N$. Assume, for each category, that ages at issue are $x = 30, 30, 40$ and durations are $k = 5, 10, 5$, respectively. Funds are invested in a money market instrument whose returns ($\mathit{\bar{I}}$) follow an $MA(1)$ process with parameters $\delta_1 = 0.04376$ and $\alpha_1 = 0.08043$. The mortality decrements are the 1979-1981 U.S. Male Life Tables. Use S to denote the sum of losses for this block of business. (See Bowers, Gerber, Hickman, Jones and Nesbitt (1986) for a description of a policy's loss.)

Under the above assumptions, from Frees (1990) we have that

$$\mathrm{Var}(\ \mathrm{E}(S \mid \mathit{\bar{I}})\ ) = N^2\ (0.00861)$$

and

$\text{Var } S = N^2\ (0.00861) + N(0.13380).$

The ratio, $\rho^2_{S\mathcal{I}} = \text{Var}(\ E(S \mid \mathcal{I})\ )\ /\ \text{Var } S$, is the proportion of the pool's variability that is attributable to the interest rate risk. Figure 3.1 plots the relationship between this ratio and the group size N. Here, we see that the ratio is small for small block sizes but increases quickly as the size grows. The limiting value of the ratio is one.

Thus, this example substantiates, and sharpens, the common actuarial wisdom that the interest rate risk dominates the mortality risk. The example uses relatively homogeneous groups of policies; however, it is the assumption of independence of mortality within the pool that makes the averaging effective. Section 6 will provide an alternative viewpoint when the independence assumption is perturbed.

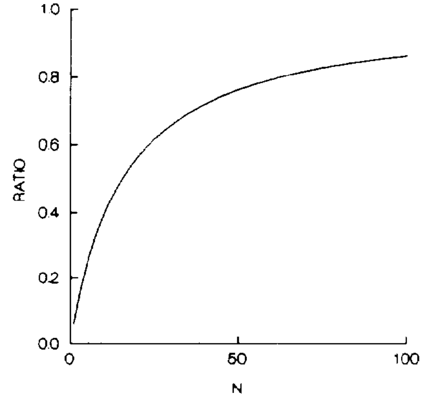

Figure 3.1 Plot of the proportion of the pool's variability attributable to interest rate risk for Example 3.2.

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The third type of risk management technique considered in this paper is financial risk management. As a first example of our risk assessment tools, we show that financial diversification is limited as a risk management tool, due to a common investment environment.

Example 3.3 Capital Asset Pricing Model

We consider here the capital asset pricing model (CAPM) of a security's return,

$$Y_i = R + \beta_i\ (M\text{-}R) + \epsilon_i. \tag{3.5}$$

Here, Y_i is the ith security's return, R is the risk-free rate of return, M is the market return, ϵ_i is the so-called "idiosyncratic" risk associated with the ith security and β_i is the slope associated with the ith security. The risk-free rate and slope parameters are non-stochastic quantities. The idiosyncratic risks are mutually independent, and independent of the market return.

To see the effectiveness of diversification, assume that a portfolio is created with w_i invested in the ith security, for $i = 1, \ldots, n$. Then, using equation (3.5), the portfolio's return can be expressed as

$$Y = \sum_{i=1}^{n} w_i Y_i = R(\sum_{i=1}^{n} w_i) + \beta_w (M-R) + \sum_{i=1}^{n} w_i e_i \tag{3.6}$$

where $\beta_w = \Sigma_{i=1}^{n} w_i \beta_i$ is the weighted average slope. The portfolio's risk is summarized by

$$\text{Var } Y = \beta_w^2 \text{ Var } M + \sum_{i=1}^{n} w_i^2 \text{ Var } e_i \tag{3.7}$$

It is not hard to construct weak conditions so that the portfolio's riskiness is less than the riskiness of an individual security, that is, Var $Y <$ Var $Y_i = \beta_i^2$ Var M + Var ϵ_i.

What effect does the market return have on the portfolio? Using equation (3.6), straightforward calculations show that $E(Y|M) = R(\Sigma_{i=1}^{n} w_i) + \beta_w(M\text{-}R)$. Thus, we have

$$\rho_{YM}^2 = \frac{\beta_w^2 \text{ Var } M}{\beta_w^2 \text{ Var } M + \sum_{i=1}^{n} w_i^2 \text{ Var } e_i} . \tag{3.8}$$

To interpret equation (3.8), note that under minimal conditions we have that ρ_{YM}^2 tends one (for example, Var ϵ_i is bounded and $\Sigma_{i=1}^{n} w_i^2$ tends to zero). This indicates, for large portfolios, that most of risk is due to market risk. Although this fact is well-known to investment analysts, equation (3.8) provides a measure to quantify the adjective "most."

Section 4 The Risk of a Common Disaster

Example 3.3 (CAPM) shows that pooling, or diversification, of different risks may be of limited value in the presence of a common random variable. This is well-known for investment risks where alternative risk management strategies, such as hedging, have been developed. This section investigates another application where different risks share a common random variable that we call a "disaster." Examples include epidemics in life insurance and annuities, flooding, earthquakes and hurricanes in homeowners' insurance, or even a nuclear holocaust in virtually all areas of insurance!

In accordance with the theme of this paper, the purposes of this section is to quantify the risk due to a common disaster. To manage this risk, pooling is ineffective. Risk managers should look to risk exchanges (such as reinsurance), contract exclusions (such as earthquake exclusions in homeowners' insurance) or other techniques for handling this source of risk.

We will suppose that Z is the time to a disaster that is common to all policyholders. This disaster may affect policyholders as follows.

Suppose that the ith policyholder is insured for amount B_i and that the time until the insured event is X_i. With the benefit amount and the time until failure, the insurer's liability is $Y_i = B_i f_i(X_i)$, where $f_i(.)$ is a known function that depends on the terms of the contract. For example, we may use $f_i(x) = \exp(-\delta x) = v^x$ for whole life insurance where δ represents a constant force of interest. Similarly, we may use $f_i(x) = \exp(-\delta x)I(x \leq m)$ for an m-year term insurance policy or $f_i(x) = I(x \leq m)$ for an m period property and casualty policy (without discounting). See Bowers et al (1986, Chapters 3 and 4) for additional examples.

To include the possibility of a common disaster, we first assume that each policy depends on T_i, a time to failure without regard to the disaster. For a whole life policy this may be the future remaining lifetime of an individual or for a homeowner's policy this may be the frequency of a fire. Policies are assumed to be subject to T_i and to disasters. Further, in the event of a disaster, we use c_i to be a variable that indicates whether the ith policyholder succumbs to disaster. With these elements, the time to failure is defined by

$$X_i = \begin{cases} T_i & \text{if } c_i=0 \\ \min(T_i, Z) & \text{if } c_i=1 \end{cases} \tag{4.1}$$

where $\{c_i\}$, $\{T_i\}$ and Z are mutually independent sets of random variables. Thus, if the policy fails before disaster occurs so that $T_i < Z$, then failure occurs at T_i regardless of the outcome of c_i. However, if the policy does not fail before disaster occurs so that $T_i \geq Z$, then some policyholders are affected by the common disaster ($c_i = 1$) although others are not ($c_i = 0$). For simplicity, assume that these events are independent and occur with probability $q_i = \text{Prob}(c_i = 1)$.

Our interest lies in assessing the riskiness of the pool of policies whose liability is denoted by

$$Y = \sum_{i=1}^{n} Y_i = \sum_{c_i=0} B_i f_i(T_i) + \sum_{c_i=1} B_i f_i(\min(T_i, Z)) \ . \tag{4.2}$$

Because of the model formulation, the common disaster component is described by the variables $D = \{c_1, \ldots, c_n, Z\}$. Thus, we may describe the proportion of variability due to the common disaster as $\rho^2_{YD} = \text{Var}(\, E(Y \mid D)\,) / (\text{Var } Y)$.

Before assessing the riskiness due to a disaster, we first examine the expected liability. It turns out that disasters, even relatively rare ones, can have a large effect on the expected liability. This is important because traditionally insurance pricing has been based on net premium principles (see, for example, Bowers et al, 1986, Chapter 6). For an illustration of a pool where a relatively infrequent disaster has large effect on the pool's expected liability, consider the following example.

Example 4.1 Pool of Whole Life Policies

Assume that the insurer promises to pay $1 to a policyholder age x upon death. Without disaster, the future remaining lifetime, T, has probability density function ${}_tp_x\,\mu_{x+t}$. Assuming a constant force of interest δ, the expected value of future liability is

$$\mathrm{E}\exp(-\delta T) = \int_0^\infty e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt = \bar{A}_x@(\delta)\,.$$

Here, the notation $\bar{A}_x\ @(\delta)$ means to calculate the net single premium of a whole life insurance policy payable at the moment of death, discounted at constant force of interest δ (see, for example, Bowers et al, 1986, Chapter 4). For simplicity, we have assumed that the policy is paid-up.

Assume that the time until disaster, Z, has an exponential distribution with parameter λ. Thus, with survival function $\mathrm{Prob}(\min(T, Z) > t) = {}_tp_x\,e^{-\lambda t}$, we have the probability density function of $\min(T, Z)$ is $-\frac{\partial}{\partial t}\mathrm{Prob}(\min(T, Z) > t) = {}_tp_x\,e^{-\lambda t}(\mu_{x+t} + \lambda)$. With the relationship $\bar{A}_x = 1 - \delta\,\bar{a}_x$, this yields

$$\mathrm{E}\exp(-\delta\min(T, Z)) = \int_0^\infty e^{-\delta t}\,{}_tp_x\,e^{-\lambda t}(\mu_{x+t} + \lambda)\,dt = \bar{A}_x@(\delta+\lambda) + \lambda\,\bar{a}_x@(\delta+\lambda)$$

$$= \frac{\delta}{\delta+\lambda}\bar{A}_x@(\delta+\lambda) + \frac{\lambda}{\delta+\lambda}\,.$$

Thus, using equation (4.1), we have that the expected liability is

$$\mathrm{E}\exp(-\delta X) = (1-q)\,\bar{A}_x\ @(\delta) + q\left(\frac{\delta}{\delta+\lambda}\ \bar{A}_x@(\delta+\lambda) + \frac{\lambda}{\delta+\lambda}\right).$$

To assess the affect of λ on the expected liability, it is instructive to compare expected liabilities in the case of complete disaster ($q = 1$, as for example, a nuclear holocaust) to the case of no disaster ($q = 0$). Thus, we examine

$$\mathrm{Ratio} = \frac{\frac{\delta}{\delta+\lambda}\bar{A}_x@(\delta+\lambda) + \frac{\lambda}{\delta+\lambda}}{\bar{A}_x@(\delta)}$$

as a function of λ.

Figure 4.1 provides the ratio of expected liabilities over several values of λ. Here, the net single premiums were calculated using 1979-1981 U.S. Male Life Tables with a uniform distribution of death approximation (see Bowers et al, 1986, Chapters 3 and 4). Thus, for example, for expected time to disaster $\lambda^{-1} = (0.0004)^{-1} = 250$ years, the ratio is approximately 1.10 at $i = 10\%$. Given the competitive market for whole life insurance, a 10% increase in expected liabilities is important, especially considering that the expected time until disaster is 250 years. ■

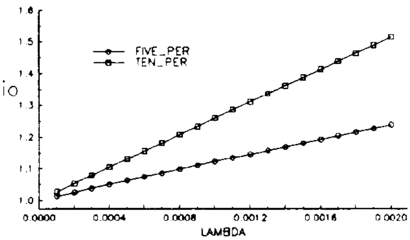

Figure 4.1. Plot of the ratio of expected insurance liabilities over various values of λ. Here, λ^{-1} is the expected time to disaster. The upper line with square plotting symbols corresponds to $i=10\%$. The lower line with circular plotting symbols corresponds to $i=5\%$.

Although disasters are important in the calculation of a pool's expected liability, they are even more important with respect to variance calculations. In Example 4.1, we only needed to calculate the expected liability of a single policy. This is because the expectation of the sum of liabilities equals the sum of expected liabilities, even under dependence induced by a common disaster. However, this is not true when considering variances. To illustrate this, we consider the following example.

Example 4.2 Pool of Property and Casualty Policies

For simplicity, we consider a pool of identical policies with identical risk distributions. Further, suppose that each policy pays \$1 for failure by time m, that is, $B_i = 1$ and $f_i(x) = I(x \le m)$. With these specifications, the pool's liability is given by equation (4.2). We assume that the risk distributions $\{T_i\}$ are i.i.d. We use the notation $\text{Prob}(T_i \le m) = {}_mq_T$, $\text{Prob}(Z \le m) = {}_mq_Z$ and $\text{Prob}(c_i = 1) = q$ to denote probabilities of failure. With these notations, straightforward calculations show that

$$\text{E}\, Y = n\,(\,{}_mq_T + q\ {}_mp_T\,{}_mq_Z\,),$$

where ${}_mp_T = 1 - {}_mq_T$. As is Example 4.1, we examine the ratio

$$\frac{\text{E}\ Y\ @(q=1)}{\text{E}\ Y\ @(q=0)} = 1 + {}_mq_Z\,\frac{1 - {}_mq_T}{{}_mq_T}\,.$$

Recall that we interpret this to be the ratio of expected liabilities under complete disaster ($q=1$) to common disaster ($q=0$). For this example, we see that we expect this ratio to be close to one if ${}_mq_Z$ is small compared to ${}_mq_T$, that is, if the probability of a disaster is small compared to the probability of a "natural" disaster.

Straightforward yet tedious calculations show that

$$\text{Var}(\,\text{E}(Y\,|\,D)\,) = n^2\,q^2\,{}_mp_T^2\,{}_mp_Z\,{}_mq_Z + n\,q(1-q)\,{}_mp_T^2\,{}_mq_Z$$

and

$$\text{Var}(\,Y\,) = n^2\,q^2\,{}_mp_T^2\,{}_mp_Z\,{}_mq_Z + n\,(\,{}_mq_T + q\,{}_mp_T\,{}_mq_Z - q^2\,{}_mp_T^2\,{}_mq_Z\,{}_mp_Z\,)\,.$$

Thus, as in Example 3.2, we see that the coefficient associated with n^2 dominates the pool's variability for large values of n. Thus, as $n \to \infty$, the ratio ρ^2_{YD} =

$\mathrm{Var}(\,\mathrm{E}(Y \mid D)\,)/\mathrm{Var}(\,Y\,) \rightarrow 1$. We interpret this to mean that, for large pools, most of the risk is due to the common disaster.

Section 5. Ordering of Risks and Assessing Relative Uncertainty

Measures of relative importance of different causal factors can be developed from many different vantage points. Methods of sensitivity analysis, called *comparative statics* in economics, were introduced in Section 1. The purpose of this section is to provide background material for the approach used in the paper for assessing relative importance of sources of insurance risks.

This section is organized as follows. To justify variance as a measure of risk, we begin by introducing "risk premia" from the economics of uncertainty literature. Within this framework, the variance measure can be obtained by examining several special cases. Further, economic risk premia provide a type of partial ordering of risks that is appealing in our discussions of risk attribution. Although not discussed in this paper, another method for quantifying uncertainty is through the entropy and information theory approach (see, for example, Brockett, 1991). The variance measure can also be obtained as a special case from this framework. These two approaches provide frameworks for generalizing the risk measure.

Because much of the work on assessing relative importance has arisen in studies of regression analysis methodology, in Subsection 5.2 we review this line of thought. In regression analysis, measures of relative importance are often based on functions related to variability. Not only does this methodology provide an additional motivation for examining the variability measure, it also suggests a method for investigating multivariate risks decomposition, the subject of Section 6.

Section 5.1 Ordering of Risks

In economic theory, decision makers express preferences concerning alternative bundles of wealth using utility functions (see, for example, Bowers et al, 1986, Chapter 1). As an important variation, risk managers choose among stochastic risks by maximizing expected utility. These utility functions are often referred to as Von Neumann-Morganstern utility functions. Von Neumann and Morganstern (1947) were the first to provide a consistent set of axioms that showed expected utility maximization to be the consequence of "rational" decision-making.

Arrow-Pratt Risk Premia

Under mild conditions on the utility functions (such as concavity), it can be shown that a risk manager is risk-averse. A risk manager with utility function u(.) is said to be risk-averse (at wealth W)

if $u(W) > E\,u(W+Y)$ for all risks Y where $E\,Y = 0$ and $\mathrm{Var}\,Y > 0$. Thus, if a risk manager is to take on risk Y in addition to the certain wealth level W, then it seems reasonable to ask what level of compensation is required. The Arrow-Pratt risk premium π is this additional level of compensation, defined by

$$u(W-\pi) = E\,u(W+Y). \tag{5.1}$$

From equation (5.1), it is apparent that the risk premium π is a function of wealth W, the utility function u and the distribution of the risk. Some special cases will help to interpret the risk premium π. Throughout, we assume that $E\,Y = 0$. This can always be done through relabeling W, if necessary.

Example 5.1 Quadratic Utility

Consider the utility function $u(x) = bx - cx^2$. First note that $u(W-\pi) = b(W-\pi) - c\,(W-\pi)^2$ and $E\,u(W+Y) = b\,W - c(\mathrm{Var}\,Y + W^2)$. From equation (5.1), we have

$$(W-\pi)^2 - (b/c)(W-\pi) = \mathrm{Var}\,Y + W^2 - W\,b/c$$

so

$$\pi^2 - \pi(2W - b/c) = \mathrm{Var}\,Y\,.$$

Completing the squares yields

$$(\pi - (W - b/(2c)))^2 = \mathrm{Var}\,Y + (W - b/(2c))^2\,.$$

Because cardinal utility functions are invariant up to affine (location and scale) transformations, we may choose b such that $W = b/(2c)$. With this choice, we have

$$\pi^2 = \mathrm{Var}\,Y\,. \tag{5.2}$$

■

Example 5.2 Normally Distributed Risks

Suppose that Y is normally distributed with mean zero and variance σ^2. For simplicity, further suppose that u^{-1} exists and that both $u(\cdot)$ and $u^{-1}(\cdot)$ are increasing. Then, from equation (5.1), we have

$$\pi = W - u^{-1}\left(\int_{-\infty}^{\infty} u(W+y)\,\frac{e^{-y^2/(2\sigma^2)}}{\sqrt{2\pi}\,\sigma}\,dy \right) \tag{5.3}$$

A straightforward calculus exercise shows that π is an increasing function of σ^2, although it still depends on W and u(·). For an illustration where this is easy to see, consider Example 5.3.

■

Example 5.3 Exponential Utility

Suppose that we define $u(x) = (1-e^{-ax})/a$. Thus, from equation (5.1),

$$(1-e^{-a(W-\pi)})/a = u(W-\pi) = E\,u(W+Y) = (1-e^{-aW}\,E\,e^{-aY})/a.$$

This yields $\pi = -(\ln (E\,e^{-aY}))/a$. For the case of $Y \sim N(0,\sigma^2)$, this yields

$$\pi = \frac{1}{a} \ln e^{a^2\sigma^2/2} = \frac{a}{2} \sigma^2 .$$

Pratt Asymptotic Approximation

Pratt (1964) provided an approximation for π that is simpler to calculate than the definition in equation (5.1). It is based on two Taylor-series expansions of the form

$$u(W) - \pi\, u'(W) \approx u(W-\pi) = E\,u(W+Y) \approx E(\, u(W) + Y u'(W) + Y^2/2\, u''(W)\,) \tag{5.4}$$

Pratt argued that these expansions could be justified by considering distributions with variances that are small relative to wealth W.

Equating the left and right-hand side of display (5.4) yields the approximate risk premium,

$$\pi_P = (r(W)/2)\ \mathrm{Var}\ Y, \tag{5.5}$$

where $r(W) = -u''(W)/\,u'(W)$ is known as the Arrow-Pratt absolute risk-aversion function. Thus, if a risk manager is comparing two risks, then a higher risk premia will be sought for the risk with the higher variance. As in Examples 5.1 and 5.2, we see that the variance measure summarizes risk in several important special cases.

One advantage of the Arrow-Pratt risk premia defined in equation (5.1) is that any two risks can be compared. That is, a risk manager will choose the risk with the smaller risk premium. Thus, this choice mechanism provides a complete ordering of risks. However, the Arrow-Pratt risk premia uses just one summary measure of a risk's distribution, and other aspects may be relevant.

More generally, several criteria have been established to provide ordering of risks. See, for example, Levy (1992) for an overview of the area of stochastic dominance and Kass et al. (1994) for an actuarial perspective. Here, we focus on a specific ordering called *weakly less risky* that has found broad applications in the financial economics literature.

Weakly Less Risky Ordering

Consider two random variables X and Y. We say that X is *weakly less risky* (WLR) than Y if

$$E\, u(X) \geq E\, u(Y) \quad \text{for each } u \in \mathbf{U}_* . \tag{5.6}$$

Here, $\mathbf{U}_*$ is the class of utility functions where both $u(X)$ and $u(Y)$ are integrable for $u \in \mathbf{U}_*$. Thus, WLR provides a partial ordering between two random variables. The ordering is only partial because it is possible that neither (i) X is WLR than Y nor (ii) Y is WLR than X are true. Using equation (5.1), we see that if $W + X$ is *WLR* than $W + Y$, then $\pi(X) \leq \pi(Y)$ for each increasing $u(.) \in \mathbf{U}_*$. Thus, the partial ordering of the *WLR* principle is preserved when examining risk premia.

Recall that if $u(W) > E\, u(W + Y)$, then the consumer is said to be risk averse at W. If the relationship holds for all wealth levels W, then the consumer is said to be *globally risk-averse*. For global risk aversion, a necessary and sufficient condition is that u(.) is strictly concave (see, for example, Ingersoll, 1987, p. 37). Thus, we will work with $\mathbf{U}$, defined to be the class of strictly concave, increasing utility functions. Recall that a function u(.) is strictly concave if $u(tx + (1-t)y) > t\, u(x) + (1-t)\, u(y)$ for $0<t<1$ and all x, y. If $u''(x)<0$ for all x, then u(.) is strictly concave.

The following is an important characterization of the WLR ordering. A necessary and sufficient condition for X to be WLR than Y with respect to $\mathbf{U}$ is that there exists a random variable ϵ such that

$$Y =^d X + \epsilon \quad \text{and} \quad E(\epsilon \mid X=x) = 0 \;\text{ for all } x. \tag{5.7}$$

Here, the symbol $=^d$ means equal in distribution. This characterization is due to Rothschild and Stiglitz (1970). See, for example, Ingersoll (1987, p.119) for a proof. Henceforth, when we say X is WLR than Y, we implicitly mean with respect to $\mathbf{U}$. If display (5.7) holds, then it is well known that X and ϵ are uncorrelated although they may not be independent.

Now consider the case where X is a factor that may be used to help understand the risk Y. In general, two random variables X and Y need not be ordered through the WLR principle. However, we can always find a function of X that is WLR than Y.

PROPOSITION 5.1

Consider a stochastic element X and a random variable Y that are defined on a common probability space such that $E \mid Y \mid < \infty$. Define $T(x) \equiv E(Y \mid X=x)$. Then, $T(X)$ is WLR than Y.

Further, if there exists another function T_* such that (i) T_* has a unique inverse and (ii) $T_*(X)$ is WLR than Y, then $T(X) = T_*(X)$ with probability one.

PROOF: Define $\epsilon = Y - T(X)$. With this definition, the equations in display (5.2) trivially hold and thus $T(X)$ is WLR than Y.

Now suppose that $T_*(X)$ is WLR than Y. Then, by (5.2), there exists a random variable ϵ_* such that $Y =^d T_*(X) + \epsilon_*$ and $E(\epsilon_* \mid T_*(X)=x)=0$ for all x. Thus,

$$T(x) \equiv E(Y \mid X=x) = E(T_*(X)+\epsilon_* \mid X=x)$$
$$= E(T_*(X) \mid X=x) = T_*(x) \text{ for all } x, \text{ with probability one.}$$

This is true because $E(\epsilon_* \mid X=x) = E(\epsilon_* \mid T_*(X)=y)=0$ for all $y=T_*(x)$. ■

The requirement of a "common probability space" is a mild one and can always be accomplished by redefining the random elements (see for example, Serfling, 1980, Theorem 1.6.3, or Kass et al, 1994, Theorem 1.2). The element X may be a random variable, a random vector or an element taking values in a more general space. The only requirement is that it represent the domain of a real-valued transformation T(.).

Proposition 5.1 shows that if we wish to assess the effect of X on Y, then $T(X) = E(Y \mid X)$ is the appropriate function, at least in terms of the WLR ordering. Alternatively, we will see in Subsection 5.2 that there are several measures that can be used for assessing the effect that X has on Y. The function $T(X)$ is appropriate for summarizing the effect due to X for two reasons. First, the risk $T(X)$ accounts for nearly all the uncertainty in Y in the sense that residual, Y-$T(X)$, is uncorrelated with X. More precisely, we have that $E(Y-T(X) \mid X=x) = 0$ for all x, a condition stronger than zero correlation although not as strong as statistical independence. Second, and more importantly, all risk managers whose preferences can be represented by $u(.) \in U_*$ will prefer $T(X)$ to Y. Moreover, we have that T(.) is essentially unique, at least if it has a unique inverse. To summarize the uncertainty of $E(Y \mid X)$, Examples 5.1 and 5.2 and the Pratt asymptotic approximations suggest the use of $Var(E(Y \mid X))$. Of course, other classes of utility functions will suggest other generalizations.

Section 5.2 Regression Methodologies

Questions of relative importance of explanatory factors arise naturally in regression analysis. Here the focus is on modeling a stochastic element Y in terms of one or more explanatory factors. The basic, and most widely applied, model assumes that Y can be expressed as a linear combination of the explanatory variables, $X_1, \ldots, X_k$, and an unobserved "error" term e. That is,

$$Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k + e \tag{5.8}$$

where $\beta_0, \beta_1, \ldots, \beta_k$ are fixed, yet unknown, parameters. The error term is generally assumed to be independent of the explanatory variables and is used to represent the "inherent variability" in Y. Hence, with the linear representation in equation (5.8), it is natural to wish to describe the relative influence that an explanatory variable has on Y.

As described by Kruskal (1987), the following are the most widely cited measures for assessing the impact of X_j on Y. Here, we assume that $\{X_1, \ldots, X_k, Y\}$ are random variables.

1. ρ_{Yj} - the correlation between X_j and Y, without regard to the other explanatory variables.
2. β_j - the jth regression coefficient. This is defined to be the partial derivative of the conditional expected value of Y with respect to X_j. It is interpreted to be the change in the (conditional) expected value of Y per unit change in X_j, holding the other explanatory variables fixed.
3. β_j^* - the standardized regression coefficient. Defined to be equal to β_j ((Var X_j)/(Var Y))$^{1/2}$, that is, the jth regression coefficient standardized to be unitless.
4. β_j E X_j - the jth regression coefficient time the expected value of X_j.
5. ρ_j^2 - the jth coefficient of partial determination.

The first measure only captures relationships between Y and X_j. The second through fourth are based on small changes in the (conditional) expected value of Y. The fifth measure is based purely on variation and is described more fully below.

Denote the variance of Y by σ_Y^2 and let $\mathbf{X} = (X_1, \ldots, X_k)'$. Further, define $\sigma_{Y\mathbf{X}} = \text{Cov}(Y, \mathbf{X})$ and $\Sigma_{\mathbf{X}} = \text{Var}\ \mathbf{X}$. Then it turns out that (Anderson, 1958, Section 2.5.2),

$$\rho_{Y\mathbf{X}} = \max_\alpha \text{Corr}(Y, \alpha'\mathbf{X}) = \frac{\sqrt{\sigma_{Y\mathbf{X}}' \Sigma_{\mathbf{X}}^{-1} \sigma_{Y\mathbf{X}}}}{\sigma_Y}\ . \tag{5.9}$$

Thus, $\rho_{Y\mathbf{X}}$ is called the multiple correlation coefficient. Further, $\rho_{Y\mathbf{X}}^2$ is the (population) coefficient of determination. For further interpretation, assume that Y, $\mathbf{X}$ are jointly normally distributed. Then, Y |

$\mathbf{X}$ is normally distributed with mean $\mathrm{E}\,(Y \mid \mathbf{X}) = \mathrm{E}\,Y + \sigma_{Y\mathbf{X}}'\,\Sigma_{\mathbf{X}}^{-1}\,(\mathbf{X} - \mathrm{E}\,\mathbf{X})$ and variance $\mathrm{Var}\,(Y \mid \mathbf{X}) = \sigma_Y^2 - \sigma_{Y\mathbf{X}}'\,\Sigma_{\mathbf{X}}^{-1}\,\sigma_{Y\mathbf{X}}$. Thus, $1 - \mathrm{Var}\,(Y \mid \mathbf{X})\,/\,\mathrm{Var}\,Y = \rho_{Y\mathbf{X}}^2$, that is, the coefficient of determination is one minus the proportion of variability still remaining after conditioning on $\mathbf{X}$. Another interpretation is

$$\rho_{Y\mathbf{X}}^2 = \frac{\mathrm{Var}\,(\,\mathrm{E}(Y \mid \mathbf{X})\,)}{\mathrm{Var}\,Y} \tag{5.10}$$

which is consistent with the measures introduced in Section 3.

More generally, let $\mathbf{X}_{(j)} = (X_1, \ldots, X_j)'$, $\sigma_{Y(j)} = \mathrm{Cov}(Y, \mathbf{X}_{(j)})$, $\Sigma_{(j)} = \mathrm{Var}\,\mathbf{X}_{(j)}$ and $\sigma_{Y \mid (j)} = \sigma_Y^2 - \sigma_{Y(j)}'\,\Sigma_{(j)}^{-1}\,\sigma_{Y(j)}$, for $j = 1, \ldots, k$. Define $\rho_{Yj \mid (j-1)}^2 = 1 - \sigma_{Y \mid (j)}/\sigma_{Y \mid (j-1)}$, $j = 2, \ldots, k$ and $\rho_{Y1}^2 = 1 - \sigma_{Y \mid (1)}/\sigma_Y^2$. With this notation, we may state the well-known result (see, for example, Anderson, 1958, Section 2.5),

$$1 - \rho_{Y\mathbf{X}}^2 = (1 - \rho_{Y1}^2)(1 - \rho_{Y2 \mid 1}^2) \ldots (1 - \rho_{Yk \mid (k-1)}^2) \tag{5.11}$$

Equation (5.11) provides a decomposition of the coefficient of determination into interpretable components. Assuming normality, it can be checked that $\mathrm{Corr}(Y, X_j \mid X_1, \ldots, X_{j-1}) = \rho_{Y,j \mid (j-1)}$, the correlation between Y and X_j, conditional on $X_1, \ldots, X_{j-1}$. Thus, $\rho_{Y,k \mid (k-1)}$ is called a partial correlation coefficient and $\rho_{Y,k \mid (k-1)}^2 = \rho_k^2$ is called a coefficient of partial determination.

Equation (5.11) provides a satisfactory decomposition of $\rho_{Y\mathbf{X}}^2$ when the explanatory variables are (i) hierarchical or (ii) orthogonal. When they are hierarchical, because $\rho_{Yj \mid (j-1)}^2$ can be interpreted as the proportion of Ys variability explained by $X_1, \ldots, X_j$, it is reasonable to build $\rho_{Y\mathbf{X}}^2$ according to equation (5.11). Equation (5.11) makes an assumption about the ordering of the explanatory variables. For example, in general we have $\rho_{Y2}^2 \neq \rho_{Y2 \mid 1}^2$.

However, if the explanatory variables are orthogonal, several simplifications arise. In this case, $\mathrm{Var}\,\mathbf{X}$ is a diagonal matrix and equation (5.11) yields

$$\rho_{Y\mathbf{X}}^2 = \sum_{j=1}^{k} \frac{\mathrm{Cov}(Y, X_j)^2}{\mathrm{Var}\,X_j\;\mathrm{Var}\,Y} = \sum_{j=1}^{k} \rho_{Y,j}^2 \,.$$

Thus, the total proportion of variability can be directly decomposed into individual components. In a more tedious fashion, this can be checked using equation (5.10) and the relation

$$\rho_{Yj \mid (j-1)}^2 = \frac{1 - \rho_{Y1}^2 - \ldots - \rho_{Yj}^2}{1 - \rho_{Y1}^2 - \ldots - \rho_{Y,j-1}^2} \,.$$

When the explanatory variables are neither hierarchical nor orthogonal, several measures have been proposed to determine the relative importance of each factor. Overviews and broad discussions of the interpretation of relative importance in the social sciences can be found in Pedhazur (1981, Chapter 5), Williams (1978) and Kruskal and Majors (1989). To overcome this asymmetry of the relation in equation (5.11), averaging techniques have been proposed by Kruskal (1987), Pratt (1987) and Genizi(1993). Other methods have been recently described by Chevan and Sutherland (1991) and Budescu (1993). This fundamental decomposition is the subject of ongoing research.

Section 6 Risk Attributable to Several Sources

Section 6.1 Multivariate Risk Decomposition

Although the Section 3.1 discussion was given in terms of a single source of risk, the extension to multiple risks is immediate. Thus, for a risk Y, we define

$$\mathrm{R}_Y(X_1, X_2, \ldots, X_k) = \mathrm{Var}(\,\mathrm{E}(Y \mid X_1, X_2, \ldots, X_k)\,)$$

to be the risk of Y attributable to the multiple sources $X_1, X_2, \ldots, X_k$. The case of a single source of risk, presented in Section 3.1, corresponds to $k=1$.

As described in Section 2.3, it is desirable to have a measure of riskiness due to one source, after having controlled for other sources. Suppose that we are interested in the riskiness of Y attributable to the p variables $X_{k+1}, \ldots, X_{k+p}$, after having controlled for the effects of k variables $X_1, \ldots, X_k$. We define this measure to be

$$\mathrm{R}_Y(X_{k+1}, \ldots, X_{k+p} \mid X_1, \ldots, X_k) = \mathrm{R}_Y(X_1, \ldots, X_{k+p}) - \mathrm{R}_Y(X_1, X_2, \ldots, X_k)\,. \tag{6.1}$$

This measure of additional riskiness is always nonnegative. Indeed, straight-forward calculations show that

$$\mathrm{R}_Y(X_{k+1}, \ldots, X_{k+p} \mid X_1, \ldots, X_k) = \mathrm{E}(\;\mathrm{Var}(\,\mathrm{E}(Y \mid X_1, \ldots, X_{k+p}) \mid X_1, X_2, \ldots, X_k)\,) \geq 0. \tag{6.2}$$

We often will examine the case of $p = 1$,

$$\mathrm{R}_Y(X_{k+1} \mid X_1, X_2, \ldots, X_k) = \mathrm{R}_Y(X_1, \ldots, X_{k+1}) - \mathrm{R}_Y(X_1, \ldots, X_k), \tag{6.3}$$

interpreted to be the riskiness of Y attributable to X_{k+1}, after controlling for the effects of $X_1, \ldots, X_k$.

For example, equation (6.3), together with (6.2), yields the following ordering of risks:

$$\mathrm{R}_Y(X_1) \;\leq\; \mathrm{R}_Y(X_1, X_2) \;\leq \ldots \leq\; \mathrm{R}_Y(X_1, X_2, \ldots, X_k) \;\leq\; \mathrm{R}_Y \, . \tag{6.4}$$

Here, $\mathrm{R}_Y = \mathrm{Var}\, Y$ is the overall risk.

Risks attributable to different sources are not additive. That is,

$$\mathrm{R}_Y(X_1,\ldots,X_k) \neq \sum_{j=1}^{k} \mathrm{R}_Y(X_j)$$

in general. However, if the sources of risk are hierarchical, then it is appropriate to use the decomposition

$$\mathrm{R}_Y(X_1,\ldots,X_k) = \sum_{j=1}^{k} \mathrm{R}_Y(X_j \mid X_1,\ldots,X_{j-1}) \, . \tag{6.5}$$

Here, $\mathrm{R}_Y(X_j \mid X_1, \ldots, X_{j-1})$ evaluated at $j=1$ is defined to be $\mathrm{R}_Y(X_1)$. To demonstrate the nonadditivity of risks from different sources, we return to a previous illustration.

Example 3.1 - Continued

Recall that $Y = X\,C$, where X and C are independent. We have established that $\mathrm{Var}(\, \mathrm{E}(Y \mid X)\,) = \mathrm{R}_Y(X) = \mathrm{Var}(\, X\,)\,(\mathrm{E}\ C)^2$. Similarly, $\mathrm{R}_Y(C) = \mathrm{Var}(\, C\,)\,(\mathrm{E}\ X)^2$. Straightforward calculations show that

$$\mathrm{Var}\ Y = \mathrm{R}_Y = \mathrm{R}_Y(X, C) = \mathrm{Var}(\,X\,)\mathrm{Var}(\,C\,) + \mathrm{Var}(\,X\,)(\mathrm{E}\ C)^2 + \mathrm{Var}(\,C\,)(\mathrm{E}\ X)^2.$$

Thus, $\mathrm{R}_Y(X) \leq \mathrm{R}_Y(X, C)$ and $\mathrm{R}_Y(C) \leq \mathrm{R}_Y(X, C)$, in accordance with equation (6.4). Further, if $\mathrm{Var}(\,X\,)\mathrm{Var}(\,C\,) > 0$, then

$$\mathrm{R}_Y(X) + \mathrm{R}_Y(C) < \mathrm{R}_Y(X, C).$$

In this application, because X represents the interest rate risk, we interpret it to be more "fundamental" than C, the random claim amount. Thus, we use

$$\mathrm{R}_Y(X) = \mathrm{Var}(\,X\,)\,(\mathrm{E}\ C)^2$$

and

$$\mathrm{R}_Y(C \mid X) = \mathrm{Var}(\,X\,)\mathrm{Var}(\,C\,) + \mathrm{Var}(\,C\,)(\mathrm{E}\ X)^2$$

to decompose R_Y, the overall risk. ∎

In general, the order of attribution matters, even for independent sources of risk. To illustrate in the above example, we have $\mathrm{R}_Y(C \mid X) \neq \mathrm{R}_Y(C)$. In the special case of linear independent sources of risk, the order of attribution is unimportant. To see this, consider $Y = X_1 + X_2 + Z$, where X_1, X_2

and Z are mutually independent. Then, straight-forward calculations show that $R_Y(X_1) = \text{Var } X_1$, $R_Y(X_2) = \text{Var } X_2$, $R_Y(X_1, X_2) = \text{Var } X_1 + \text{Var } X_2$ and

$$R_Y(X_1 \mid X_2) = R_Y(X_1, X_2) - R_Y(X_2) = \text{Var } X_1 = R_Y(X_1).$$

Section 6.2 Pool of *m*-Year Term Policies

To assess the effect of several factors on a risk, this subsection considers a pool of policies that are subject to (i) mortality risk, (ii) risk of a common disaster and (iii) risk of a common investment environment. Specifically, the mortality risk is denoted by $M = \{T_1, \ldots, T_n\}$, where T_i is the future lifetime of the ith policyholder. As in Section 4, the common disaster component is described by the variables $D = \{c_1, \ldots, c_n, Z\}$, where c_i indicates whether the ith policyholder succumbs to disaster and Z is the time until disaster. As in Example 3.2, $\mathcal{I} = \{\Delta_1, \Delta_2, \ldots\}$ describes the money market returns so that $v(k) = \Pi_{s=1}^{k} \exp(-\Delta_s)$ is a k-period discounting function.

The risks are n, m-year, term policies that, for convenience, we assume are each issued to a male life (x) age 30. The following additional assumptions are made.

Assumptions:

1. We assume that $\mathcal{I}$, D and M are mutually independent. Further, for convenience we assume that $\{c_i\}$ and $\{T_i\}$ are i.i.d., as in Section 4. Because policies lifetimes $\{T_i\}$ are i.i.d., we assume that all lives are age x at contract initiation.
2. For convenience, we assume that time to disaster follows an exponential distribution with parameter λ, as in Section 4.
3. Interest and mortality distributions are the same as in Example 3.2.
4. Policies are payable at the end of year of death.

Similar to equation (4.2), we use $Y = \Sigma_{i=1}^{n} Y_i$ for the pool risk, where

$$Y_i = \begin{cases} v([T_i]+1)\ I(T_i < m) & \text{if } c_i = 0 \\ v([\min(T_i, Z)]+1)\ I(\min(T_i, Z) < m) & \text{if } c_i = 1 \end{cases} \tag{6.6}$$

Here, the square brackets [.] denote the greatest integer function. For simplicity, we define $T_i^* = [T_i]$ and $Z^* = [Z]$. To reduce the notational complexity, we drop the star (*) notation.

Table 6.1 and Figures 6.1-6.3 assess the effects of the three basic risk factors for a pool of term policies. The tables provides values for (i) $\rho^2_{Y\not{I}} = R_Y(\not{I})/R_Y$, the proportion of risk due to the interest environment, (ii) $\rho^2_{YD} = R_Y(D)/R_Y$, the proportion of risk due to a common disaster, (iii) $\rho^2_{YM} = R_Y(M)/R_Y$, the proportion of risk due to mortality and (iv) $\rho^2_{Y\not{I}D} = R_Y(\not{I}, D)/R_Y$, the proportion of risk due to the interest environment and a common disaster. The tables show the effects of n, the number of policies in the pool, m, the term of the policy, q, the probability of a policyholder being affected by disaster and λ, the parameter controlling the expected time until disaster.

Table 6.1, and the corresponding Figure 6.1, illustrate the impact on n and q on the risk factors. Here, we consider only $m = 5$ year term policies with the expected time until disaster equal to $\lambda^{-1} = (0.02)^{-1} = 50$ years. For policies without disaster corresponding to $q = 0$, we see that for most pool size, mortality dominates as a risk factor. It is only for the extremely large pool size, $n = 10{,}000$, that the common interest environment becomes an important factor for this short term policy. Interestingly, in the case of complete disaster corresponding to $q = 1$, mortality has almost no impact as a risk factor. Even for moderately large values of q, we see that the disaster component dominates the risk. Again, because of the short term nature of the policy, we see that the disaster component dominates the interest component.

TABLE 6.1. Relative Importance of Risk Sources for a Pool of $m = 5$ Year Term Policies. The Expected Time to Disaster is $\lambda^{-1} = 50$ years.

q	n	E Y	$\rho^2_{Y\not{I}}$	ρ^2_{YD}	ρ^2_{YM}	$\rho^2_{Y\not{I}D}$	$R_Y^{1/2}$
0.00	1	0.007	0.01	0.00	98.08	0.01	0.08
	10	0.074	0.12	0.00	97.97	0.12	0.26
	50	0.371	0.62	0.00	97.49	0.62	0.58
	100	0.742	1.23	0.00	96.89	1.23	0.82
	10000	74.249	55.49	0.00	43.66	55.49	12.16
0.02	1	0.009	0.02	18.30	78.23	18.65	0.09
	10	0.091	0.15	20.63	75.82	21.16	0.29
	50	0.454	0.65	29.46	66.68	30.66	0.68
	100	0.909	1.13	37.89	57.95	39.73	1.03
	10000	90.851	4.21	91.83	2.15	97.76	53.68
0.20	1	0.024	0.04	72.06	20.92	73.43	0.14
	10	0.240	0.19	86.25	9.41	88.05	0.66
	50	1.201	0.27	94.49	2.73	96.53	2.75
	100	2.403	0.29	96.07	1.45	98.16	5.35
	10000	240.268	0.30	97.84	0.02	99.90	518.77
1.00	1	0.090	0.17	98.05	0.00	100.00	0.26
	10	0.904	0.17	97.99	0.00	100.00	2.59
	50	4.522	0.17	97.99	0.00	100.00	12.96
	100	9.043	0.17	97.99	0.00	100.00	25.91
	10000	904.342	0.17	97.99	0.00	100.00	2591.28

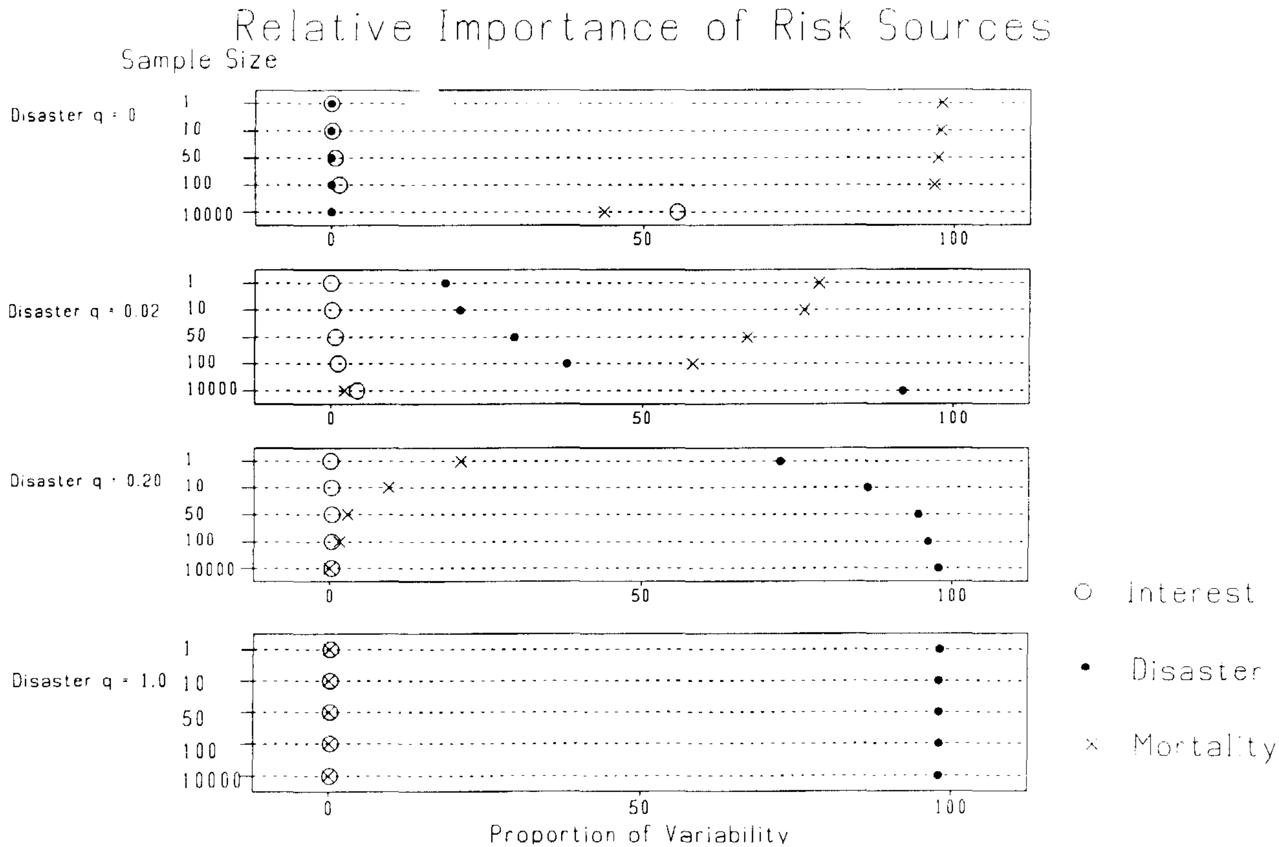

Figure 6.1. Relative Importance of Risk Sources for a Pool of m = 5 Year Term Policies. The Expected Time to Disaster is λ^{-1} = 50 years.

Figure 6.2, and the corresponding Table B.1 in the appendix, illustrate the impact on m and q on the risk factors. Here, we consider only $n = 50$ policies with the expected time until disaster equal to $\lambda^{-1} = (0.02)^{-1} = 50$ years. Not surprisingly, we see that as the term m increases, the importance of the interest environment increases. As in Figure 6.1, Figure 6.2 shows that as q increases, the importance of the disaster component increases. Interestingly, for moderate levels of $q = (0.02)$, the mortality component does not seem to be severely affected by the term of the policy. Only when the term increases to a large level ($m = 50$) does the mortality component shrink dramatically.

Figure 6.3, and the corresponding Table B.2 in the appendix, illustrate the impact on q and λ on the risk factors. Here, we consider only $n = 50$, $m = 1$ year term policies. As anticipated, Figure 6.3 shows that such a short term means that the interest component is negligible, over all values of q and λ. It is interesting to note that the disaster component still dominates for $q = 1$, even in the case when the time to disaster is $\lambda^{-1} = (0.002)^{-1} = 500$ years!

Section 7 Summary

Actuaries manage insurance risks through (i) classical pooling techniques, (ii) risk transference techniques including reinsurance and (iii) financial risk management techniques such as hedging. These broad categories, and a plethora of special cases and variations, of risk management techniques exist to enable actuaries and other financial analysts to cope with the many sources of risk that exist in the world today. These risk management tools are designed to provide relief from specific sources of risk. The purpose of this paper is to introduce a measure that identifies the relative importance of a risk source. With a measure to understand the importance of a factor, the risk manager will be in a position to decide upon the appropriate risk management tool.

The measure was shown to be intuitively appealing when assessing the effectiveness of basic risk management techniques including risk exchange, pooling and financial risk management. In particular, an example illustrated how a common investment environment dominates when pooling mortality risks, thus substantiating the common actuarial wisdom that investment dominates mortality risks. We also showed how catastrophe risks could be modeled and their impact assessed. Catastrophes, or common disasters, are similar to investment risks in that they represent factors that are common to all policies and hence cannot be reduced through pooling techniques. The presence of an important catastrophe risk requires other techniques, such as policy limitations and reinsurance.

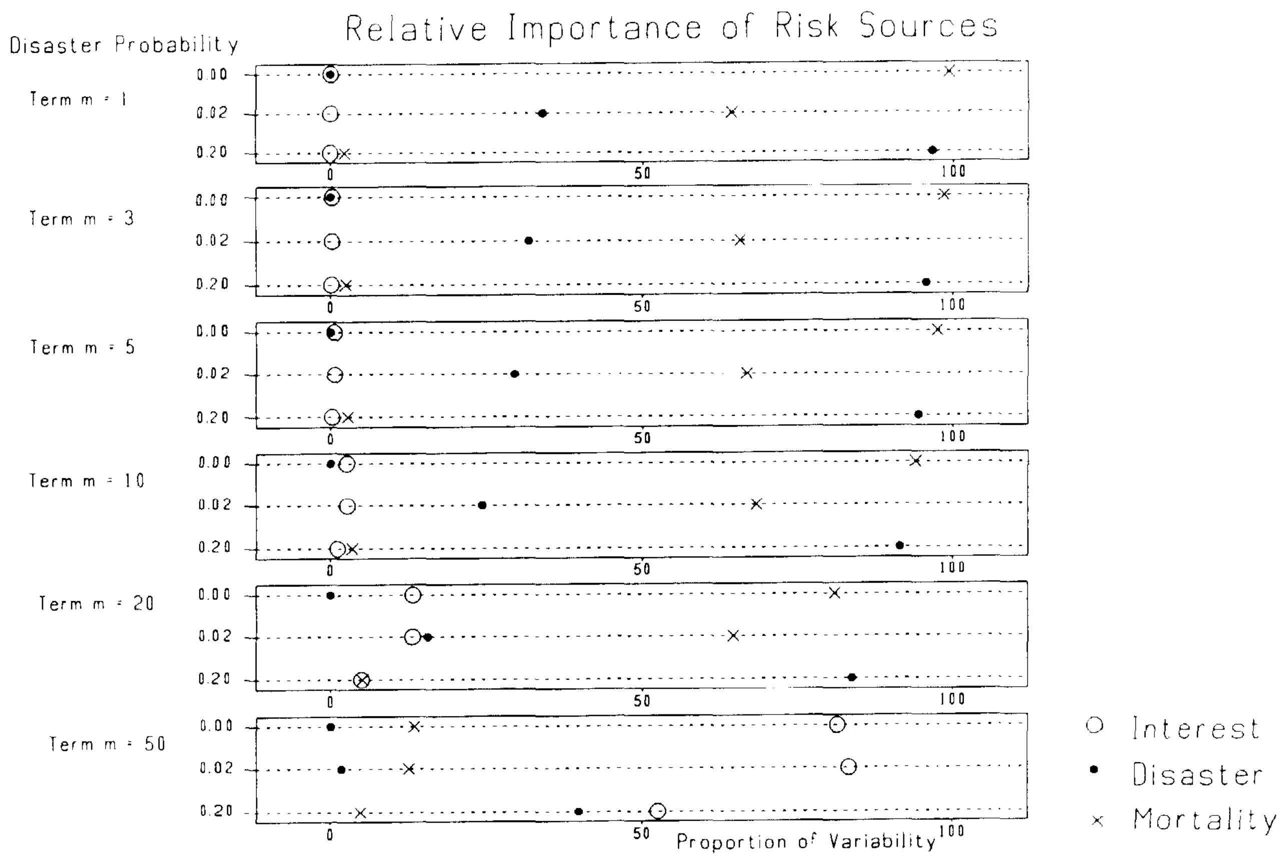

Figure 6.2. Relative Importance of Risk Sources for a Pool of $n=50$ Term Policies. The Expected Time to Disaster is λ^{-1} = 50 years.

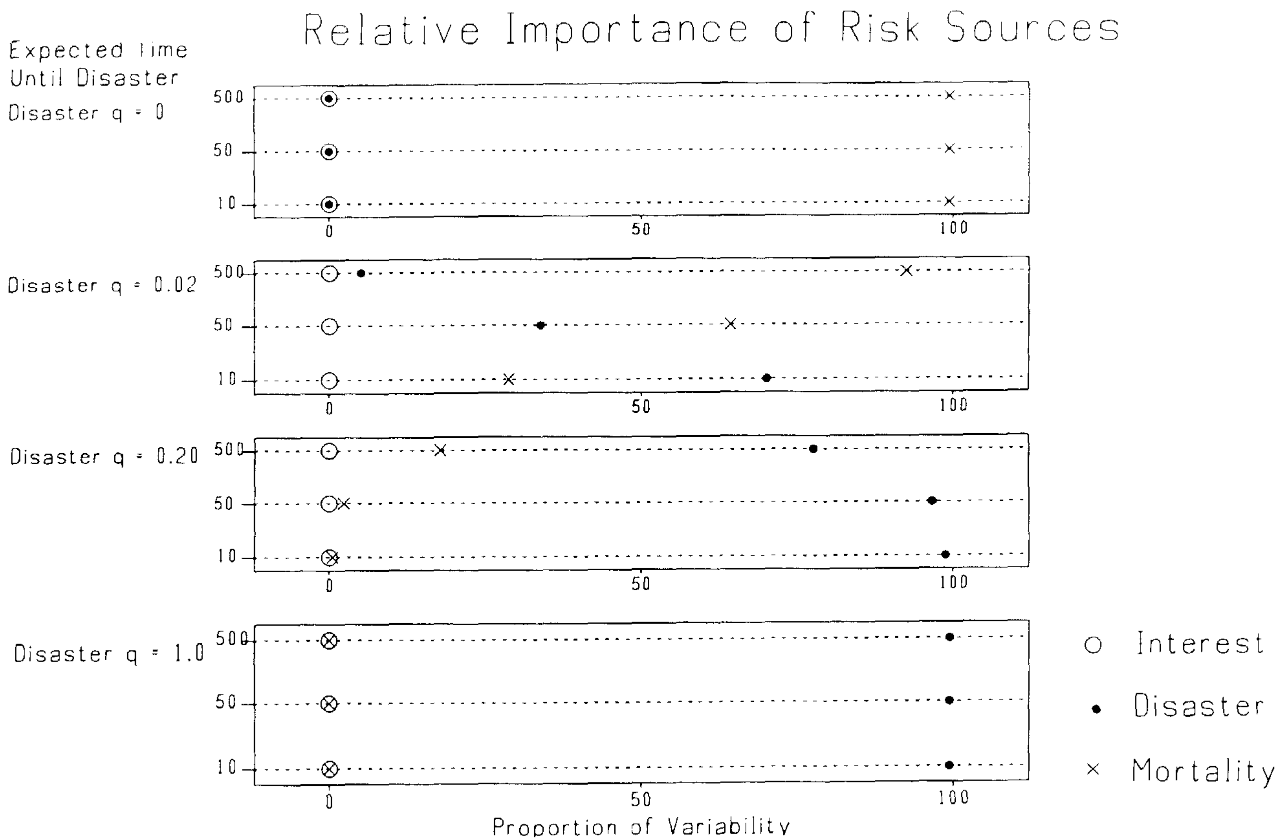

Figure 6.3. Relative Importance of Risk Sources for a Pool of $n=50$, $m=1$ Year Term Policies.

Our measure of relative importance arises from both the statistics and economics literatures. The statistics literature addresses primarily linear risks that are not necessarily monetary. However, this literature provides the richest source of prior investigations on questions of relative importance. The economics literature provides important motivation for identifying a variable, $T(X) = \mathrm{E}\,(Y \mid X)$, as the "source" of a factor in a risk Y, through the weakly less risky ordering. The idea argued in Section 5.2 is that all rational decision-makers would prefer $T(X)$ to Y and, thus, $T(X)$ captures all the important information in Y that is due to X. To assess the risk of $T(X)$, we have used the variance functional. An important area of future research is to examine the usefulness of other measures to summarize risk.

Our measure of relative importance extends naturally to the multivariate situation, where several factors may affect a risk simultaneously. The paper discussed the importance of factor hierarchies. To illustrate the relative importance measure in a multivariate situation, we considered a pool of policies that is subject to mortality, catastrophe and a common investment environment. Here, the new measure substantiated our intuition of relative importance in situations reflecting different mixes of pool sizes, term limits, relative frequency of disaster and probability of the occurrence of disaster.

Acknowledgements

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Part 1. Calculation of $\mathrm{E}\,(Y_i \mid \mathcal{F}, D)$

For $c_i = 0$, we have

$$\mathrm{E}\,(\mathrm{v}(T+1)\,\mathrm{I}(T < m) \mid \mathcal{F}, D) = \mathrm{E}\,(\mathrm{v}(T+1)\,\mathrm{I}(T < m) \mid \mathcal{F}) = \Sigma_{k=1}^{m}\, \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x\,.$$

Here, ${}_{k\,|}\mathrm{q}_x = \mathrm{Prob}\,(T = k)$ for a life aged x.

For $c_i = 1$, we consider $\mathrm{E}\,(\mathrm{v}(\min(T, Z) + 1)\,\mathrm{I}(\min(T, Z) < m) \mid \mathcal{F}, D)$. If $Z \geq m$, this reduces to the case of $c_i = 0$. If $Z < m$, this reduces to:

$$\mathrm{E}\,(\mathrm{v}(\min(T, Z) + 1) \mid \mathcal{F}, D) = \Sigma_{k=1}^{Z}\, \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x + {}_Z\mathrm{p}_x\, \mathrm{v}(Z+1). \tag{A.1}$$

Putting these calculations together, we have:

$$\mathrm{E}\,(Y_i \mid \mathcal{F}, D) = \big(\mathrm{I}(c_i = 0) + \mathrm{I}(c_i = 1)\mathrm{I}(Z \geq m)\big)\,\big(\Sigma_{k=1}^{m}\, \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x\big) \tag{A.2}$$

$$+\ \mathrm{I}(c_i = 1)\mathrm{I}(Z < m)\,\big(\Sigma_{k=1}^{Z}\, \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x + {}_Z\mathrm{p}_x\, \mathrm{v}(Z+1)\big)$$

Part 2. Calculation of $\mathrm{E}\,(Y_i \mid \mathcal{F})$

Using the law of iterated expectations, we have $\mathrm{E}\,(Y_i \mid \mathcal{F}) = \mathrm{E}\,(\,\mathrm{E}\,(Y_i \mid \mathcal{F}, D) \mid \mathcal{F})$. Thus, in equation (A.2), the difficult term to evaluate corresponds to the case where $c_i = 1$ and $Z < m$. Thus, we consider

$$\mathrm{E}\,\Big(\mathrm{I}(Z < m)\,\Big(\sum_{k=1}^{Z} \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x + {}_Z\mathrm{p}_x\, \mathrm{v}(Z+1)\Big) \;\Big|\; \mathcal{F}\Big)$$

$$= (1-e^{-\lambda}) \sum_{s=0}^{m-1} e^{-\lambda s} \Big(\sum_{k=1}^{s} \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x + {}_s\mathrm{p}_x\, \mathrm{v}(s+1)\Big)$$

with $\mathrm{Prob}(Z = s) = (1-e^{-\lambda})\, e^{-\lambda s}$. Interchanging the order of summation yields

$$(1-e^{-\lambda}) \sum_{s=0}^{m-1} e^{-\lambda s}\ {}_s\mathrm{p}_x\, \mathrm{v}(s+1) + (1-e^{-\lambda}) \sum_{s=1}^{m-1} \sum_{k=1}^{s} e^{-\lambda s}\, \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x$$

$$= (1-e^{-\lambda}) \sum_{k=1}^{m} e^{-\lambda(k-1)}\ {}_{k-1}\mathrm{p}_x\, \mathrm{v}(k) + (1-e^{-\lambda}) \sum_{k=1}^{m-1} \sum_{s=k}^{m-1} e^{-\lambda s}\, \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x$$

$$= (e^{\lambda}-1) \sum_{k=1}^{m} e^{-\lambda k}\ {}_{k-1}\mathrm{p}_x\, \mathrm{v}(k) + \sum_{k=1}^{m-1} \mathrm{v}(k)\ {}_{k-1\,|}\mathrm{q}_x\, (e^{-\lambda k} - e^{-\lambda m})$$

$$= \sum_{k=1}^{m} \mathrm{v}(k)\, \mathrm{f}_1(k, m),$$

where $\mathrm{f}_1(k, m) = (e^{\lambda}-1)\, e^{-\lambda k}\ {}_{k-1}\mathrm{p}_x + {}_{k-1\,|}\mathrm{q}_x\, (e^{-\lambda k} - e^{-\lambda m})$.

Thus, from equation (A.2) and the notation $q = \mathrm{Prob}(c = 1)$, we have

$$\mathrm{E}\,(Y_i \mid \mathcal{I}) = (1 - q + q\, e^{-\lambda m}) \sum_{k=1}^{m} v(k)\; {}_{k-1\,|}q_x + q \sum_{k=1}^{m} v(k)\, f_1(k, m)$$

$$= \sum_{k=1}^{m} v(k)\, f_2(k, m), \tag{A.3}$$

where

$$f_2(k, m) = (1 - q + q\, e^{-\lambda m})\; {}_{k-1\,|}q_x + q\, f_1(k, m)$$

$$= (1 - q)\; {}_{k-1\,|}q_x + q\, e^{-\lambda k} \left((e^{\lambda}-1)\; {}_{k-1}p_x + {}_{k-1\,|}q_x\right). \tag{A.4}$$

Part 3. Calculation of $\mathrm{E}\,(Y_i \mid D)$

Again, using the law of iterated expectations, we have $\mathrm{E}\,(Y_i \mid D) = \mathrm{E}\,(\,\mathrm{E}\,(Y_i \mid \mathcal{I}, D) \mid D)$. For the $MA(1)$ model, from Proposition 1 of Frees (1990), we have $\mathrm{E}\, v(k) = C_1 \exp(-k\,\delta_1)$. Thus, from equation (A.2), we have

$$\mathrm{E}\,(Y_i \mid D) = C_1 \Big(\big(\mathrm{I}(c_i = 0) + \mathrm{I}(c_i = 1)\mathrm{I}(Z \geq m)\big) \big(\Sigma_{k=1}^{m} \exp(-k\,\delta_1)\; {}_{k-1\,|}q_x\big)$$

$$+\ \mathrm{I}(c_i = 1)\mathrm{I}(Z < m) \big(\Sigma_{k=1}^{Z} \exp(-k\,\delta_1)\; {}_{k-1\,|}q_x + {}_{Z}p_x \exp(-(Z+1)\delta_1)\big) \Big)$$

$$= C_1 \Big(\Sigma_{k=1}^{m} \exp(-k\,\delta_1)\; {}_{k-1\,|}q_x \tag{A.5}$$

$$+\ \mathrm{I}(c_i = 1)\mathrm{I}(Z < m) \big({}_{Z}p_x \exp(-(Z+1)\delta_1) - \Sigma_{k=Z+1}^{m} \exp(-k\,\delta_1)\; {}_{k-1\,|}q_x \big) \Big).$$

Part 4. Calculation of $\mathrm{E}\,(Y_i \mid M)$

From equation (6.6), we have

$$\mathrm{E}\,(Y_i \mid M) = C_1 \big(\mathrm{I}(T_i < m)\,(1 - q) \exp(-(T_i+1)\delta_1) + q\, \mathrm{E}\,(\exp(-(\min(T_i, Z)+1)\delta_1)\mathrm{I}(Z<m) \mid T_i) \big)$$

$$= C_1 \Big(\mathrm{I}(T_i < m)\,(1 - q) \exp(-(T_i+1)\delta_1) + q\,(1-e^{-\lambda}) \sum_{s=0}^{m-1} \exp(-\lambda s-(\min(T_i, s)+1)\delta_1) \Big). \tag{A.6}$$

Part 5. Calculation of $\mathrm{E}\, Y_i$

From equation (A.3), we have immediately

$$\mathrm{E}\, Y_i = C_1 \sum_{k=1}^{m} \exp(-k\,\delta_1)\, f_2(k, m), \tag{A.7}$$

where f_2 is defined in equation (A.4).

Part 6. Calculation of $R_Y(\mathcal{I})$

For the $MA(1)$ model, from Proposition 2 of Frees (1990), we have E $v(k)^2 = C_2 \exp(-k\,\alpha_1)$ and, for $s < r$, E $v(s)v(r) = C_3 \exp(-s\,\alpha_1 - (r-s)\delta_1)$. Using this and equation (A.3) yields

$$R_Y(\mathcal{I}) = \mathrm{Var}(\,E(Y \mid \mathcal{I})\,) = \mathrm{Var}\,(n \sum_{k=1}^{m} v(k)\, f_2(k,m)\,)$$

$$= C_2 n^2 \sum_{k=1}^{m} \exp(-k\,\alpha_1)\, f_2(k,m)^2 + 2C_3 n^2 \sum_{r=2}^{m} \sum_{s=1}^{r-1} \exp(-s\,\alpha_1 - (r-s)\delta_1) f_2(s,m) f_2(r,m)\,.$$

$$-(\,C_1 \sum_{k=1}^{m} \exp(-k\,\delta_1)\, f_2(k,m)\,)^2\,.$$

Part 7. Calculation of $R_Y(D)$

To calculate $R_Y(D) = \mathrm{Var}(\,E(Y \mid D)\,)$, we first note from equation (A.5) that $E(Y_i \mid D)$ still depends on i. Thus, we use the law of total variation to isolate the common disaster and the succumbing to disaster that is individual-specific, as follows.

First, using equation (A.5), denote the conditional expectation

$$E\,(Y \mid D) = C_1\,(\,n\,A_1 + \sum_{i=1}^{n} I(c_i = 1) f_3(Z,m)\,)\,.$$

Here, $A_1 = \Sigma_{k=1}^{m} \exp(-k\,\delta_1)\; {}_{k-1\,|}q_x$ is the pure premium for an m-year term policy at δ_1 force of interest and

$$f_3(s,m) = I(s < m)\,(\,{}_s p_x \exp(-(s+1)\delta_1) - \sum_{k=s+1}^{m} \exp(-k\,\delta_1)\; {}_{k-1\,|}q_x\,)\,. \qquad \text{(A.8)}$$

Now, with the law of total variation, we have

$$R_Y(D) = \mathrm{Var}(\,E(Y \mid D)\,) = \mathrm{Var}(\,E(\,E(Y \mid D) \mid Z)) + E(\,\mathrm{Var}(\,E\,(Y \mid D) \mid Z)\,). \qquad \text{(A.9)}$$

The first term on the right-hand side of equation (A.9) is

$$\mathrm{Var}(\,E(\,E(Y \mid D) \mid Z)) = C_1^2\,\mathrm{Var}(\,E(\,n\,A_1 + \sum_{i=1}^{n} I(c_i = 1) f_3(Z,m)) \mid Z\,)$$

$$= C_1^2\,\mathrm{Var}(\,n\,A_1 + n\,q\,f_3(Z,m)\,) = C_1^2\, n^2\, q^2\,\mathrm{Var}(\,f_3(Z,m)\,).$$

The second term on the right-hand side of equation (A.9) is

$$E(\ \mathrm{Var}(\ E\ (Y \mid D) \mid Z)\ ) = C_1^2\ E(\ \mathrm{Var}(n\,A_1\ +\ \sum_{i=1}^{n} I(c_i = 1) f_3(Z,m)\ \mid Z)\ )$$

$$= C_1^2\ n\ q\ (1 - q)\ E(\ f_3(Z,m)^2\ ).$$

Thus,

$$R_Y(D) = C_1^2\ n\ q\ (\ n\ q\ \mathrm{Var}(\ f_3(Z,m)\ ) + (1 - q)\ E(\ f_3(Z,m)^2\ )\ ). \qquad (A.10)$$

Part 8. Calculation of $R_Y(M)$

From equation (A.6), we have

$$R_Y(M) = \mathrm{Var}(\ E(Y \mid M)\ ) = C_1^{\,2}\ n\ \mathrm{Var}(\ f_4(T,m,\delta_1)\ ),$$

where

$$f_4(t,m,\beta) = (1\text{-}q)\ I(t<m)\ \exp(-(t+1)\beta) + q\ (1\text{-}e^{-\lambda}) \sum_{s=0}^{m-1} \exp(-\lambda s\text{-}(\min(t,s)+1)\beta). \qquad (A.11)$$

Part 9. Calculation of $R_Y(\mathcal{I}, D)$

Similar to equation (A.5), from equation (A.2) we have

$$E\ (Y \mid \mathcal{I}, D) = n\ \Sigma_{k=1}^{m}\ v(k)\ {}_{k\text{-}1\mid}q_x + \sum_{i=1}^{n} I(c_i = 1) I(Z < m)\ (\ {}_Zp_x\ v(Z+1) - \sum_{k=Z+1}^{m} v(k)\ {}_{k\text{-}1\mid}q_x\ ) \qquad (A.12)$$

Similar to equation (A.9), we condition on $\mathcal{I}$, Z and use the law of total variation. The first term is

$$E(\ \mathrm{Var}(\ E(Y \mid \mathcal{I}, D) \mid \mathcal{I}, Z\ )) = n\ q\ (1\text{-}q)\ E\ I(Z < m)({}_Zp_x\ v(Z+1) - \sum_{k=Z+1}^{m} v(k)\ {}_{k\text{-}1\mid}q_x\ )^2\ .$$

$$= n\ q\ (1\text{-}q)\ (1\text{-}e^{-\lambda}) \sum_{s=0}^{m-1} e^{-\lambda s}\ E\ ({}_sp_x\ v(s+1) - \sum_{k=s+1}^{m} v(k)\ {}_{k\text{-}1\mid}q_x\ )^2\ .$$

$$= n\ q\ (1\text{-}q)\ (1\text{-}e^{-\lambda}) \sum_{s=0}^{m-1} e^{-\lambda s}\ E\ (\ ({}_sp_x\ v(s+1))^2 + (\ \sum_{k=s+1}^{m} v(k)\ {}_{k\text{-}1\mid}q_x\ )^2 - 2\ {}_sp_x\ v(s+1) \sum_{k=s+1}^{m} v(k)\ {}_{k\text{-}1\mid}q_x\ )$$

$$= n\, q\, (1-q)\, (1-e^{-\lambda}) \sum_{s=0}^{m-1} e^{-\lambda s} \Big(\big(C_2\, {}_s p_x^{\,2} \exp(-(s+1)\alpha_1) + C_2 \sum_{k=s+1}^{m} \exp(-(k+1)\alpha_1)\, ({}_{k-1\,|}\,q_x)^2$$

$$+ 2\, C_3 \sum_{j=s+1}^{m} \sum_{k=s+1, k>j}^{m} \exp(-(j+1)\alpha_1 - (k-j)\delta_1))\; {}_{j-1\,|}\,q_x\; {}_{k-1\,|}\,q_x \tag{A.13}$$

$$- 2\, C_2\, {}_s p_x\; {}_{s\,|}\,q_x \exp(-(s+1)\alpha_1) - 2\, C_3\, {}_s p_x \sum_{k=s+2}^{m} {}_{k-1\,|}\,q_x \exp(-(s+1)\alpha_1 - (k-(s+1))\delta_1) \Big).$$

The second term is Var(E(E(Y | $\mathcal{I}$, D) | $\mathcal{I}$, Z)) = Var(E(Y | $\mathcal{I}$, Z)) = E(E(Y | $\mathcal{I}$, Z))^2 - (E Y)^2. Now, E Y can be determined directly from (A.7). From equation (A.2), we have

$$\mathrm{E}\,(Y_i \mid \mathcal{I}, Z) = (1 - q + q\mathrm{I}(Z \geq m))\, (\Sigma_{k=1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x)$$

$$+ q\, \mathrm{I}(Z < m)\, (\Sigma_{k=1}^{Z} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x + {}_Z p_x\, \mathrm{v}(Z+1))$$

$$= (\Sigma_{k=1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x) + q\, \mathrm{I}(Z < m)\, \big({}_Z p_x\, \mathrm{v}(Z+1) - \sum_{k=Z+1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x \big)\,. \tag{A.14}$$

Thus, for the first part, we have

$$\mathrm{E}(\, \mathrm{E}(Y \mid \mathcal{I}, Z)\,)^2 = n^2\, \mathrm{E}\big(\sum_{k=1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x + q\, \mathrm{I}(Z < m)\, \big({}_Z p_x\, \mathrm{v}(Z+1) - \sum_{k=Z+1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x \big) \big)^2$$

$$= n^2\, \mathrm{E}\big(\sum_{k=1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x \big)^2 + n^2 q^2\, \mathrm{E}\; \mathrm{I}(Z < m)\, \big({}_Z p_x\, \mathrm{v}(Z+1) - \sum_{k=Z+1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x \big)^2$$

$$+ 2\, n^2\, \mathrm{E}\big(\sum_{k=1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x\; q\, \mathrm{I}(Z < m)\, \big({}_Z p_x\, \mathrm{v}(Z+1) - \sum_{k=Z+1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x \big) \big)\,. \tag{A.15}$$

The second term on the right-hand side of equation (A.15) can be determined directly from equation (A.13). The first term on the right-hand side of equation (A.15) can be expressed as

$$n^2\, \mathrm{E}\big(\sum_{k=1}^{m} \mathrm{v}(k)\; {}_{k-1\,|}\,q_x \big)^2 \tag{A.16}$$

$$= n^2 \big(\sum_{k=1}^{m} C_2 \exp(-k\alpha_1)\; {}_{k-1\,|}\,q_x^{\,2} + 2\, C_3 \sum_{s=1}^{m} \sum_{r=1, s<r}^{m} \exp(-s\alpha_1 - (r-s)\delta_1)\; {}_{s-1\,|}\,q_x\; {}_{r-1\,|}\,q_x \big)\,.$$

The third term on the right-hand side of equation (A.15) can be expressed as

$$2\,n^2\,q\,(1-e^{-\lambda})\sum_{s=0}^{m-1} e^{-\lambda s}\,E\Big(\sum_{k=1}^{m} v(k)\;{}_{k-1\,|}q_x\,\big({}_sp_x\,v(s+1) - \sum_{j=s+1}^{m} v(j)\;{}_{j-1\,|}q_x\big)\Big)$$

$$= 2\,n^2\,q\,(1-e^{-\lambda})\Big(E\sum_{k=1}^{m}\sum_{j=1}^{m} v(k)\,e^{-\lambda(j-1)}\,v(j)\;{}_{k-1\,|}q_x\;{}_{j-1}p_x \;-\; E\sum_{k=1}^{m} v(k)\;{}_{k-1\,|}q_x\sum_{s=0}^{m-1}\sum_{j=s+1}^{m} v(j)\,e^{-\lambda s}\;{}_{j-1\,|}q_x\Big)$$

$$= 2\,n^2\,q\,(1-e^{-\lambda})\Big(E\sum_{k=1}^{m}\sum_{j=1}^{m} v(k)\,e^{-\lambda(j-1)}\,v(j)\;{}_{k-1\,|}q_x\;{}_{j-1}p_x \;-\; E\sum_{k=1}^{m} v(k)\;{}_{k-1\,|}q_x\sum_{j=1}^{m}\sum_{s=0}^{j-1} v(j)\,e^{-\lambda s}\;{}_{j-1\,|}q_x\Big)$$

$$= 2\,n^2\,q\,\Big(E\sum_{k=1}^{m}\sum_{j=1}^{m} v(k)\,(e^{\lambda}-1)\,e^{-\lambda j}\,v(j)\;{}_{k-1\,|}q_x\;{}_{j-1}p_x \;-\; E\sum_{k=1}^{m}\sum_{j=1}^{m} v(k)\;{}_{k-1\,|}q_x\,v(j)\,(1-e^{-\lambda j})\;{}_{j-1\,|}q_x\Big)$$

$$= 2\,n^2\,q\,\Big(E\sum_{k=1}^{m}\sum_{j=1}^{m} v(k)\,e^{-\lambda j}\,v(j)\;{}_{k-1\,|}q_x\,\big((e^{\lambda}-1)\,{}_{j-1}p_x + {}_{j-1\,|}q_x\big) - E\Big(\sum_{k=1}^{m} v(k)\;{}_{k-1\,|}q_x\Big)^2\Big). \qquad (A.17)$$

The second term on the right-hand side of (A.17) is expressed by (A.16). The first term on the right-hand side of (A.17) can be expressed as

$$2\,n^2\,q\,E\sum_{k=1}^{m}\sum_{j=1}^{m} v(k)\,e^{-\lambda j}\,v(j)\;{}_{k-1\,|}q_x\,\big((e^{\lambda}-1)\,{}_{j-1}p_x + {}_{k-1\,|}q_x\big)$$

$$= \;2\,n^2\,q\,\Big(C_2\sum_{k=1}^{m}\exp(-k\alpha_1)\exp(-k\lambda)\;{}_{k-1\,|}q_x\,\big((e^{\lambda}-1)\,{}_{k-1}p_x + {}_{k-1\,|}q_x\big)$$

$$+\;C_3\sum_{s=1}^{m}\sum_{r=1,s<r}^{m}\exp(-s\alpha_1-(r-s)\delta_1)\exp(-s\lambda)\;{}_{r-1\,|}q_x\,\big((e^{\lambda}-1)\,{}_{s-1}p_x + {}_{s-1\,|}q_x\big)$$

$$+\;C_3\sum_{s=1}^{m}\sum_{r=1,s>r}^{m}\exp(-r\alpha_1-(s-r)\delta_1)\exp(-s\lambda)\;{}_{r-1\,|}q_x\,\big((e^{\lambda}-1)\,{}_{s-1}p_x + {}_{s-1\,|}q_x\big)\Big).$$

Part 10. Calculation of R_Y

We condition on $\mathcal{I}$, Z and use the law of total variation to get

$$R_Y = \mathrm{Var}\,Y = E(\,\mathrm{Var}(Y\mid \mathcal{I}, Z)\,) \;+\; \mathrm{Var}(\,E(Y\mid \mathcal{I}, Z)\,)$$

where $\mathrm{Var}(\,E(Y\mid \mathcal{I}, Z)\,)$ was computed in Part 9. For the first term on the right-hand side, by the independence we have

$$E(\,\mathrm{Var}(Y\mid \mathcal{I}, Z)\,) = n\,E(\,\mathrm{Var}(Y_i\mid \mathcal{I}, Z)\,) \qquad (A.18)$$

To calculate this, first note that

$$\mathrm{E}(\ \mathrm{Var}(Y_i \mid \mathcal{F}, Z)\ ) = \mathrm{E}(\ \mathrm{E}(Y_i^2 \mid \mathcal{F}, Z) - (\mathrm{E}(Y_i \mid \mathcal{F}, Z))^2\ ) . \tag{A.19}$$

Equation (A.15) provides an expression for $\mathrm{E}((\mathrm{E}(Y_i \mid \mathcal{F}, Z))^2)$. From equation (6.6), we have

$$\mathrm{E}(Y_i^2 \mid \mathcal{F}, Z) = \mathrm{E}(\mathrm{I}(T_i < m)(v(T_i+1)\mathrm{I}(c_i=0) + v(\min(T_i,Z)+1)\mathrm{I}(Z<m)\mathrm{I}(c_i=1))^2 \mid \mathcal{F}, Z)$$

$$= \mathrm{E}(\ \mathrm{I}(T_i<m)\ (v(T_i+1)^2\mathrm{I}(c_i=0) + v(\min(T_i,Z)+1)^2\mathrm{I}(Z<m)\mathrm{I}(c_i=1)) \mid \mathcal{F}, Z)$$

$$= \mathrm{E}(\ \mathrm{I}(T_i<m)\ (v(T_i+1)^2(1-q) + v(\min(T_i,Z)+1)^2\mathrm{I}(Z<m)q) \mid \mathcal{F}, Z) .$$

Thus,

$$\mathrm{E}(\ \mathrm{E}(Y_i^2 \mid \mathcal{F}, Z)\ ) = (1-q)\ \mathrm{E}\ (\mathrm{I}(T_i<m)(v(T_i+1)^2) + q\ \mathrm{E}(\ v(\min(T_i,Z)+1)^2\mathrm{I}(Z<m)\ )$$

$$= (1-q)\ C_2 \sum_{k=1}^{m} \exp(-k\alpha_1)\ {}_{k-1\,|}q_x + q\ C_2\ (1-e^{-\lambda}) \sum_{s=0}^{m-1}\ \mathrm{E}\ \exp(-\lambda s-(\min(T_i,s)+1)\alpha_1)\ ) .$$

$$= C_2\ \ \mathrm{E}\ f_4(T_i,m,\alpha_1)$$

where f_4 is defined in equation (A.11). This, (A.18) and (A.19), is sufficient for the calculation of $\mathrm{E}(\ \mathrm{Var}(Y \mid \mathcal{F}, Z)\ )$ and hence R_Y.

TABLE B.1 Relative Importance of Risk Sources for a Pool of $n=50$ Term Policies. The Expected Time to Disaster is $\lambda^{-1} = 50$ years.

m	q	E Y	ρ^2_{YI}	ρ^2_{YD}	ρ^2_{YM}	ρ^2_{YID}	$R_Y^{1/2}$
1	0.00	0.073	0.04	0.00	99.41	0.04	0.27
	0.02	0.092	0.04	33.89	64.21	34.12	0.32
	0.20	0.262	0.02	96.62	2.26	97.17	1.41
3	0.00	0.221	0.24	0.00	98.53	0.24	0.45
	0.02	0.274	0.25	31.64	65.56	32.28	0.54
	0.20	0.752	0.11	95.58	2.48	96.86	2.28
5	0.00	0.371	0.62	0.00	97.49	0.62	0.58
	0.02	0.454	0.65	29.46	66.68	30.66	0.68
	0.20	1.201	0.27	94.49	2.73	96.53	2.75
10	0.00	0.768	2.59	0.00	93.97	2.59	0.80
	0.02	0.910	2.67	24.36	68.29	27.82	0.91
	0.20	2.187	1.06	91.47	3.43	95.59	3.34
20	0.00	1.718	13.29	0.00	80.94	13.29	1.13
	0.02	1.928	13.21	15.62	64.48	29.65	1.24
	0.20	3.825	4.95	83.76	5.09	93.35	3.63
50	0.00	6.654	84.32	0.00	13.47	84.32	3.04
	0.02	6.875	83.26	1.81	12.61	85.14	3.10
	0.20	8.861	52.45	39.72	4.78	93.71	4.28

TABLE B.2 Relative Importance of Risk Sources for a Pool of $n=50$, $m=1$ Year Term Policies.

q	λ	E Y	ρ^2_{YI}	ρ^2_{YD}	ρ^2_{YM}	ρ^2_{YID}	$R_Y^{1/2}$
0.00	0.002	0.073	0.04	0.00	99.41	0.04	0.27
	0.020	0.073	0.04	0.00	99.41	0.04	0.27
	0.100	0.073	0.04	0.00	99.41	0.04	0.27
0.02	0.002	0.075	0.04	4.97	92.55	5.04	0.27
	0.020	0.092	0.04	33.89	64.21	34.12	0.32
	0.100	0.164	0.06	70.08	28.73	70.53	0.48
0.20	0.002	0.092	0.02	77.43	17.61	77.87	0.50
	0.020	0.262	0.02	96.62	2.26	97.17	1.41
	0.100	0.982	0.06	98.75	0.52	99.35	2.94
1.00	0.002	0.169	0.00	99.45	0.00	100.00	2.14
	0.020	1.019	0.01	99.44	0.00	100.00	6.67
	0.100	4.617	0.06	99.40	0.00	100.00	14.05