

Certain Limits in the Theory of Annuities

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Abstract

The article provides a complete and rigorous analysis based on calculus for the following topics in the theory of compound interest.

- i) the monotone convergence properties of the rates of interest and discount payable m times per interest conversion period in relation to the force of interest;
- ii) the monotone convergence properties of the present value and accumulated value of an ordinary annuity with m payments per interest conversion period in relation to those of a continuously paying ordinary annuity;
- iii) a method for approximating the present value of an ordinary annuity, or an increasing annuity with m payments per interest conversion period, by accumulating, at compound interest for the fraction $(m - 1)/2m$ of an interest conversion period, the present value of the same type of annuity with one payment per interest conversion period.

The presentation is based on the concept of the derivative and the definite integral of the accumulation function for compound interest and can be visualized graphically. The exposition is largely self-contained.

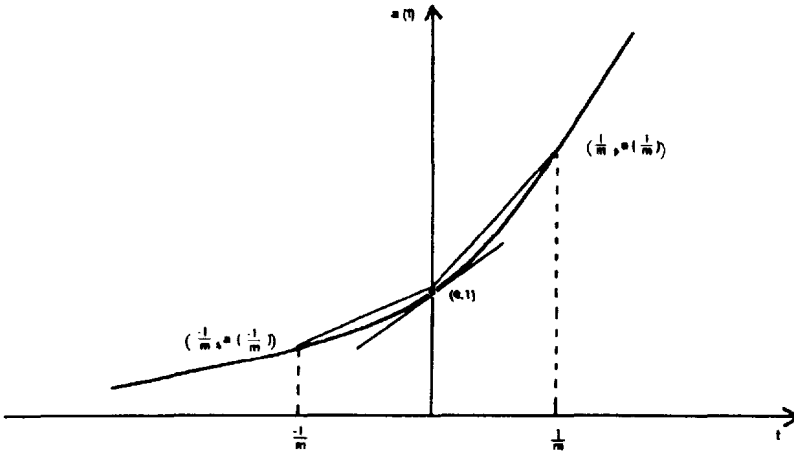
The purpose of this article is to provide a rigorous yet elementary analysis of the limiting properties for: i) certain rates of interest and discount, payable several times per interest conversion period; ii) the present value and accumulated value of related ordinary annuities immediate or due, which involve several payments per interest conversion period. In addition, a method is given for approximating the present value of an ordinary annuity or an increasing annuity with multiple payments per interest conversion period by the present value of an annuity of the same type with one payment per interest conversion period. It provides a more complete treatment of this important topic than that found in the standard texts on compound interest and annuities. The approach of presentation is based on the concept of derivative and definite integral of the exponential accumulation function and is easily visualized

geometrically. It has been used by the author for dealing with this topic in teaching an intermediate level course on the mathematics of finance to actuarial science students that has calculus as prerequisite. The exposition is largely self contained for the benefit of the reader who is only slightly acquainted with the theory of compound interest as developed in Broverman [1] or Kellison [3].

When money accumulates at compound interest, the effective rate of interest i and the effective rate of discount d per interest conversion period, usually per year, are constant over time. Furthermore, i measures the interest earned per unit of money invested at the beginning of each period, whereas d measures the interest earned per unit of accumulated value at the end of each period, and $(1 + i)(1 - d) = 1$. The nominal rates of interest $i^{(m)}$ and discount $d^{(m)}$ payable m times per interest conversion period are defined by the equations

$$(1 + i^{(m)}/m)^m = 1 + i, (1 - d^{(m)}/m)^m = 1 - d \tag{1}$$

This makes $i^{(m)}/m$ the effective rate of interest and $d^{(m)}/m$ the effective rate of discount per m th of an interest conversion period. Note $i^{(1)} = i$ and $d^{(1)} = d$. Under compound interest, the accumulation function for a \$1 at the end of time t is given by $a(t) = (1 + i)^t$. On the other hand, the present value of a \$1 due at the end of t years is given by the reciprocal of $a(t)$, the present value function or discounting factor for year t . The force of interest $\delta = a'(t)/a(t) = \ln(1 + i)$ is constant over time, and it represents the slope $a'(0)$ of the tangent to the graph of $a(t)$ at $t = 0$, as shown in the figure below.



Theorem 1

(a) $i^{(m)}$ is a decreasing sequence that converges to the force of interest δ as m increases indefinitely.

(b) $d^{(m)}$ is an increasing sequence that converges to the force of interest δ as m increases indefinitely.

(c) $d < d^{(2)} \dots < d^{(m)} \dots < \delta < \dots < i^{(m)} \dots < i^{(2)} < i$.

Proof. Consider the accumulation function $a(t) = (1+i)^t$. This is the exponential function whose graph is shown in the figure above. The graph of $a(t)$ is convex, i.e., concave up, because $a''(t) = \ln^2(1+i)(1+i)^t > 0$ for $i > -1$. From equation (1) it follows that

$$i^{(m)} = m[(1+i)^{1/m} - 1].$$

Thus, $i^{(m)}$ represents the slope of the chord, shown in the figure above which joins the point $(1/m, a(1/m))$ to the point $(0,1)$ on the graph of $a(t)$. Therefore, $i^{(m)}$ is a decreasing sequence which converges to $\ln(1+i)$, the slope $a'(0)$ of $a(t)$ at $t = 0$. This proves (a). The proof for (b) is similar. From equation (1) it follows that

$$d^{(m)} = m[1 - (1-d)^{1/m}] = m[1 - (1+i)^{-1/m}].$$

Now, $d^{(m)}$ represents the slope of the chord, which joins the point $(-1/m, a(-1/m))$ to the point $(0,1)$ on the graph of $a(t)$ and, consequently, it is an increasing sequence whose limit is $\ln(1+i)$, the slope $a'(0)$ of $a(t)$ at $t = 0$. Finally, property (c) follows directly from (a) and (b).

We now consider an annuity immediate that pays $\$(1/m)$ at the end of each m th of an interest conversion period for n periods. This is equivalent to an annuity immediate of mn payments of $\$(1/m)$, where the effective rate of interest and discount per payment period are $i^{(m)}/m$ and $d^{(m)}/m$, respectively. Therefore, the formulas for its present value $a_{\overline{mn}|}^{(m)}$ and its accumulated value $s_{\overline{mn}|}^{(m)}$ are obtained from those for an ordinary annuity $a_{\overline{n}|}$ and $s_{\overline{n}|}$ on replacing n by nm , i by $i^{(m)}/m$, and d by $d^{(m)}/m$, respectively. Thus

$$a_{\overline{mn}|}^{(m)} = \frac{1}{m} \frac{1 - (1 + i^{(m)}/m)^{-mn}}{i^{(m)}/m} = \frac{1 - (1 + i)^{-n}}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|} \quad (2)$$

$$s_{\overline{mn}|}^{(m)} = \frac{1}{m} \frac{(1 + i^{(m)}/m)^{mn} - 1}{i^{(m)}/m} = \frac{(1 + i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} s_{\overline{n}|} \quad (3)$$

Similarly, if each payment of $\$(1/m)$ is made at the beginning of each m th of an interest conversion period for n periods, then the corresponding formulas for such an annuity due are:

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m} \frac{1 - (1 + i^{(m)}/m)^{-mn}}{d^{(m)}/m} = \frac{1 - (1 + i)^{-n}}{d^{(m)}/m} = \frac{d}{d^{(m)}} \ddot{a}_{\overline{n}|} \quad (4)$$

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{1}{m} \frac{(1 + i^{(m)}/m)^{mn} - 1}{d^{(m)}} = \frac{(1 + i)^n - 1}{d^{(m)}} = \frac{d}{d^{(m)}} \ddot{s}_{\overline{n}|} \quad (5)$$

Finally we recall the formulas for the present value $\ddot{a}_{\overline{n}|}$ and accumulated value $\ddot{s}_{\overline{n}|}$ of an annuity that pays continuously at the rate of a \$1 per interest conversion period for n periods. They are

$$\ddot{a}_{\overline{n}|} = \int_0^n (1 + i)^{-t} dt = \frac{1 - (1 + i)^{-n}}{\delta} \quad (6)$$

$$\ddot{s}_{\overline{n}|} = \int_0^n (1 + i)^t dt = \frac{(1 + i)^n - 1}{\delta} \quad (7)$$

For a more extensive discussion of the theory of annuities we refer the reader to Broverman [1], chapter 2 or Kellison [4], chapters 3 and 4. The limit properties for these values listed in Theorem 2 below are now immediate consequences of those for the rates $i^{(m)}$ and $d^{(m)}$ established in Theorem 1 and the formulas in equations (2) through (7).

Theorem 2

(a) $a_{\overline{n}|}^{(m)}$ and $s_{\overline{n}|}^{(m)}$ are both increasing sequences that converge to $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$ respectively, as m increases indefinitely.

(b) $\ddot{a}_{\overline{n}|}^{(m)}$ and $\ddot{s}_{\overline{n}|}^{(m)}$ are both decreasing sequences that converge to $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$, respectively, as m increases indefinitely.

(c) $a_{\overline{n}|} < a_{\overline{n}|}^{(2)} \dots < a_{\overline{n}|}^{(m)} \dots < \bar{a}_{\overline{n}|} \dots < \ddot{a}_{\overline{n}|}^{(m)} \dots < \ddot{a}_{\overline{n}|}^{(2)} < \ddot{a}_{\overline{n}|}$

(d) $s_{\overline{n}|} < s_{\overline{n}|}^{(2)} \dots < s_{\overline{n}|}^{(m)} \dots < \bar{s}_{\overline{n}|} \dots < \ddot{s}_{\overline{n}|}^{(m)} \dots < \ddot{s}_{\overline{n}|}^{(2)} < \ddot{s}_{\overline{n}|}$

There is an alternative method for establishing Theorem 2 that is independent of the proof of Theorem 1. As a result Theorem 1 is derivable from Theorem 2 and the formulas given by equations (2) and (4). This approach is based on the observation that each pair $s_{\overline{n}|}^{(m)}$, $\ddot{s}_{\overline{n}|}^{(m)}$, and $a_{\overline{n}|}^{(m)}$, $\ddot{a}_{\overline{n}|}^{(m)}$ can be interpreted as two Riemann sums that represent a lower and an upper approximation for the area under the graph of the exponential functions $a(t)$ and $1/a(t)$, respectively, from $t = 0$ to $t = n$. For a more extensive discussion of Riemann sums and the concept of the definite integral we refer the reader to Thomas and Finney [5].

From the definition of $s_{\overline{n}|}^{(m)}$, $\ddot{s}_{\overline{n}|}^{(m)}$, and the fact that the effective accumulation factor for each m th of an interest conversion period is constant, namely $(1 + i)^{1/m}$, or, equivalently, the effective rate of interest per m th of an interest conversion period is $(1 + i)^{1/m} - 1$, we have

$$s_{\overline{n}|}^{(m)} = \sum_{k=1}^{mn} (1 + i)^{(k-1)/m} \cdot \frac{1}{m}, \quad \ddot{s}_{\overline{n}|}^{(m)} = \sum_{k=1}^{mn} (1 + i)^{k/m} \cdot \frac{1}{m}$$

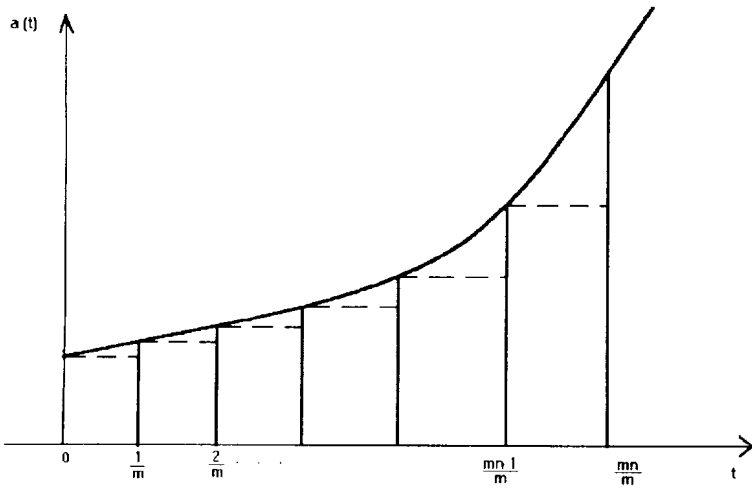


Figure 1

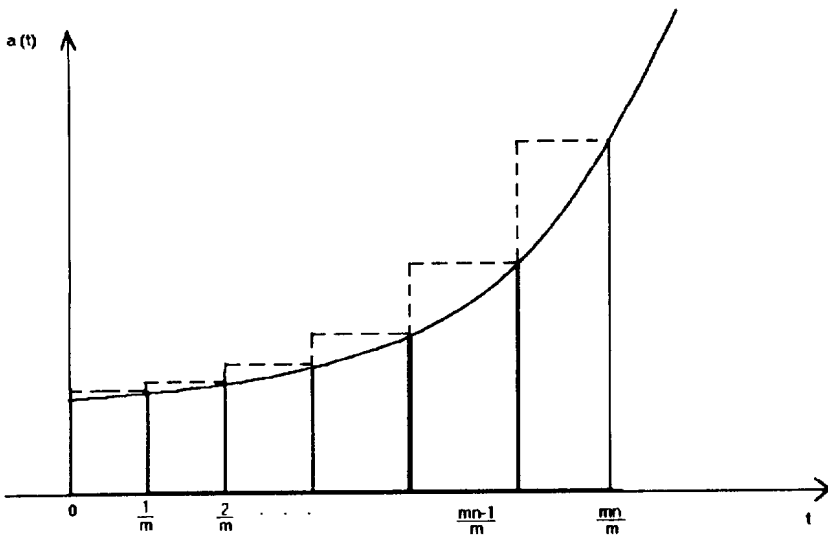


Figure 2

$s_{\bar{n}}^{(m)}$ and $\check{s}_{\bar{n}}^{(m)}$ represent the lower and upper Riemann sums for the definite integral $\bar{a}_{\bar{n}}$ of $a(t)$ over the interval $[0, n]$, as shown in figure 1 and 2 above. These Riemann sums are based on the partition $\{0, 1/m, 2/m, \dots, mn/m = n\}$ of the interval, where the mn division points are equidistant with a spacing equal to $1/m$ which decreases with m increasing.

Thus $s_{\bar{n}}^{(m)}$ is the total area of the rectangular regions in figure 1 that are included below the graph of $a(t)$, whereas $\check{s}_{\bar{n}}^{(m)}$ is the total area of the rectangular regions in figure 2 whose union includes the area under the graph of $a(t)$. Therefore, $s_{\bar{n}}^{(m)}$ is an increasing sequence, $\check{s}_{\bar{n}}^{(m)}$ is a decreasing sequence and, as m increases indefinitely, both converge to the area $\bar{a}_{\bar{n}}$ under the graph of $a(t)$ from $t = 0$ to $t = n$. This proves property (d) and the second half of (a) and (b) of Theorem 2. A similar argument, when applied to the function $1/a(t)$, yields property (c) and the rest of these properties.

An advantage of the second technique is that it can be used to obtain a better estimate for the rate of convergence of the sequences $s_{\bar{n}}^{(m)}$ and $\check{s}_{\bar{n}}^{(m)}$, $a_{\bar{n}}^{(m)}$ and $\check{a}_{\bar{n}}^{(m)}$, $i^{(m)}$ and $d^{(m)}$ to their corresponding limits.

Theorem 3

- (a) $\bar{a}_{\bar{n}} - s_{\bar{n}}^{(m)} < \frac{i}{m} s_{\bar{n}}$ and $\check{s}_{\bar{n}}^{(m)} - \bar{a}_{\bar{n}} < \frac{d}{m} \check{s}_{\bar{n}}$
 (b) $\bar{a}_{\bar{n}} - a_{\bar{n}}^{(m)} < \frac{1}{m} a_{\bar{n}}$ and $\check{a}_{\bar{n}}^{(m)} - \bar{a}_{\bar{n}} < \frac{d}{m} \check{a}_{\bar{n}}$
 (c) $\frac{1}{\delta} - \frac{1}{m} < \frac{1}{i^{(m)}} < \frac{1}{\delta} < \frac{1}{d^{(m)}} < \frac{1}{\delta} + \frac{1}{m}$

Proof. To prove (a) we observe that the total excess in area between the rectangular regions, whose respective areas are $s_{\bar{n}}^{(m)}$ and $\check{s}_{\bar{n}}^{(m)}$ as shown in figure 1a and 1b, is larger than that by which each differs from the entire area under the graph of $a(t)$ from $t = 0$ to $t = n$. That is

$$\begin{aligned} \check{s}_{\bar{n}}^{(m)} - \bar{a}_{\bar{n}} &< \check{s}_{\bar{n}}^{(m)} - s_{\bar{n}}^{(m)} \\ &= \frac{1}{m} \sum_{k=1}^{mn} [(1+i)^{k/m} - (1+i)^{(k-1)/m}] \\ &= \frac{(1+i)^n - 1}{m} = \frac{i}{m} s_{\bar{n}} = \frac{d}{m} \check{s}_{\bar{n}} \end{aligned}$$

Note that the sum telescopes to $((1+i)^n - 1)/m$, reflecting the fact that the two annuities have identical payments except those at time n and at time zero. (b) follows from (a) by discounting the inequalities in (b) or from a similar argument applied to the area under the graph of $1/a(t)$. (c) follows upon substituting the formulas given by equations (3), (5) and (7) in (a) and simplifying the resulting expressions.

Theorem 4

(a) $(1+i)^{\frac{m-1}{2m}} \leq \frac{i}{i^{(m)}} \leq (1+\frac{m-1}{2m}i)$

(b) $(1-d)^{\frac{m-1}{2m}} \leq \frac{d}{d^{(m)}} \leq (1-\frac{m-1}{2m}d)$

(c) $a_{\overline{m}}(1+i)^{\frac{m-1}{2m}} \leq a_{\overline{m}}^{(m)} \leq a_{\overline{m}}(1+\frac{m-1}{2m}i)$

(d) $\ddot{a}_{\overline{m}}(1-d)^{\frac{m-1}{2m}} \leq \ddot{a}_{\overline{m}}^{(m)} \leq \ddot{a}_{\overline{m}}(1-\frac{m-1}{2m}d)$

(e) $(Ia)_{\overline{m}}(1+i)^{\frac{m-1}{2m}} \leq (Ia)_{\overline{m}}^{(m)} \leq (Ia)_{\overline{m}}(1+\frac{m-1}{2m}i)$.

Proof. For the proof of the lower bound for $i/i^{(m)}$ in (a) we use the formula

$$(q^m - 1)/(q - 1) = 1 + q + \dots + q^{m-1}$$

for the sum of finite geometric series with $q = (1+i)^{1/m}$, and the classical inequality that relates the geometric mean of a finite set of positive real numbers q_0, q_1, \dots, q_{m-1} to their arithmetic mean, where $q_j = q^j, j = 0, 1, \dots, m-1$. That is

$$\sqrt[m]{q_0 q_1, \dots, q_{m-1}} \leq \frac{1}{m}(q_0 + q_1 + \dots + q_{m-1})$$

Thus we have

$$\begin{aligned} \frac{i}{i^{(m)}} &= \frac{1}{m} \frac{((1+i)^{1/m})^m - 1}{(1+i)^{1/m} - 1} \\ &= \frac{1}{m} \left[1 + (1+i)^{1/m} + ((1+i)^{1/m})^2 + \dots + ((1+i)^{1/m})^{m-1} \right] \\ &\geq \sqrt[m]{((1+i)^{1/m})^{1+2+\dots+(m-1)}} = (1+i)^{\frac{m(m-1)}{2m^2}} = (1+i)^{\frac{m-1}{2m}} \\ &= (1+i)^{\frac{m-1}{2m}} \end{aligned}$$

For $j = 1, 2, \dots, m-1$ we have

$$\begin{aligned} \frac{m}{ij} \left[(1+i)^{j/m} - 1 \right] &= \frac{1}{i} \int_0^1 (1+x)^{j/m-1} dx \\ &\leq \frac{1}{i} \int_0^1 1 dx = 1 \end{aligned}$$

Hence $(1+i)^{j/m} \leq 1 + \frac{j}{m} i$

$$\begin{aligned} \frac{i}{i^{(m)}} &= \frac{1}{m} \left[1 + (1+i)^{1/m} + (1+i)^{2/m} + \dots + (1+i)^{(m-1)/m} \right] \\ &\leq \frac{1}{m} \left[1 + (1+\frac{i}{m}) + (1+\frac{2i}{m}) + \dots + (1+\frac{m-1}{m}i) \right] \\ &= \frac{1}{m} \left[m + \frac{i}{m}(1+2+\dots+(m-1)) \right] \\ &= 1 + i \frac{m(m-1)}{2m^2} = 1 + \frac{m-1}{2m}i \end{aligned}$$

The proof for (b) is identical with that for (a), except that in this case $q = (1 - d)^{1/m}$.

Finally (c), (d) and (e) follow immediately from (a) and (b), since

$$a_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} a_{\overline{n}|} \quad \ddot{a}_{\overline{n}|}^{(m)} = \frac{d}{d^{(m)}} \ddot{a}_{\overline{n}|}$$

and

$$(Ia)_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} (Ia)_{\overline{n}|}$$

for an increasing annuity. This completes the proof of Theorem 4.

The usual approximation for $a_{\overline{n}|}^{(m)}$ in terms of $a_{\overline{n}|}$ found in the standard textbooks is given by

$$a_{\overline{n}|}^{(m)} \doteq a_{\overline{n}|} + \frac{m-1}{2m} (1 - (1+i)^{-n}) = a_{\overline{n}|} \left(1 + \frac{m-1}{2m} i \right). \quad (8)$$

This is the upper bound for $a_{\overline{n}|}^{(m)}$ in Theorem 4(c) and it is tantamount to approximating $a_{\overline{n}|}^{(m)}$ by accumulating the present value $a_{\overline{n}|}$, at simple interest for the fraction $\frac{m-1}{2m}$ of an interest conversion period. On the other hand, the lower bound for $a_{\overline{n}|}^{(m)}$ in Theorem 4(c) provides an approximation for $a_{\overline{n}|}^{(m)}$ by accumulating the present value $a_{\overline{n}|}$, at compound interest for the fraction $\frac{m-1}{2m}$ of an interest conversion period. That is

$$a_{\overline{n}|}^{(m)} \doteq a_{\overline{n}|} (1+i)^{\frac{m-1}{2m}} \quad (9)$$

However, an examination of the interest tables show that the lower bound $(1+i)^{(m-1)/2m}$ provides an approximation for $i/i^{(m)}$, which is far more accurate than that by the upper bound $1 + \frac{m-1}{2m}i$, since it agrees to at least three decimal places for rates between 1% and 12%. Hence, equation (9) provides a better approximation for $a_{\overline{n}|}^{(m)}$ than equation (8).

The same property is also valid for the lower bounds in (d) and (e) which provide better approximations than the corresponding upper bounds for

$$\ddot{a}_{\overline{n}|}^{(m)} \quad \text{and} \quad (Ia)_{\overline{n}|}^{(m)}.$$

On taking limits in Theorem 4(a) and (b), we get

$$(1+i)^{\frac{1}{2}} \leq \frac{i}{\delta} \leq 1 + \frac{1}{2}i \quad (10)$$

$$(1-d)^{\frac{1}{2}} \leq \frac{d}{\delta} \leq 1 - \frac{1}{2}d. \quad (11)$$

As before, a glance at the interest tables confirms that the lower bounds in (10) and (11) provide better approximations for i/δ and d/δ than the upper bounds.

Consequently, the time it takes for money to double at compound interest can be approximated as follows

$$\frac{\ln 2}{\ln(1+i)} = \frac{\ln 2}{i} \cdot \frac{i}{\delta} = \frac{.6931}{i} \sqrt{1+i}.$$

The latter is a more accurate formula than the rule of 72 used in Kellison [4], because it agrees with the exact time to almost two decimal places for interest rates between 1% and 25%.

The approximation $\frac{i}{\delta} \doteq \sqrt{1+i}$, is equivalent to $\delta \doteq \sqrt{id} < \frac{i+d}{2}$, i.e., the force of interest is approximately the geometric mean of the rates of interest and discount.

Finally, we note that on multiplying the inequalities (a) and (b) in Theorem 4, we obtain the following inequality,

$$i^{(m)}d^{(m)} \leq id.$$

This yields that $\delta^2 \leq id$ on letting m go to infinity.

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References

1. Broverman, S. A., *Mathematics of Investment and Credit*, ACTEX Publications, Inc., Winston and Avon, Connecticut, 1991.
2. Butcher, M. V. and Nesbitt, C.J., *Mathematics of Compound Interest*, Ulrich's Books, Inc, Ann Arbor, Michigan, 1971.
3. Kellison, S. G., *The Theory of Interest*, Second Edition, Richard D. Irwin, Inc., Homewood, Illinois, 1991.
4. McCutcheon, J. J. and Scott, W. F., *An Introduction to the Mathematics of Finance*, Oxford: Heineman Professional Publishing, 1986.
5. Thomas, G. B., Jr., and Finney, R. L., *Calculus and Analytic Geometry*, 8th Edition, Addison-Wesley Publishing Co., New York, 1992.