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VARIANCES OF LOSS FUNCTIONS FOR TERM, PURE ENDOWMENTS, AND REGULAR ENDOWMENTS

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This paper demonstrates that the variance of the loss function for n-year term is greater than the sum of the variances of the loss functions for n-year pure endowment and n-year regular endowment plans. The result is true for both single premiums and annual premium plans.

Theorem 1 If three random variables L_1, L_2, L_3, with means E_1, E_2, E_3 and variances var L_1, var L_2, var L_3 related so that ($L_3=L_1+L_2$) then Var $L_1 \geq$ Var L_2 + Var L_3 if Cov (L_2, L_3) ≤ 0.

Proof. Let

$$S = \text{var } L_1 - \text{var } L_2 - \text{var } L_3$$

$$\therefore S = \text{var } L_1 - \text{var } L_2 - \text{var } L_1 - \text{var } L_2 - 2\,\text{cov}(L_1, L_2)$$

$$= -2[\text{var } L_2 + \text{cov}(L_1, L_2)]$$

$$= -2E[(L_2 - E_2)^2 + (L_1 - E_1)(L_2 - E_2)]$$

$$= -2E[(L_2 - E_2)(L_2 - E_2 + L_1 - E_1)]$$

$$= -2E[(L_2 - E_2)(L_3 - E_3)]$$

$$= -2\,\text{cov}(L_2, L_3)$$

and $S \geq 0$ only if cov (L_2, L_3)≤ 0 q.e.d.

CASE 1 - Single Premium Plans

N-Year Term: $L_1 = v^T(1 - I_n) - \overline{A}^{\,1}_{x:\overline{n}|}$, where I_n is an indicator random variable equal to 1 if the future lifetime $T \geq n$.

N-Year Pure Endowment: $L_2 = v^n I_n - v^n_{\ n}p_x$

N-Year Regular Endowment: $L_3 = L_1 + L_2 = v^T(1 - I_n) + v^n I_n - \overline{A}_{x:\overline{n}|}$. The expected value of the random variables L_1, L_2, L_3 is zero.

$$L_2 L_3 = v^{2n} I_n - v^n{}_n p_x L_3 - \overline{A}_{x:\overline{n}|} L_2 - v^n_{\ n}p_x \overline{A}_{x:\overline{n}|}$$

$$\text{cov}(L_2, L_3) = E[L_2 L_3] = v^{2n}{}_n p_x - v^n{}_n p_x \cdot \overline{A}_{x:\overline{n}|} = v^n{}_n p_x (v^n - \overline{A}_{x:\overline{n}|}) \leq 0$$

$\therefore$ var $L_1 \geq$ var L_2 + var L_3. (applying theorem 1)

CASE 2 - Annual Premium Plans with $P_1 = \overline{P}(\overline{A}^{\,1}_{x:\overline{n}|}), P_2 = \overline{P}(A_{x:\overline{n}|}^{\ \ \,1}), P_3 = \overline{P}(\overline{A}_{x:\overline{n}|})$

N-Year Term : $L_1 = v^T(1 - I_n) - P_1[\overline{a}_{\overline{T}|}(1 - I_n) + \overline{a}_{\overline{n}|} I_n]$

N-Year Pure Endowment:

$$L_2 = v^n I_n - P_2[\overline{a}_{\overline{T|}}(1-I_n) + \overline{a}_{\overline{n|}} I_n]$$

$$= \frac{P_2}{\delta} v^T (1-I_n) + (1+\frac{P_2}{\delta}) v^n I_n - \frac{P_2}{\delta}$$

$$= \frac{1}{(1-\overline{A}_{x\overline{n|}})}[v^n {}_n p_x v^T (1-I_n) + (1-\overline{A}_{\overset{1}{x}\overline{n|}}) v^n I_n - v^n {}_n p_x]$$

N-Year Regular Endowment:

$$L_3 = v^T(1-I_n) + v^n I_n - P_3[\overline{a}_{\overline{T|}}(1-I_n) + \overline{a}_{\overline{n}} I_n]$$

$$= (1+\frac{P_3}{\delta}) v^T (1-I_n) + (1+\frac{P_3}{\delta}) v^n I_n - \frac{P_3}{\delta}$$

$$= \frac{1}{(1-\overline{A}_{x\overline{n|}})}[v^T(1-I_n) + v^n I_n - \overline{A}_{x\overline{n|}}]$$

$$\therefore L_2 L_3 = \frac{1}{(1-\overline{A}_{x\overline{n|}})^2}[v^n {}_n p_x v^{2T}(1-I_n) + (1-\overline{A}_{\overset{1}{x}\overline{n|}}) v^{2n} I_n - v^n {}_n p_x L_3 - \overline{A}_{x\overline{n|}} L_2$$

$$- v^n {}_n p_x \overline{A}_{x\overline{n}}]$$

$$\text{Cov}\,(L_2, L_3) = E[L_2 L_3] = \frac{1}{(1-\overline{A}_{x\overline{n|}})^2}[v^n {}_n p_x {}^2\overline{A}_{\overset{1}{x}\overline{n|}} + v^{2n} {}_n p_x (1-\overline{A}_{\overset{1}{x}\overline{n|}}) - v^n {}_n p_x \overline{A}_{x\overline{n|}}]$$

$$= \frac{v^n {}_n p_x}{(1-\overline{A}_{x\overline{n|}})^2}[{}^2\overline{A}_{\overset{1}{x}\overline{n|}} + v^n(1-\overline{A}_{\overset{1}{x}\overline{n|}}) - \overline{A}_{x\overline{n|}}]$$

$$= \frac{v^n {}_n p_x}{(1-\overline{A}_{x\overline{n|}})^2}[{}^2\overline{A}_{\overset{1}{x}\overline{n|}} + v^n {}_n q_x - v^n \overline{A}_{\overset{1}{x}\overline{n|}} - \overline{A}_{\overset{1}{x}\overline{n|}}]$$

$$\therefore E[L_2 L_3] = \frac{v^n {}_n p_x}{(1-\overline{A}_{x\overline{n|}})^2} \cdot E[(v^{2T} + v^n - v^{n+T} - v^T)(1-I_n)]$$

$$= \frac{v^n {}_n p_x}{(1-\overline{A}_{x\overline{n|}})^2} \cdot E[(v^T - 1)(v^T - v^n)(1-I_n)]$$

The righthand side cannot be positive.

$\therefore$ var $L_1 \geq$ var L_2 + var L_3 (applying theorem 1)

Q.E.D.

<u>Conclusion</u>

Intuitively actuaries expect term insurance to exhibit high variances. This paper confirms that intuition and shows that for both single premium and annual premium plan the variance for the term insurance is greater than the sum of the variances for the pure endowment insurance and the regular endowment insurance.