ACTUARIAL RESEARCH CLEARING HOUSE
1995 VOL. 2

Variance of Whole Life Discounted Benefit Random Variable v^T under De Moivre's Law

John Mereu

It will probably come as no surprise to the reader to learn that the variance of v^T is a function of age, that there is some age where the variance is maximized, and that the age where the maximum occurs depends on the interest rate. What might be surprising, however, is that the maximum variance itself does not vary with the interest rate and equals .06613989398, provided the force of interest δ is not less than approximately $\frac{3.245}{\omega}$.

I stumbled across this curiosity accidentally while playing with an APL program that computed the variance of v^T with age and interest rate as parameters. A mathematical proof is derived below.

As given in *Actuarial Mathematics*, by Bowers, *et al.*

$$\text{Var } v^T = \frac{{}^2\bar{a}_{\overline{\omega-x}|}}{\omega - x} - \left(\frac{\bar{a}_{\overline{\omega-x}|}}{\omega - x}\right)^2$$

$$= \frac{1-e^{-2\delta n}}{2\delta n} - \left(\frac{1-e^{-\delta n}}{\delta n}\right)^2,$$

substituting $n = \omega - y$. Then letting $y = \delta n$ we have

$$\text{Var }(v^T) = V(y) \quad = \frac{1-e^{-2y}}{2y} - \left(\frac{1-e^{-y}}{y}\right)^2.$$

V(y) is a function of only one variable, namely y. Its maximum can be approximated by using the Newton-Raphson method to find y^*, where $V'(y^*) = 0$, and then computing $V(y^*)$.

It can be shown that $V(y) = \left(\frac{1}{2y} - \frac{1}{y^2}\right) + 2e^{-y}\left(\frac{1}{y^2}\right) - e^{-2y}\left(\frac{1}{2y} + \frac{1}{y^2}\right),$

$$V'(y) = \left(-\frac{1}{2y^2} + \frac{2}{y^3}\right) + e^{-y}\left(\frac{-2}{y^2} - \frac{4}{y^3}\right) + e^{-2y}\left(\frac{1}{y} + \frac{5}{2y^2} + \frac{2}{y^3}\right),$$

and $$V''(y) = \left(\frac{1}{y^3} - \frac{6}{y^4}\right) + e^{-y}\left(\frac{2}{y^2} + \frac{8}{y^3} + \frac{12}{y^4}\right) + e^{-2y}\left(-\frac{2}{y} - \frac{6}{y^2} - \frac{9}{y^3} - \frac{6}{y^4}\right).$$

Starting with a trial value y_0, the Newton-Raphson Method (below) generates a sequence of values y_n that converges on y^*.

Newton Raphson: $y_n = y_{n-1} - \dfrac{V'(y_{n-1})}{V''(y_{n-1})}$

The value of y^* is found to be 3.245 304 482...,which leads to $V(y^*)$=.06613989398... .

Conclusion

Because the variance of v^T under De Moivre's Law can be represented as a function of a single variable $y=\delta(\omega-x)$, an inherently two-dimensional search can be simplified to a one dimensional one. The variance is maximized when $y=y^*$. Since $\delta(\omega-x)=y$, the age x^* of maximization is given by $\omega - \frac{y^*}{\delta}$, which varies with the interest rate.

If $\delta < \frac{y^*}{\omega}$, however, then x^* turns out to be negative, and a value of $y=y^*$ is not allowable. For these lower interest rates the maximum variance will occur when $x=0$. The variance there is $V(\omega\delta) < V(y^*)$.