m-Year Step-Wise Level Percent Financing; Life Insurance and Life Annuity Variances
by
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ABSTRACT

As the title indicates, this paper consists of two parts. The first concerns the financing of public systems, the second is a challenging introduction to variances encountered in life insurance and life annuity theory. The first part emerges from work by som. actuarial students and faculty at the University of Michigan during the past five years. The second part was stimulated by some developments in the preparation of the second edition of Actuarial Mathematics. As usual, matters are not fully settled in our minds, and we seek your assistance. Further applications of the first part (and its sources) can be made, and we leave much unfinished work on the second part to you.

I. FINANCING OF OLD-AGE, SURVIVORS AND DISABILITY INSURANCE (OASDI) AND OTHER PUBLIC SYSTEMS

1.1 Introduction to OASDI Financing

Basic actuarial work for the financing of OASDI is done each year by the Office of the Actuary, Social Security Administration, and is communicated in the annual reports of The Board of Trustees, Federal Old-Age, Survivors and Disability Insurance Trust Funds. The Office provides projections for the next 75 years of the annual outgo for benefits and administration, of the taxable payroll, and of the contribution income. The projections are prepared carefully on three sets of assumptions, Low Cost, Intermediate, and High Cost. At present 12.4 percent of taxable payrolls is the main source of (new money) income to OASDI. Under present projections, the income appears adequate to finance OASDI for a term of approximately 20 years.

For the past 5 years, some 15 actuarial students, with Research Experience for Undergraduates grants, Cecil Nesbitt and colleagues have studied the long-term financing of OASDI. Our starting point has been the long-term projections, and other data, co-operatively supplied by the Office of the Actuary. In such studies, we have become increasingly concerned about the actuarial balances based on summarized cost rates for extended terms of years.

To calculate such cost rates for a term of m years, one considers the discounted value of the annual outgoes for OASDI, adjusted for the fund on hand at the beginning of the m-year term,

and for the targeted fund at the end of the term. We denote the result as the _m-year summarized obligation_ for financing of OASDI. Since the current annual outgo approximates $400 billion, and is rising, one can imagine how large a sum the m-year summarized obligation must be.

To complete the process of calculating an m-year level contribution rate applicable to increasing taxable payrolls, one computes the _m-year summarized taxable payrolls_ by discounting the payrolls during the m-years to the beginning of the term. Then, except for final adjustments to take account of taxation of benefits, and other matters, the m-year summarized level percent cost rate is the quotient of the m-year summarized obligation, divided by the m-year summarized taxable payrolls.

Our studies have shown that if the term of years is 20 or more, then under present conditions the summarized level percent cost rate (applied to the increasing taxable payrolls) will provide less than interest (in then current year dollars) on the initial summarized obligation. This is strikingly confirmed by the Office of the Actuary's computation of the 75-year summarized level percent cost rate for the open group of OASDI participants for the next 75 years.

Here, the summarized obligation at the beginning of the 75-year term (without adjustments for beginning and ending funds) is $20,300 billions, a tremendous fund (see Office of the Actuary, Table 23-1, 04-03-95, Open Group Operations and Intermediate Basis). With adjustments, the summarized obligation approximates $20,070 billions.

The 75-year summarized cost rate in Table III.D1, p. 194, 1995 OASDI Trustees' Report is 15.44 percent. This cost rate provides for the adjustments.

Interest on the 75-year summarized obligation, payable continuously at the annual rate of 0.06196, amounts in year 1995 to $(0.06196)(20,070) = \$1,244$ billions.

The taxable payroll for 1995 (on the Intermediate basis) is $2,960 billions, (1995 OASDI Trustees' Report, Table III.B1, p. 177). On such basis, funding the summarized obligation starts with $(0.1544)(2,960) = \$457$ billions. This is only about one third of the $1,244 billions of interest required above on the 75-year obligation. Thus little progress would be made to fund the 75-year obligation in 1995.

The difficulty of trying to finance OASDI for 75 years is compounded by moving the process forward one year at the end of the year, and thereby considering a new huge obligation on which one pays much less than interest in the early years. By setting up obligations on which one

pays less than interest (in then current year dollars), year after year, leaves one in the dark as to what financing is being accomplished over the long-term.

1.2 A Suggested Process

To our minds, a more reasonable process is to recognize the summarized obligation for a much shorter term of $m = 12$ or 16 years. The aim would be to completely amortize the mo⌐. modest obligation within the 12 or 16 years, and at the end of the term to have a targeted reserve fund available for the succeeding term. The actual experience in the m-year term might produce year-end reserves decreasing toward less than one year's outgo; or, under favorable conditions, the year-end reserves might increase to more than the targeted fund. To control the situation, if at a year-end the reserve is decreasing toward less than one year's outgo, or, on the other hand, being greater than 2 years' discounted outgo, automatic review should be invoked. At that time, a new balance should be agreed upon for financing benefits for a new term of years. For that new term, an initial reserve of at least one year's outgo should be available.

To reach agreement on financing for the new term, would entail careful projections of benefit costs for that term, and, indeed, might include some adjustments of the benefits as future experience would indicate. But up to the time of the review, OASDI benefits would have been paid for, the m-year obligation would have been amortized appropriately, and a reserve maintained at least equal to the following year's outgo. In other words, we would see clearly what funding was being achieved.

We regard this as a realistic form of Social Security financing that can adapt as experience develops in the future. It provides a clear actuarial process for a limited term of years, and regards financing as a vehicle for paying Social Security benefits, and having no other purpose. For example, OASDI financing should not be confused with classical pension funding which can build up reserves of 20 to 40 years of benefit outgo. Such classical pension funding of OASDI might invite many abuses, and would also entail the sacrifice of much freedom of the American public. Beyond 1 or 2-year reserve financing, the participants should rely on the important third branch of the United States Government, namely, the Judicial System. The full majesty and

power of the law should be brought to bear as a guarantee of benefit payments. This is something for the future to develop.

1. 3 Further Developments

A. OASDI Financing

In prior papers, we have developed the more mathematical concept of n-year roll-forward reserve financing, and have seen the need to support such financing by m-year level percent contribution rates.[*] In this statement, one goes directly to the m-year step-wise level percent contribution rates, adjusted for the fund at the beginning and the end of the m years. We propose that year-end reserves shall be kept within the lower bound of one year's outgo and the upper bound of two-years' discounted outgoes. This m-year step-wise level contribution rate approach is more actuarial, more flexible, more easily understood, and more in keeping with the present practice of the Office of the Actuary, than the n-year roll-forward reserve concept. Both concepts have merit, however, and are closely related.

At any rate, the m-year step-wise level percent contribution rate applied to increasing taxable payrolls, for $m = 12$ or 16, is an appealing way to face the problems of financing the obligations of OASDI. This approach is close to what is now done for the Trustees' reports with, however, two simple but important modifications. First, the summarized obligation and summarized taxable payrolls are calculated for the next 12 or 16 years rather than the next 75 years. Second, the contribution rate is calculated for a fixed (not moving) term, and is maintained at that level for the full 12 or 16 years unless a review is invoked beforehand. The approach may also commend itself, under proper conditions, to the financing of other large public systems, as would also be the case for n-year roll-forward reserve financing.[†]

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[*] See the 1994.1, 1995.1, and 1996.1 issues of Actuarial Research Clearing House (ARCH), published by the Society of Actuaries, 475 N. Martingale Rd., Suite 800, Schaumburg, IL 60173-2226. A copy of the full 1995 paper (outlined in ARCH 1996.1) may be obtained by request sent to *Cecil Nesbitt, University of Michigan, Department of Mathematics, Ann Arbor, Michigan 48109-1109.*

[†] This statement is prepared for the 8th Annual Conference of the National Academy of Social Insurance, January 25, 26, 1996, and for part of a talk on unfinished ideas to the Michigan Actuarial Society, March 25, 1996. It is now offered to you for review in the future.

You will be hearing about how 1 or 2-year roll-forward reserve financing of a Bangladesh system could be set in motion through the paper by John Beekman and Md Humayun Kabir. We shall be interested in what develops.

C.   Health Insurance for Lansing, Michigan retirees from City Employment

Alan Sonnanstine and Andrew Cohen, among illustrations of a number of funding methods of such insurance, have shown how n-year roll-forward reserve financing for $n = 2, 3, 4,$ 5, and 10 might be initiated. Again, we shall be interested in what develops.

II.   LIFE INSURANCE AND LIFE ANNUITY VARIANCES

2.1 Introduction

In Section 5.3 of the textbook Actuarial Mathematics [13] there is a discussion of the aggregate payment technique, and the current payment technique. For continuous cases, the aggregate payment technique uses the random variable T, denoting the future lifetime of (x), and for a discrete case, the random variable K, denoting the curtate future lifetime of (x). Thereby, the approach can be applied for the valuation of death benefits, but also for annuity benefits. The current payment technique has been thought of in connection with life annuity purposes, and for these purposes usually provides simpler valuation processes. We shall call the aggregate payment technique the Life Insurance Approach, and the current payment technique the Life Annuity Approach.

At this stage, we believe the Life Insurance Approach makes sense for single premium whole life insurance, and the Life Annuity Approach does the same for single premium whole life annuities. The alternative approach seems more difficult in each of these cases.

One wonders whether for the current payment technique, there is a pair of random variables, S for continuous cases, J for a discrete case, which can be used for both life insurance and life annuity purposes, but may be particularly appropriate for the latter. It now appears that such a pair can be defined for the cited simple cases. We consider:

1. S is a random variable over the domain $[0, \omega - x)$.

2. S has the probability distribution function, $\overset{\circ}{e}_{x\,\overline{\pi}}\big/\overset{\circ}{e}_x$, where

$$\overset{\circ}{e}_{x\,\overline{\pi}} = \int_0^s {}_u p_x\, du \tag{2.1.1}$$

It follows from (2.1.1) that S has the probability density function,

$${}_s p_x\big/\overset{\circ}{e}_x$$

For a discrete case, J may be defined as:
   1. J is a random variable over the domain 0, 1, 2, ...
   2. J has probability distribution function, $\ddot{e}_{x\,\overline{\pi}}\big/\ddot{e}_x$, where

$$\ddot{e}_{x\,\overline{\pi}} = \sum_{h=0}^{j-1} {}_h p_x \tag{2.1.2}$$

Both T and S are defined for each element of the domain $[\,0\,,\,\omega - x\,)$. Both K and J are random variables over the domain of non-negative integers up to $\omega - 1 - x$, but by means of step-functions increasing by a unit at the beginning of each year, may also be defined for the whole domain $[\,0\,,\,\omega - x\,)$. But T and K have well-recognized interpretations as full or curtate future lifetimes, while S and J are new concepts involving survival time.

    Initially, we tried to think that S denotes the minimum future survival time, provided survival of (x) for at least s years is required. Similarly, J is the minimum curtate survival time, provided that survival of (x) to at least the beginning of the future year j is stipulated. We had a hard time convincing others of these labored interpretations. Now we think of S as closely related to the current payment technique for annuities and, as such, denoting the future survival time required to qualify for the annuity payment at time s. There are some difficulties here. We have defined S as a random variable over the real numbers $[\,0\,,\,\omega - x\,)$. For this purpose, we are using a possibly unorthodox probability space for which elementary events (survival to durations s) are not "mutually exclusive" but are "successively distinct." Do you agree with our definition of S as a random variable with distribution function $\overset{\circ}{e}_{x\,\overline{\pi}}\big/\overset{\circ}{e}_x$?

    Similarly, J denotes the future survival time to qualify for the annuity payment at the beginning of future year j. Again we are using a possibly unorthodox probability space for which elementary events (survival to beginning of future years j) are not "mutually exclusive" but are "successively distinct". Do you agree with our definition of J as a random variable with distribution function $\ddot{e}_{x\,\overline{\pi}}\big/\ddot{e}_x = \sum_{h=0}^{j-1} {}_h p_x \bigg/ \sum_{h=0}^{\infty} {}_h p_x$? A minor problem is that we use a notation

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beginning with 0 to denote survival durations, which we believe is consistent with notation in the textbook [13].

A first challenge is whether you can provide better interpretations of the random variables S and J. For annuity purposes, they provide more direct formulas, for the expected value and variance of the annuity, than do the future lifetime variables T and K. Whether S and J can be adapted for other purposes remains to be seen.

2.2 Expected Values - Continuous Case

In this section, we consider moments of random variable functions of T and S, and utilize the probability density functions of T and S to calculate the moments of the r.v. functions. Readers are familiar with the formulas:

$$E[T] = \overset{0}{e_x} \tag{2.2.1}$$

$$E[v^T] = \overline{A}_x \tag{2.2.2}$$

$$E[\overline{a}_{\overline{T}|}] = E\left[\frac{1-v^T}{\delta}\right] = \frac{1}{\delta}\left[1 - \overline{A}_x\right] = \overline{a}_x \tag{2.2.3}$$

Newer formulas are:

$$E[S] = \int_0^{\omega-x} s\left(_s p_x \Big/ \overset{0}{e_x}\right) ds = \int_0^{\omega-x} s\, l_{x+s}/T_x ds = \int_0^{\omega-x} s\left(-dT_{x+s}/T_x\right) = \int_0^{\omega-x} T_{x+s}\, ds/T_x$$

$$= \int_0^{\omega-x} l_{x+s}\, \overset{0}{e_{x+s}}\, ds \Big/ \int_0^{\omega-x} l_{x+s} ds$$

$$= \text{weighted average of } \overset{0}{e_{x+s}} \text{ values, with } l_{x+s} \text{weights.} \tag{2.2.4}$$

$$E\left[\overset{0}{e_x}\, v^s\right] = \int_0^{\omega-x} \overset{0}{e_x}\, v^s\left(_s p_x \Big/ \overset{0}{e_x}\right) ds = \overline{a}_x \tag{2.2.5}$$

$$E\left[1 - \delta\overset{0}{e_x}\, v^s\right] = 1 - \delta\, E\left[\overset{0}{e_x}\, v^s\right] = 1 - \delta\overline{a}_x = \overline{A}_x \tag{2.2.6}$$

Note that (2.2.3), by use of (2.2.2), gets to expected value $\overline{a}_x$, while (2.2.5) gets the same expected value directly. On the other hand, (2.2.2) yields $\overline{A}_x$ directly, while (2.2.6) depends on (2.2.5).

2.3 Variances - Continuous Case

$$\mathrm{Var}\ \left(v^{\scriptscriptstyle T}\right) = \int_0^{\omega-x} v^{2t}\,_tp_x\mu_{x+t}\,dt - \overline{A}_x^2 = {}^2\overline{A}_x - \overline{A}_x^2 \tag{2.3.1}$$

where ${}^2\overline{A}_x$ is calculated with v^2 in place of v.

$$\mathrm{Var}\ \left[\overline{a}_{\overline{T}|}\right] = \mathrm{Var}\ \frac{1-v^{\scriptscriptstyle T}}{\delta} = \frac{1}{\delta^2}\,\mathrm{Var}\!\left(v^{\scriptscriptstyle T}\right) = \left({}^2\overline{A}_x - \overline{A}_x^2\right)\!/\delta^2 \tag{2.3.2}$$

$$\mathrm{Var}\ \left[\overset{o}{e}_x\,v^{\scriptscriptstyle S}\right] = \int_0^{\omega-x}\left(\overset{o}{e}_x\right)^{\!2} v^{2s}\ \left(\,_sp_x\big/\overset{o}{e}_x\right)\!ds - \overline{a}_x^2$$

$$= \overset{o}{e}_x\ {}^2\overline{a}_x - \overline{a}_x^2 \tag{2.3.3}$$

where ${}^2\overline{a}_x$ is calculated with v^2 in place of v.

$$\mathrm{Var}\ \left[1-\delta\overset{o}{e}_x\,v^{\scriptscriptstyle S}\right] = \delta^2\,\mathrm{Var}\!\left[\overset{o}{e}_x\,v^{\scriptscriptstyle S}\right] = \delta^2\!\left[\overset{o}{e}_x\ {}^2\overline{a}_x - \overline{a}_x^2\right] \tag{2.3.4}$$

The random variable functions, $v^{\scriptscriptstyle T}$ and $1-\delta\overset{o}{e}_x v^{\scriptscriptstyle S}$, have the same expected values $\overline{A}_x$ [see (2.2.2) and (2.2.6)] but their variances, as given by (2.3.1) and (2.3.4), may not agree. We say that moments calculated by use of the r.v. T follow the Life Insurance Approach, while moments calculated by use of the r.v. S result from the Life Annuity Approach. To compare the variances around $\overline{A}_x$, we consider

$$\delta^2\!\left[\overset{o}{e}_x\ {}^2\overline{a}_x - \overline{a}_x^2\right] \overset{<}{\underset{>}{=}} {}^2\overline{A}_x - \overline{A}_x^2$$

which simplifies to

$$\frac{{}^2\overline{a}_x}{\overline{a}_x} \overset{<}{\underset{>}{=}} \frac{2}{2+\delta\overset{o}{e}_x} \tag{2.3.5}$$

Since the variances around $\overline{a}_x$, as given by (2.3.3) and (2.3.2) are merely a factor $\left(1/\delta^2\right)$ times the variances around $\overline{A}_x$, as given by (2.3.4) and (2.3.1), the same criterion (2.3.5) applies for comparisons of them.

2.4 Expected values for a discrete case

Here, we consider the random variable K, as defined in the textbook Actuarial Mathematics, and the new random variable J, together with the r.v. functions v^{K+1}, $\ddot{a}_{\overline{K+1}|}$, and $(1+e_x)v^J$, $1-d(1+e_x)v^J$. For expected values, we have

$$E\left[v^{K+1}\right] = \sum_{k=0}^{r} v^{k+1}\,{}_k p_x q_{x+k} = A_x \qquad (2.4.1)$$

$$E\left[\ddot{a}_{\overline{K+1}|}\right] = \sum_{k=0}^{x}\left[\frac{1-v^{K+1}}{d}\right] = \frac{1-A_x}{d} = \ddot{a}_x \qquad (2.4.2)$$

$$E\left[(1+e_x)v^J\right] = \sum_{j=0}^{\infty}(1+e_x)v^J\,\frac{{}_j p_x}{1+e_x} = \ddot{a}_x \qquad (2.4.3)$$

$$E\left[1-d(1+e_x)v^J\right] = 1-d\ddot{a}_x = A_x \qquad (2.4.4)$$

Note that (2.4.1) provides A_x directly while (2.4.4) requires the relation $1-d\ddot{a}_x = A_x$, and (2.4.3). For annuities, (2.4.3) is the direct formula.

2.5 Variances - Discrete Case

Corresponding to the expected value formulas in Section 2.4, we obtain

$$\text{Var}\left[v^{K+1}\right] = \sum_{k=0}^{\infty} v^{2(k+1)}\,{}_k p_x q_{x+k} - A_x^2 = {}^2A_x - A_x^2 \qquad (2.5.1)$$

where ${}^2A_x$ is calculated with v^2 in place of v.

$$\text{Var}\left[\ddot{a}_{\overline{K+1}|}\right] = \text{Var}\left[\frac{1-v^{K+1}}{d}\right] = \frac{1}{d^2}\,\text{Var}\left[v^{K+1}\right] = \frac{1}{d^2}\left[{}^2A_x - A_x^2\right] \qquad (2.5.2)$$

$$\text{Var}\left[(1+e_x)v^J\right] = \sum_{j=0}^{\infty}(1+e_x)^2 v^{2j}\,\frac{{}_j p_x}{1+e_x} - \ddot{a}_x^2 = (1+e_x)\,{}^2\ddot{a}_x - \ddot{a}_x^2 \qquad (2.5.3)$$

$$\text{Var}\left[1-d(1+e_x)v^J\right] = d^2\,\text{Var}\left[(1+e_x)v^J\right] = d^2\left[(1+e_x)\,{}^2\ddot{a}_x - \ddot{a}_x^2\right] \qquad (2.5.4)$$

One may show that the variances by use of $J \overset{<}{\underset{>}{=}}$ the variances by use of K according as

$$\frac{{}^2\ddot{a}_x}{\ddot{a}_x} \overset{<}{\underset{>}{=}} \frac{2}{2+de_x} \qquad (2.5.5)$$

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2.6 Summary

A second challenge in this part is to decide whether the random variables T and K are better for life insurance than the random variables S and J. In other words, is the Life Insurance Approach better than the Life Annuity Approach? How significant are the differences? A full answer may entail review of all life insurance functions for example, benefit premiums and reserves, and loss functions.

A third challenge is to decide whether the random variables S and J are better than the random variables T and K for life annuities. Here, the Life Annuity Approach may be better than the Life Insurance Approach. How significant are the differences? A full answer may entail review of all forms of life annuities.

It should be noted that, in our Life Insurance Approach and Life Annuity Approach, only the future lifetime, or the future survival time are randomized. There are many who consider investment return a random variable, and that is a further source of uncertainty. Thereby, models become more complex and more difficult for public understanding and utilization, in this third approach to uncertainty.

There is still a fourth approach which we incline to call the actuarial adaptive approach. This consists of a thorough analysis of the benefits being considered, regular determination of basic assumptions to estimate the obligations involved if these assumptions are realized, regular analysis of the experience being encountered, and steps to make such adjustments as may be necessary. This involves development of trust by the decision-makers in the actuary's methods and integrity, rather than placing faith in complex models.

Unfortunately, none of these four approaches can foretell the future.

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