ACTUARIAL RESEARCH CLEARING HOUSE
1995 VOL. 2

EXPECTED INTERNAL RATE OF RETURN

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ABSTRACT: This paper discusses a problem in corporate finance, the problem of selecting from among a group of possible economic projects. This problem most certainly involves future contingent events and is a natural for consideration by actuaries interested in expanding into non-traditional practice areas.

Current texts on Corporate Finance usually favor discounted present value methods over internal rate of return methods. In part this is due to the problem of non-uniqueness of internal rate of return. However, as we point out, whenever non-uniqueness occurs there is also a problem with the discounted present value method. In this paper an internal rate of return method developed by Terchroew, Robichek, and Montalbano (see Kellison's *Theory of Interest*, 2nd ed., pg. 158) is used as the basis for a stochastic model for solving the above selection problem.

EXPECTED INTERNAL RATE OF RETURN

INTRODUCTION

The major role of the actuary is to control the financial consequences of future contingent events. The actuary traditionally performs this role for an insurance company or in the field of employee pensions. However, this major role clearly has application in other areas and for the past few years actuarial organizations have been educating current and prospective membership about the opportunities for applying actuarial skills in financial areas outside of the traditional ones. All large corporations are faced with financial decisions whose success depends on future contingent events. The mathematical/financial tools for dealing with these decisions are becoming more sophisticated and it is probably fair to say that in many corporations the financial officers are not equipped to use these tools. Thus the opportunity for a mutually beneficial relationship exists between the actuarial profession and the (non-insurance) corporate world.

The purpose of the current paper is to discuss an aspect of corporate finance. Our interest is in the area of capital budgeting in which a corporation is faced with the problem of deciding whether to proceed with an economic project.

1. RANKING ECONOMIC PROJECTS

By an economic project we mean a finite sequence of cash flows, $C = \{c_0, c_1, c_2, \ldots, c_n\}$ with c_0 negative and at least one of the c_i positive. Denote the set of all such projects by I. We think of the c_i as the net periodic cash flows generated by an economic project undertaken by a corporation. Faced with the problem of choosing from a number of possible projects, the corporation must rank them by some method. A common method of doing this is to use the discounted present value of the sequence of cash flows at some discount factor, v. This method will provide a ranking but different choices of v will yield different rankings and one must provide a rationale for the choice of discount factor. A second method for ranking projects is to use internal rate of return (IRR). Letting $x_0 = 1 + i_0$, i_0 is an IRR for the project C if $x_0 \geq 0$ and is a root of the equation:

$$c_0x^n + c_1x^{n-1} + \ldots + c_n = 0. \qquad (1)$$

Here we have the problem that such an equation may have multiple real roots or it may have none. In the case where (1) has a single non-negative root we will call the corresponding i_0 the classical IRR. In most of the modern corporate finance textbooks internal rate of return methods are downplayed in favor of discounted present value methods for evaluating investment projects. The argument is that the latter give a definite answer to the question of whether a project should be accepted. That is, accept if the present value of projected cash flow is positive. However, whenever the cash flow sequence has a non-unique IRR, the present value method is also anomalous. For example in Exhibit I, a project is rejected if a rate of return i_1 is used for discounting and yet accepted if a still higher rate i_2 is used. This is certainly counter to intuition and suggests that perhaps internal rate of return methods should not be discarded prematurely.

EXHIBIT 1

Henceforth we shall refer to the classical IRR as C_IRR. Pure loan projects are examples of projects having a C_IRR. These are projects having $c_0 < 0$ and all other c_i non-negative. The class of all such projects is simply ordered by the C_IRR (treating as equivalent, projects having the same IRR). Moreover, the C_IRR provides a simple ordering on the class of all projects *having* a C_IRR. The ordering on either of these sets determines a partial ordering on the collection of all projects described above. There is substantial literature devoted to the attempt to extend the partial ordering given by the classical IRR to the set I. Mathematically, there is no problem. Any partial ordering on a set, S, may be extended to a total ordering on S by transfinite induction. The problem is to do this in a way which has economic meaning.

2. AN AXIOMATIC TREATMENT

In [3], [4] Promislow and Spring give an axiomatic framework for extending the notion of internal rate of return to all projects of the type considered here. Their axioms describe a function whose values correspond to the accumulation factor $1 + i$ rather than to the rate i. Briefly stated their axioms are:

> Continuity: The internal rate of return should be a continuous function of the c_i.
>
> Monotonicity: The internal rate of return is non-decreasing with respect to each c_i.
>
> Normalization: For pure loan projects, the internal rate of return should agree with the classical IRR.

A function $m: I \rightarrow [0,\infty)$ satisfying the above axioms is called an **internal rate of return function**.

Promislow and Spring develop a theory termed the Theory of C-spaces which is then used to obtain internal rate of return functions. They show that solutions of the problem being considered here previously obtained by Arrow and Levhari [1] and by Teichroew, Robichek, and Montalbano [5], [6] can be gotten by these methods and they also obtain a new version of the internal rate of return which provides a ranking for I. We consider each of these solutions in the next section.

3. THREE REALIZATIONS

We first describe the method of Arrow and Levhari. Given a project $C = \{c_0, c_1, c_2, \ldots, c_n\}$ let

$$\psi_p(x) = c_0 + c_1/x + c_2/x^2 + \ldots + c_p/x^p \qquad 0 \le p \le n. \tag{2}$$

Note that $\psi_p(x)$ is the discounted present value of the truncated project consisting of the first $p + 1$ cash flows at discount factor $1/x$. The function

$$\psi(x) = \sup(\psi_0(x),\psi_1(x),...,\psi_n(x)) \tag{3}$$

is a continuous decreasing function and has exactly one real root x_0. The Arrow-Levhari IRR is defined to be $i_0 = x_0 - 1$. We will refer to this as the A_IRR. As Promislow-Spring show this method does yield an internal rate of return function and thus gives an extension to I of the classical IRR on the set of pure loan projects. However this does not solve our economic problem. The root so obtained is an IRR for a truncated project and truncation may not be a feasible option. Let us consider two simple examples. Let

$$C_1 = \{-1,5,-11,7\} \qquad C_2 = \{-1,5,-11,15\}. \tag{4}$$

Each of these projects has a classical IRR. For C_1 it is 0 and for C_2 it is 2. The A_IRR is 4 for both projects. Though the A_IRR extends the partial ordering given by the C_IRR on pure loan projects, it does not extend the partial ordering given on the set of projects having a C_IRR.

The method of Teichroew, Robichek, and Montalbano is treated in Kellison's *Theory of Interest* [2]. The main idea of this method is to consider two states for the project depending upon whether the balance of the cash flow stream is negative or positive. The project earning rate applies only in the former state. When the project has a positive balance a market rate applies. To describe the method we define a collection of operators T_c^d acting on functions $f\colon [0,\infty) \to R$ where R is the set of real numbers and d and c are real with $d > 0$. This operator is defined by:

$$T_c^d f(x) = \begin{cases} df(x) + c & f(x) \geq 0 \\ xf(x) + c & f(x) < 0. \end{cases} \tag{5}$$

In our first use of this operator, d will be the (constant) market interest rate. We will thus delete the superscript d. We then recursively define balance functions B_k for the project C by $B_0(x) = c_0$ and

$$B_k(x) = T_{c_k}^{d} B_{k-1}(x) \qquad k = 1,2,\ldots,n. \tag{6}$$

Thus $B_k(x)$ is the balance of the project after k years using accumulation factor x in years when the balance is negative at the beginning of a year and accumulation factor d otherwise. Note that since c_0 is negative, the B_k are monotone decreasing continuous functions. They are all piecewise polynomials. Hence if $B_n(0) > 0$ there is a unique root x_0 of $B_n(x)$. The Teichroew, Robichek, Montalbano IRR which we will refer to as the T_IRR is then defined to be $i_0 = x_0 - 1$. If $B_n(0) \leq 0$ the T_IRR is -1. Again, as shown by Promislow and Spring, this method yields an internal rate of return function and thus extends the classical IRR on pure loan projects. Consider the examples:

$$C_1 = \{-5,6.5,-2.5,2\} \qquad C_2 = \{-5,-1,1,8\}. \tag{7}$$

Neither is a pure loan project but both have classical IRR. For C_1, C_IRR = .166 and for C_2, C_IRR = .16. The T_IRR for these projects for market rates $d = 1.08$ and $d = 1.25$ are:

	1.08	1.25
C_1	.157	.173
C_2	.160	.160

(8)

Although C_1 has a C_IRR, the T_IRR is not equal to the C_IRR. That is, the T_IRR does not extend the ranking given by the C_IRR on the set of all projects having a C_IRR. If we are choosing projects on the basis of the T_IRR we would prefer C_2 to C_1 at market rate 1.08 but would choose C_1 over C_2 at market rate 1.25. In itself this is not a criticism of the model. The choice of a project *should* be dependent on external conditions. The problem is that a fixed market rate is assumed for what might be a long term project. In the next section of this paper we will give a version of this method which incorporates a stochastic market interest rate model. Before doing this we discuss an internal rate of return function obtained by Promislow and Spring.

To obtain the Promislow-Spring IRR (P_IRR) let μ be the Lebesgue measure on the line. For f a real valued function on $(0,\infty)$ let X_f^+ be the set on which $f(x)$ is positive. Then the project C has P_IRR equal to $\mu(X_\psi^+) - 1$ where ψ is the present value function of the project C. That is:

$$\psi(x) = c_0 + c_1/x + c_2/x^2 + \dots + c_n/x^n. \tag{9}$$

For a pure loan project this function is monotone decreasing and has a unique root x_0. Note that x_0 equals $\mu(X_\psi^+)$ in this case and hence P_IRR = C_IRR. Moreover, whenever C has a C_IRR the root x_0 equals $\mu(X_\psi^+)$ so, unlike the A_IRR and the T_IRR, the P_IRR extends the C_IRR on the set of all projects having a classical internal rate of return. The P_IRR ranks projects by the measure of the sets of rates for which the project has a positive present value. Now suppose that the two curves in Exhibit 2, labelled C_1 and C_2 are graphs of present value functions for projects C_1 and C_2. The P_IRR selects C_1 over C_2. However, it may well be that C_2 is the better economic choice. Perhaps, as hinted by this picture there is an internal rate of return function given by integrating the present value function over some subset of $(0,\infty)$.

EXHIBIT 2

To illustrate the P_IRR, consider the project:

$$C = \{-1,3.8,1.25,-14.85,11.7\}. \tag{10}$$

The present value function is:

$$\psi(x) = -1 + 3.8/x + 1.25/x^2 - 14.85/x^3 + 11.7/x^4 \tag{11}$$

and the accumulated value at the end of the project life is:

$$x^4\,\psi(x) = -(x - 1.3)(x - 1.5)(x - 3)(x + 2). \tag{12}$$

The P_IRR is thus $(3 - 1.5 + 1.3) - 1 = 1.8$.

4. A STOCHASTIC IRR MODEL

We modify the model of Teichroew, Robichek and Montalbano by permitting the market interest rate in year k to be a random variable, d_k. We then have balance functions

$$B_k = T_{c_k}^{d_k} B_{k-1}(x) \qquad k = 1,2,\ldots,n. \tag{13}$$

The monotonicity is not effected by allowing random market interest rates so if $B_n(0)$ is positive B_n will have a unique positive root. Consider the case of four cash flows. That is $C = \{c_0,c_1,c_2,c_3\}$. With interest factors $d_1, d_2, d_3,$ we have the final balance function:

$$B_3(x) = T_{c_3}^{d_3} T_{c_2}^{d_2} T_{c_1}^{d_1} B_0(x). \tag{14}$$

This function is piecewise polynomial with up to three different polynomials defining it depending upon whether the end of period balances are positive or negative. That is, the function will be linear-quadratic-cubic or quadratic-cubic or pure cubic. Also the root may occur in any of the pieces of B_3. Thus there are six different graphs arising (see Exhibit 3). Also, Graphs 2 through 6 each arise in two different forms of the function B_3. That is, there are 11 cases to consider. We are not including the cases where $B_3(0)$ is negative, giving $x_0 = -1$.

EXHIBIT 3

As more cash flows are added the picture becomes more complicated exponentially. Thus calculating an expected internal rate of return is a somewhat delicate programming problem. We give the results of a number of calculations using several interest rate scenarios. For each of these we use an initial market rate $d_0 = 1.08$. In each case the expected IRR is gotten using 50 simulations of the projects life. The stochastic interest rate scenarios are given to illustrate the model and are not proposed as models to be used in actual practice. We compare the rankings of projects given by each of the stochastic models and by the models described earlier.

We use the following interest rate scenarios:

I. In each year the market interest rate will increase by 25% or decrease by 20% each with probability .5.

II. In each year the market interest rate will increase by 25% or decrease by 20% with probabilities .75 and .25 respectively.

III. In each year the market interest rate will increase by 25% or decrease by 20% with probabilities .25 and .75 respectively.

IV. For the first [n/2] years the interest rate will remain fixed or will rise by 50% with probabilities .25 and .75 respectively. For the remaining project years the rate remains the same or decreases by 50% with probabilities .25 and .75 respectively.

V. For the first [n/2] years the interest rate will remain fixed or will decrease by 50% with probabilities .25 and .75 respectively. For the remaining project years the rate remains the same or increases by 50% with probabilities .25 and .75 respectively.

Now consider the following four projects. Each consists of six cash flows.

$$C_1 = \{-1, 3.15, -4.29, 4.2885, -3.29, 1.1385\}$$
$$C_2 \ \{-1, 3.11, -2.465, 3.895, -6.346, 2.72\} \qquad (15)$$
$$C_3 = \{-1, 9.3, -32.77, 54.075, -41.245, 11.55\}$$
$$C_4 = \{-1, 2.49, -1.476, .9955, -2.4945, 1.4715\}.$$

None of these has a classical internal rate of return. In Exhibit 4 we give the IRR computed by each of the methods discussed and include in parentheses the ranking of the project determined by the method.

EXHIBIT 4

Note that the various methods for computing an IRR give rise to seven different rankings of the four projects. Also observe that EII > EI > EIII and EIV > EV. This is expected since the T_IRR is an increasing function of the market rate d and hence the stochastic models with higher probability of greater market rates yield higher internal rates of return.

We next consider a set of eleven four cash flow projects representing each of the eleven branches alluded to above. That is, at a fixed market rate of 1.08 each of the possible branches is included among these projects.

$$
\begin{array}{ll}
P_1 = \{-1200,-2345,-1234,5500\} & P_2 = \{-2300,-320,345,560\} \\
P_3 = \{-1200,-2000,5000,-1100\} & P_4 = \{-1230,120,-1340,2700\} \\
P_5 = \{-2000,5000,-8000,3000\} & P_6 = \{-2345,1345,-1234,3000\} \\
P_7 = \{-2000,5000,-2000,1000\} & P_8 = \{-3000,13000,-2000,-2000\} \\
P_9 = \{-10000,8000,2000,3000\} & P_{10} = \{-2000,3000,2000,-2000\} \\
P_{11} = \{-1000,5000,4000,-8000\} &
\end{array}
\qquad (16)
$$

In Exhibit 5 we list the IRRs and rank these projects according to the classical IRR (for those which have a classical IRR) and the T_IRR, A_IRR, P_IRR, and the stochastic model labelled I above.

EXHIBIT 5

In projects P3, P8, and P10 the accumulated value polynomial has only one positive root greater than one and hence had we used this for the condition for existence for a classical IRR these projects would have IRRs of .1891, 3.133, and .7446 respectively. A financial calculator may well use this latter condition and hence give these values for IRR for these projects. For the stochastic model used here the ranking agrees with that given by the T_IRR. There are severe differences with the Arrow and Promislow rankings however. Also the stochastic IRR ranks those projects which have a positive classical IRR in the same order as does the classical IRR. Those with a negative classical IRR get reversed.

For a final illustration we consider some ten cash flow projects. The projects are:

$$D1 = \{-1,2,-2,1,-1,3,-2,1,-2,1\}$$
$$D2 = \{-1,4,1,2,-8,-3,5,-3,2,1\} \quad (17)$$
$$D3 = \{-1,-1,-1,3,-2,2,-1,-3,3,2.5\}$$

For each of these projects, the present value of the cash flows at interest factor 1.08 is close to zero. These present values are -.0194, .2079, -.0199 for D1, D2, D3 respectively. Thus if the discounted present value criterion is used and a return of 8 percent is required, projects D1 and D3 will be rejected while D2 will be accepted. Using an internal rate of return criterion in a stochastic interest rate environment may give a different choice of project to pursue. With our scenarios, only EV fails to rank D2 first. Among the three projects only D3 has a classical IRR. It is .07844 and is also the IRR given for this project by each of the methods. The present value function for D1 has positive roots .6702 and 1 with the latter being a root of multiplicity two. The present value function for D2 has positive roots .9735, 1 and 4.2362. These are approximate except for the root 1. Exhibit 6 shows that there is considerable fluctuation in expected IRR given different stochastic interest rate scenarios. In the case of D3 this is not the case. This is not because D3 has a classical IRR but because the balance functions remain negative until the end of the project and hence the stochastic market rates do not enter.

EXHIBIT 6

5. CONCLUSION

The purpose of this paper has been to develop a meaningful method for extending the partial ordering of pure loan projects given by the classical internal rate of return to a much larger class of investment projects. While there already exist a number of ways for doing this it is our belief that the use of a stochastic interest rate model improves the Teichroew, Robichek, Montalbano model. To actually carry out the computations with or without the stochastic structure is somewhat complex as indicated in Exhibit 3. The main contribution here has been to carry out the computations and illustrate the dependence of ranking on the interest rate scenario chosen.

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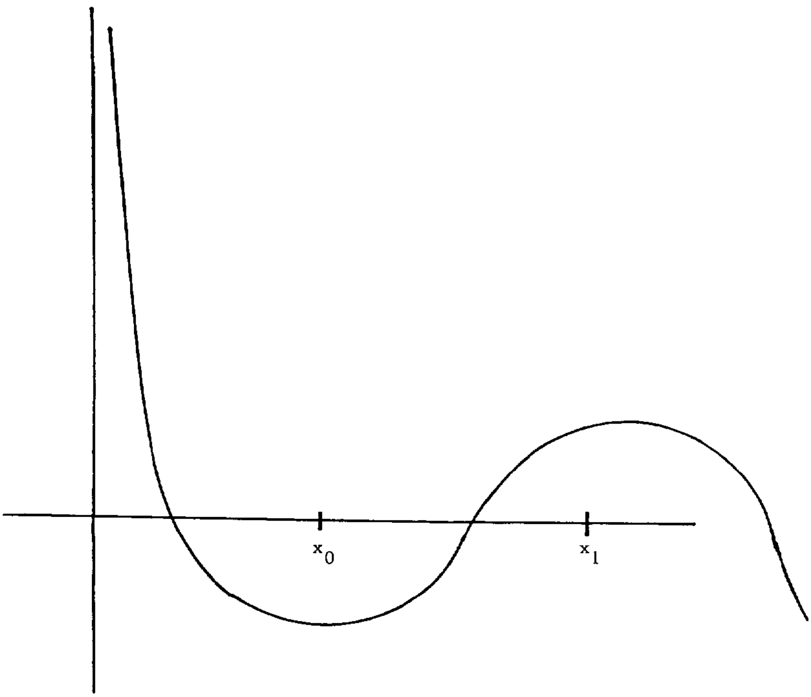
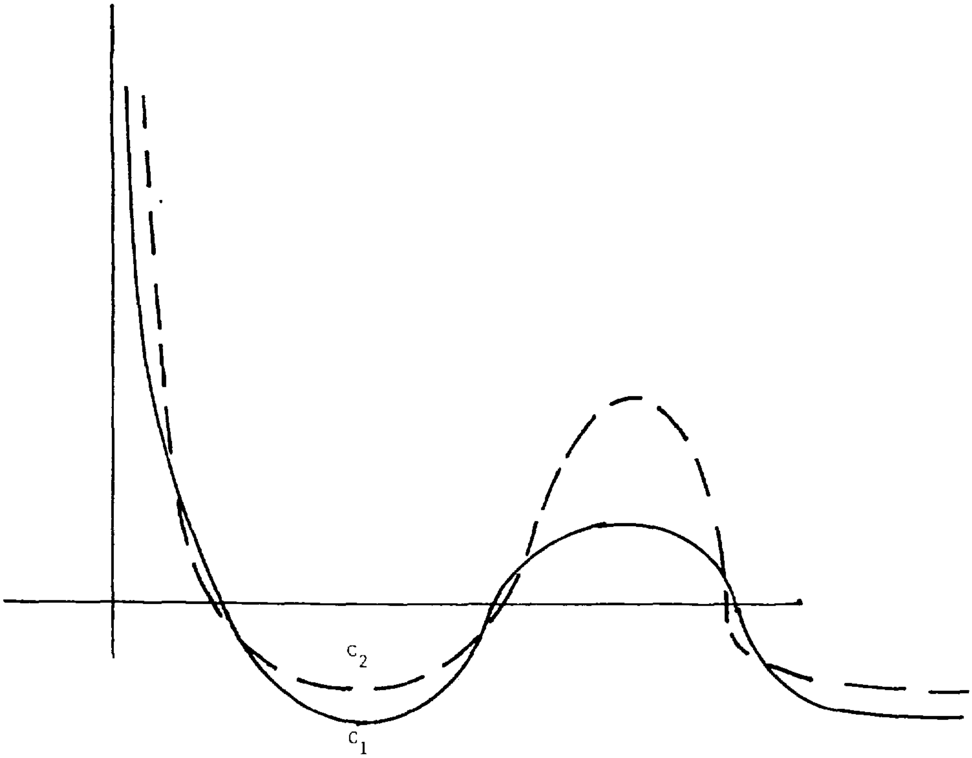
5. Teichroew, Daniel, Alexander A. Robichek, and Michael Montalbano, "Mathematical Analysis of Rates of Return Under Certainty," *Management Science*, Vol. 11, No. 3, January, 1965.

6. ____________, "An Analysis of Criteria for Investment and Financing Decisions Under Certainty," *Management Science*, Vol. 12, No. 3, November, 1965.



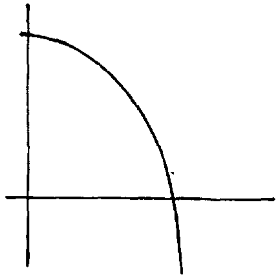
Present value function for a project with multiple IRR's. Discounted present value method suggests rejection of project at rate x_0 yet accepting at the higher rate x_1.

EXHIBIT 1

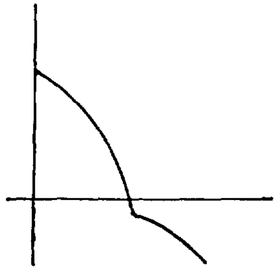


C_1 has higher P_IRR but C_2 has higher present value for most rates.

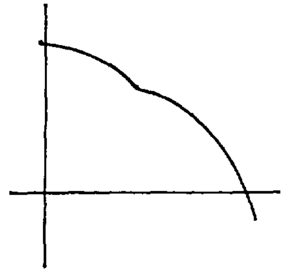
EXHIBIT 2



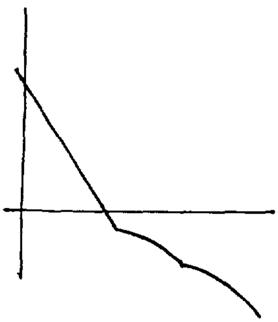
1. Pure Cubic



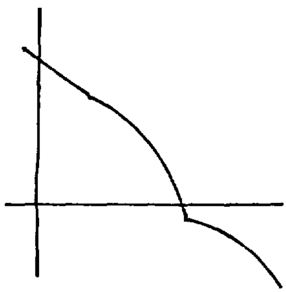
2. Quadratic-Cubic
X_0 a Root of Quadratic



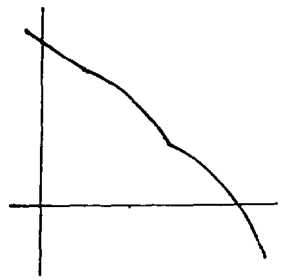
3. Quadratic-Cubic
X_0 a Root of Cubic



4. Linear-Quadratic-Cubic
X_0 a Root of Linear



5. Linear-Quadratic-Cubic
X_0 a Root of Quadratic



6. Linear-Quadratic-Cubi
X_0 a Root of Cubic

EXHIBIT 3

PROJ	P_IRR	A_IRR	T_IRR	EI	EII	EIII	EIV	EV
C1	-.05(4)	2.15(2)	.08(1)	.083(4)	.105(3)	.068(2)	.152(3)	.034(1)
C2	.983(2)	2.10(3)	-.062(4)	.10(1)	.129(1)	.074(1)	.232(1)	.023(4)
C3	1.1(1)	8.3(1)	.0795(2)	.088(2)	.106(2)	.059(4)	.174(2)	.026(3)
C4	.31(3)	1.49(4)	.0791(3)	.087(3)	.101(4)	.063(3)	.148(4)	.031(2)

EXHIBIT 4

PROJECT	C_IRR	P_IRR	A_IRR	T_IRR	EI
P1	.0724(4)	.0724(7)	.0724(9)	.0724(8)	.0724(8)
P2	-.3442(6)	-.3442(10)	-.3442(11)	-.3442(11)	-.3442(11)
P3	----	-.0590(9)	.3700(6)	.1698(6)	.1702(6)
P4	.0503(5)	.0503(8)	.0530(10)	.0503(9)	.0503(9)
P5	-.5000(7)	-.5000(11)	1.5000(3)	-.2779(10)	-.2767(10)
P6	.1215(3)	.1215(5)	.1215(8)	.1215(7)	.1215(7)
P7	1.1420(1)	1.1420(3)	1.4900(4)	.8299(3)	.8303(3)
P8	----	2.6190(2)	3.3333(2)	2.1533(1)	2.1473(1)
P9	.1833(2)	.1833(4)	.1833(7)	.1833(5)	.1833(5)
P10	----	.1000(6)	1.0000(5)	.5498(4)	.5461(4)
P11	----	3.4500(1)	4.6900(1)	.8934(2)	.8394(2)

EXHIBIT 5

PROJ	P_IRR	A_IRR	T_ITT	EI	EII	EIII	EIV	EV
D1	-.330(3)	1.00(2)	.074(3)	.077(3)	.122(2)	.047(3)	.178(2)	.019(3)
D2	3.21(1)	3.34(1)	.114(1)	.109(1)	.169(1)	.089(1)	.352(1)	.036(2)
D3	.078(2)	.078(3)	.078(2)	.078(2)	.078(3)	.078(2)	.078(3)	.078(1)

EXHIBIT 6