

Estimating Long-term Returns in Stochastic Interest Rate Models

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Abstract

This paper addresses the evaluation of long-term returns $R(t, r)$ when the short interest rate $r(t)$ is modeled by a general diffusion process:

$$dr(t) = \alpha(t, r(t))dt + \sigma(t, r(t))dZ$$

where $Z(t)$ is a Brownian motion and where $\alpha(t, r(t))$ and $\sigma(t, r(t))$ are the instantaneous drift and variance, respectively, of the processes $r(t)$. By deriving the long-term return dynamics and invoking the Feynman-Kac formula, the long-term return is represented as the solution of a partial differential equation. A finite difference method is derived for the valuation of the long-term return. Numerical examples and applications are also addressed.

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1. Introduction

Long-term return is the concern of insurance companies in reserving, investment decision making, ruin analysis as well as product pricing. Interest rate models can be used to price interest rate derivative securities and to hedge investment risk. Previous work to study long term return in the stochastic interest rate environment includes Deelstra and Delbaen (1995). Deelstra and Delbaen studied the convergence in law of the long-term return when the short interest rate modeled by an extension of the CIR model. Assuming short rate $r(t)$ follows

$$dr_t = (2\beta r_t + \delta_t)dt + g(r_t)dZ_t, \quad (1.1)$$

With $(Z_t)_{t \geq 0}$ a Brownian motion, $\beta < 0$ and $g : R \rightarrow R^+$ a Lipschitz function vanishing at zero.

They proved that under certain conditions, the following convergence almost everywhere holds:

$$\frac{1}{t} \int_0^t r_\tau d\tau \rightarrow \frac{-\hat{\delta}}{2\beta}$$

with $\frac{1}{t} \int_0^t \delta_\tau d\tau \rightarrow \hat{\delta}$ almost everywhere.

In general, however, no closed formula for long term stochastic interest rate has been given.

In this paper, we study the long term interest rate in a general stochastic setting. We developed a partial differential equation to estimate the expected long term interest rate. The numerical method to solve the partial differential equation (PDE) is also presented.

The paper is organized as follows: In next section, we give a general description about the stochastic long term return. We derive the partial differential equation for estimating the expected long term return. Section 3 presents fully explicit finite difference scheme for the numerical solution of the associated PDE. In Section 4, some examples are given for the estimation. The conclusion and discussion are presented in section 5.

2. Problem Formulation

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the filtration $\{\mathcal{F}_t, t \geq 0\}$, an increasing family of sub-sigma-algebras of \mathcal{F} .

At some future time T , $T > 0$, the long-term return of interest rate over $[0, T]$ is represented by $R(T, r)$.

Let $r(t)$ represent the instantaneous short interest rate at t .

If $r(t)$, $0 \leq t \leq T$ is known, then

$$R(T, r) = \frac{1}{T} \int_0^T r(\tau) d\tau$$

The purpose of this study is to forecast $R(T, r)$ in a general stochastic setting.

We assume that $r(t)$ follows a diffusion process described by

$$dr(t) = \alpha(t, r)dt + \sigma(t, r)dZ, \quad (2.1)$$

α = instantaneous mean of the interest rate

σ^2 = instantaneous diffusion variance of the interest rate

$Z(t)$ = standard Brownian motion.

At any time $t \in [0, T]$, the average return over $[t, T]$, $t \in [0, T]$ is defined as

$$A(t, T, r) = \frac{1}{T-t} \int_t^T r(\tau) d\tau .$$

Notice that

$$R(T, r) = A(0, T, r) \quad (2.2)$$

Define \mathcal{L} as

$$\mathcal{L} = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial r^2} + \alpha \frac{\partial}{\partial r} + \frac{\partial}{\partial t} \quad (2.3)$$

The following theorem states that the expected long-term return dynamics

are determined by a partial differential equation

THEOREM 1 Assume that $\alpha(t, r), \sigma(t, r)$ are continuous and satisfy

$$\|\alpha(t, r)\|^2 + \|\sigma(t, r)\|^2 \leq K^2(1 + \|r\|^2),$$

for every $0 \leq t \leq \infty, x \in R^+$, where K is a positive constant.

Then

(i) $v(t, r) = E^r [A(t, T, r)]$ satisfies the Cauchy problem

$$\mathcal{L}v + \frac{r}{T-t} + \frac{v}{T-t} = 0; \quad \text{in } [0, T) \times \mathcal{R}, \quad (2.4)$$

subject to the boundary condition

$$\lim_{t \rightarrow T} u(t, r) = r_T; \quad r \in \mathcal{R}. \quad (2.5)$$

(ii) The closed-form solution of the long-term interest rate is given by

$$R(T, r) = \frac{1}{T} \int_0^T \int_{\mathcal{R}} G(0, r; \tau, \xi) \xi d\xi d\tau, \quad (2.6)$$

where $G(t, r; \tau, \xi)$ is the transition probability density for the process $r(t)$ determined by (2.1); i.e.,

$$P[r(\tau), \text{ given that } r(t) = r \in A] = \int_A G(t, r; \tau, \xi) d\xi, \quad (2.7)$$

for all the Borel set A .

Proof First we define $S(t, r)$ and $u(t, r)$ as

$$S(t, r) = \int_t^T r(\tau) d\tau$$

$$u(t, r) = E^r [S]$$

According to Feynman-Kac formula, (see, for example, Karatzas and Shreve, 1991)

$$u(t, r) = E^r \left[\int_t^T r(\tau) d\tau \right] dt \quad (2.8)$$

is the solution of

$$\mathcal{L}u + r = 0 \quad (2.9)$$

and

$$S(T, r) = 0. \quad (2.10)$$

Let G be the Green's function, then u could be solved by:

$$u(t, r) = \int_0^T \int_{\mathcal{R}} G(0, r; \tau, \xi) \xi d\xi d\tau. \quad (2.11)$$

Since

$$A(t, T, r) = \frac{1}{T-t} S(t, r)$$

$$\begin{aligned} \mathcal{L}v &= \frac{1}{T-t} \mathcal{L}u + \frac{-1}{(T-t)^2} u \\ &= \frac{-r}{T-t} + \frac{-v}{T-t} \end{aligned} \quad (2.12)$$

and

$$\lim_{t \rightarrow T} u(t, r) = \lim_{t \rightarrow T} E^r [A(t, T, r)] = E^{rr} \left[\lim_{t \rightarrow T} R(T, t, r) \right] = r_T$$

Q.E.D.

Let $r_{max}(t)$ and $r_{min}(t)$ be the highest and lowest possible values of the expected short rate. Then we practically solving PDE (2.4) and (2.5) together with the following boundary conditions:

$$v(t, r_{min}) = r_{min}(t) \in [0, T]; \quad (2.13)$$

$$v(t, r_{max}) = r_{max}(t) \in [0, T]. \quad (2.14)$$

3. Finite Difference Method

Let

$$\Delta t = \frac{T}{N}, \quad \Delta r = \frac{r_{max} - r_{min}}{M}.$$

We consider a uniform grid of $(N + 1)(M + 1)$ in the (t, r) space:

$$\{(t_n = np, r_i = ih), n = 1, 2, \dots, N; i = 1, 2, \dots, M\}$$

where $p = \Delta t$ and $h = \Delta r$ are the discrete increments in time space and short rate.

We next define grid function $V_n, n = 1, 2, \dots, N$ as

$$V_n = (v(t_n, r_1), v(t_n, r_2), \dots, v(t_n, r_M))^T \quad n = 1, 2, \dots, N; \quad (3.1)$$

and denote

$$\alpha_{i,n} = \alpha(t_n, r_i), \sigma_{i,n} = \sigma(t_n, r_i); i = 1, 2, \dots, M; n = 1, 2, \dots, N. \quad (3.2)$$

By replacing the partial derivatives by the forward finite differences, we approximate PDE (2.4) by the following implicit finite differences scheme:

$$V_n = \mathcal{B}_n^{-1}(\mathcal{A}_n V_{n+1} + b_n), \quad n = 1, 2, \dots, N - 1. \quad (3.3)$$

where

$$\begin{aligned} \mathcal{B}_n(i, i) &= 1 - \frac{1}{(N - n)} + \frac{p\sigma_{i,n}^2}{h^2}, i = 1, 2, \dots, M - 1; \\ \mathcal{B}_n(i, j) &= 0, i \neq j, i, j = 1, 2, \dots, M - 1. \end{aligned} \quad (3.4)$$

$$\begin{aligned} \mathcal{A}_n(i, i) &= 1, i = 1, 2, \dots, M - 1; \\ \mathcal{A}_n(i, i + 1) &= \frac{p\sigma_{i,n}^2}{2h^2} - \frac{p\alpha_{i,n}}{2h}, i = 1, 2, \dots, M - 2; \\ \mathcal{A}_n(i, i - 1) &= \frac{p\sigma_{i,n}^2}{2h^2} + \frac{p\alpha_{i,n}}{2h}, i = 2, 3, \dots, M - 1; \\ \mathcal{A}_n(i, j) &= 0, |i - j| > 1, i, j = 1, 2, \dots, M - 1. \end{aligned} \quad (3.5)$$

and

$$b_n(i) = \frac{ihp}{N-n}, i = 1, 2, \dots, M. \quad (3.6)$$

with local truncation error that behaves as $o(h^2 + 2h + p)$ for the expected long-term return $v(t, r)$ as h and $p \rightarrow 0$.

The boundary conditions corresponding to (2.4), (2.12) and (2.13) are

$$V_N = v(Np, ih) = r_T(ih); \quad (3.7)$$

$$V_n(0) = r_{min}(np); \quad (3.8)$$

$$V_n(M) = r_{max}(np). \quad (3.9)$$

The finite differences schemes (3.1) – (3.5) evaluate $\{V_{i,n}\}$ in the order of

$$V_N \rightarrow V_{N-1} \rightarrow V_{N-2} \rightarrow \dots \rightarrow V_1 \rightarrow V_0.$$

V_0 presents the estimate of expected long term return over $[0, T]$, $R(T, r_i)$ based on the r_i at $t = 0$.

4. Numerical examples

Example 1. (CIR model)

As a numerical example, consider the one factor CIR model where the risk neutral interest rate process is assumed to be

$$dr = (\mu - \lambda\sigma)r dt + \sigma r dZ \quad (4.1)$$

where λ is the market price of risk. The corresponding PDE (2.4) to calculate the expected average rate $v(t, r)$ becomes

$$\frac{1}{2}\sigma^2 r^2 \frac{\partial^2 v}{\partial r^2} + (\mu - \lambda\sigma)r \frac{\partial v}{\partial r} + \frac{\partial v}{\partial t} + \frac{r}{T-t} + \frac{v}{T-t} = 0; \quad in \quad [0, T) \times \mathcal{R}^+, \quad (4.2)$$

To estimate the long term return, Let $x = \ln r$ and the above PDE is equivalent to

$$\frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + (\mu - \lambda\sigma - \frac{\sigma^2}{2}) \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + \frac{e^x}{T-t} + \frac{v}{T-t} = 0; \quad in \quad [0, T) \times \mathcal{R}, \quad (4.3)$$

Applying finite differences scheme (3.3) with $\sigma_{i,n} = \sigma$ and $\alpha_{i,n} = \mu - \lambda\sigma - \frac{\sigma^2}{2}$, the expected returns over $[0, T]$ are obtained.

Figure 1. Estimated One Year Return ($\sigma = 1, \mu = 7\%$)

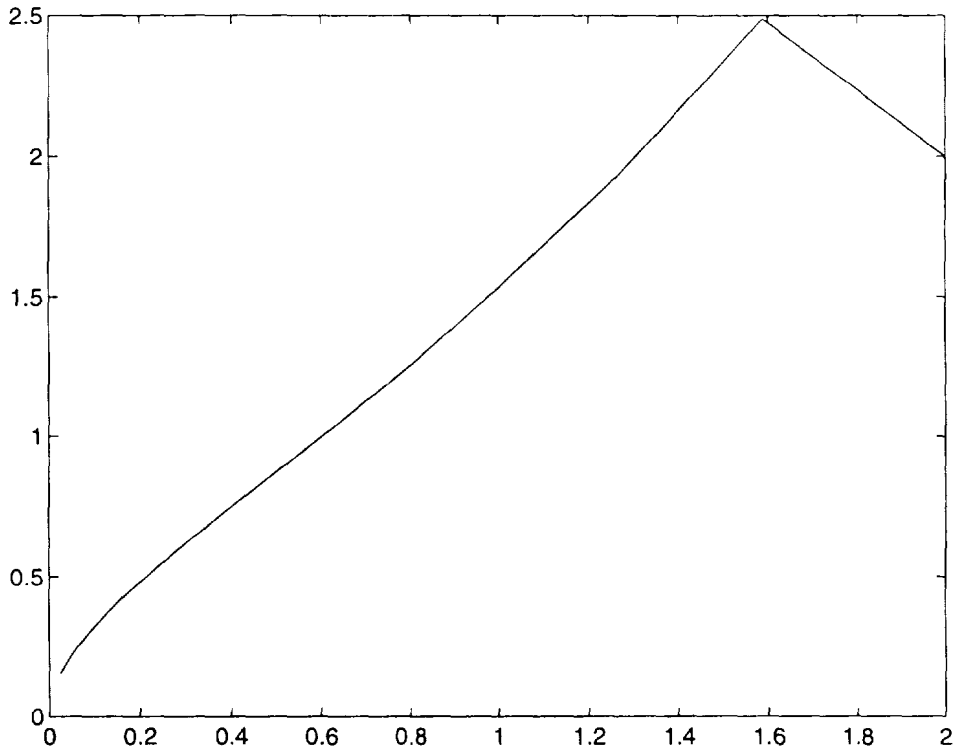


Figure 1 gives the estimated one year return corresponding to the short rate at $t = 0$.

Figure 2. Estimated One-year Return ($\sigma = 1000, \mu = 7\%$)

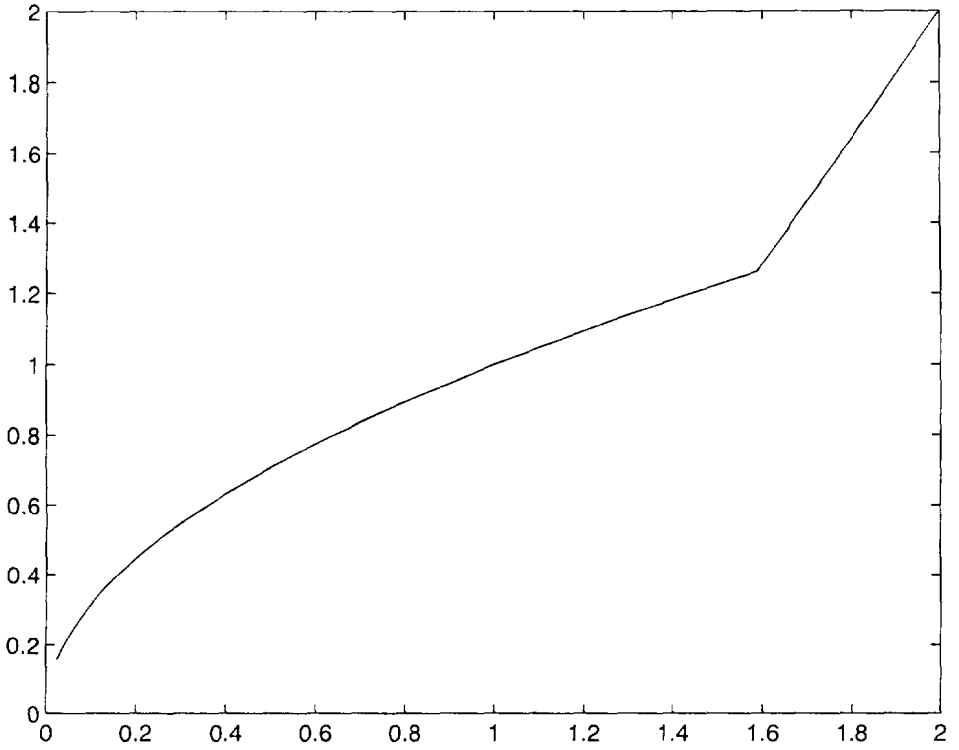


Figure 2 shows the 1-Year return when the level of volatility is increased from 1 to 1000.

Figure 3. Estimated Five-year Return ($\sigma = 1, \mu = 10\%$)

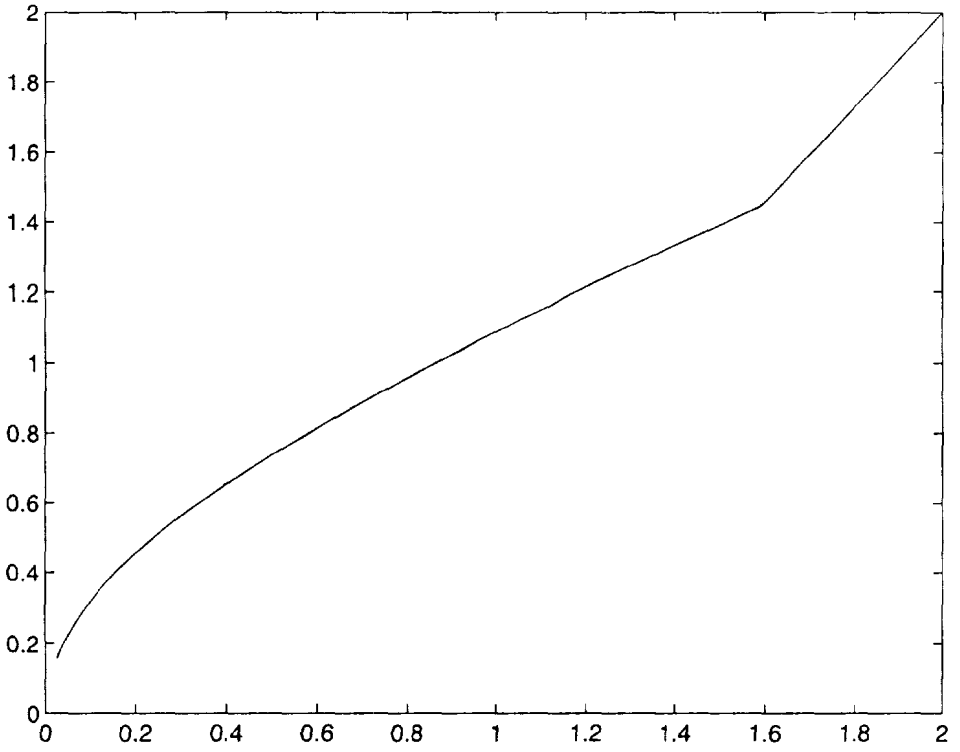


Figure 4. Estimated Ten-year Return ($\sigma = 1, \mu = 15\%$)

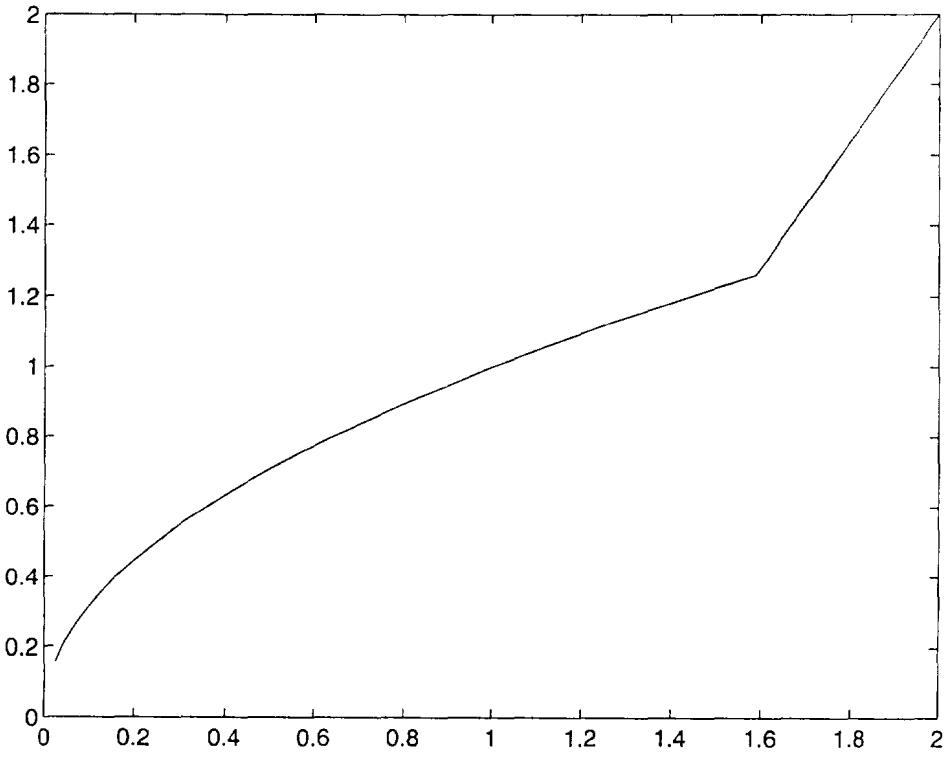
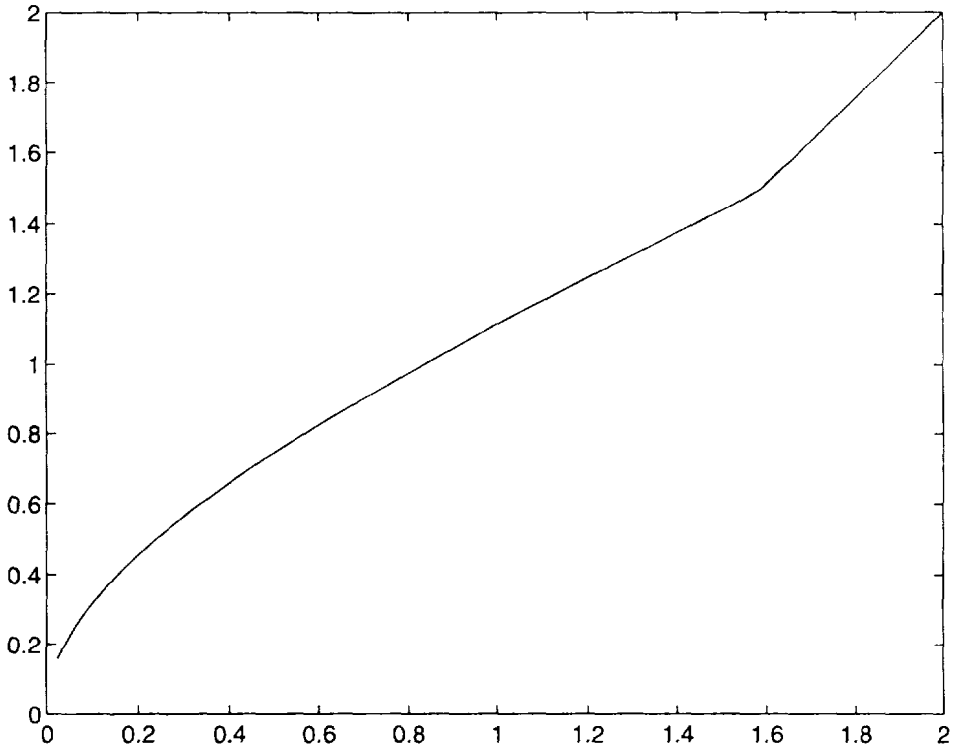
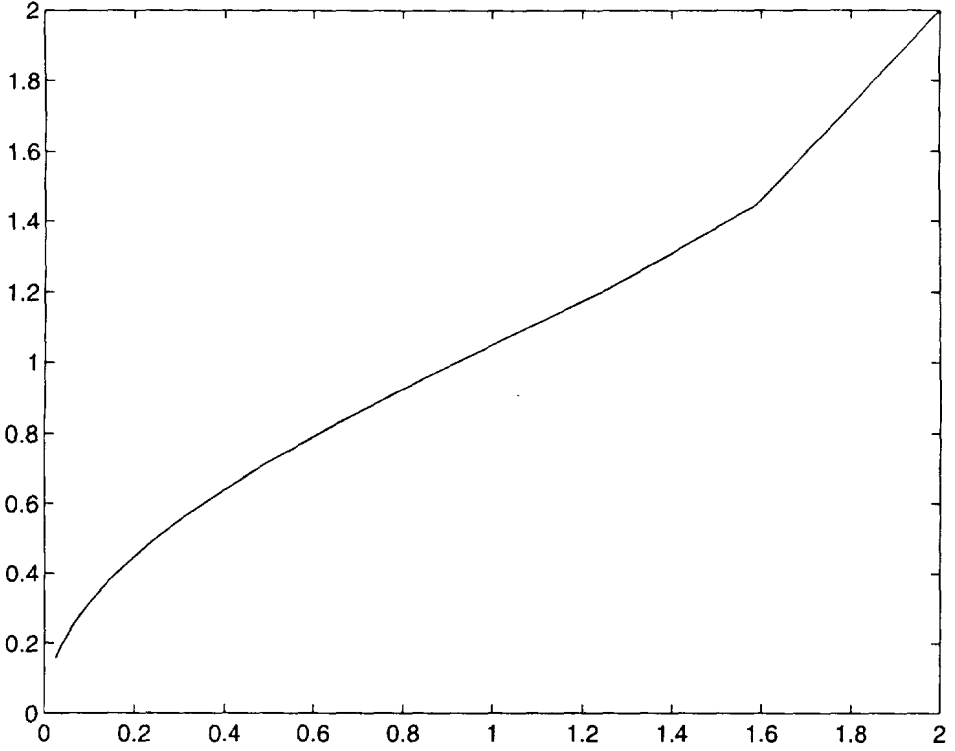


Figure 5. Estimated Twenty-year Return ($\sigma = 1, \mu = 20\%$)



Example 2. Consider a generalized CIR model where μ and σ are not constant but functions of t : $\mu = \mu_0 + kt$; $\sigma = \sigma\sqrt{(T-t)}$

Figure 6. Estimated Five-year Return ($\sigma = \sqrt{5-t}, \mu = 15\% + 0.001t.$)



Again we calculate the estimated returns over five years as shown in Figure 6. The ten years and twenty years returns are given as follows:

Figure 7. Estimated Ten-year Return ($\sigma = \sqrt{10 - t}$, $\mu = 15\% + 0.001t$.)

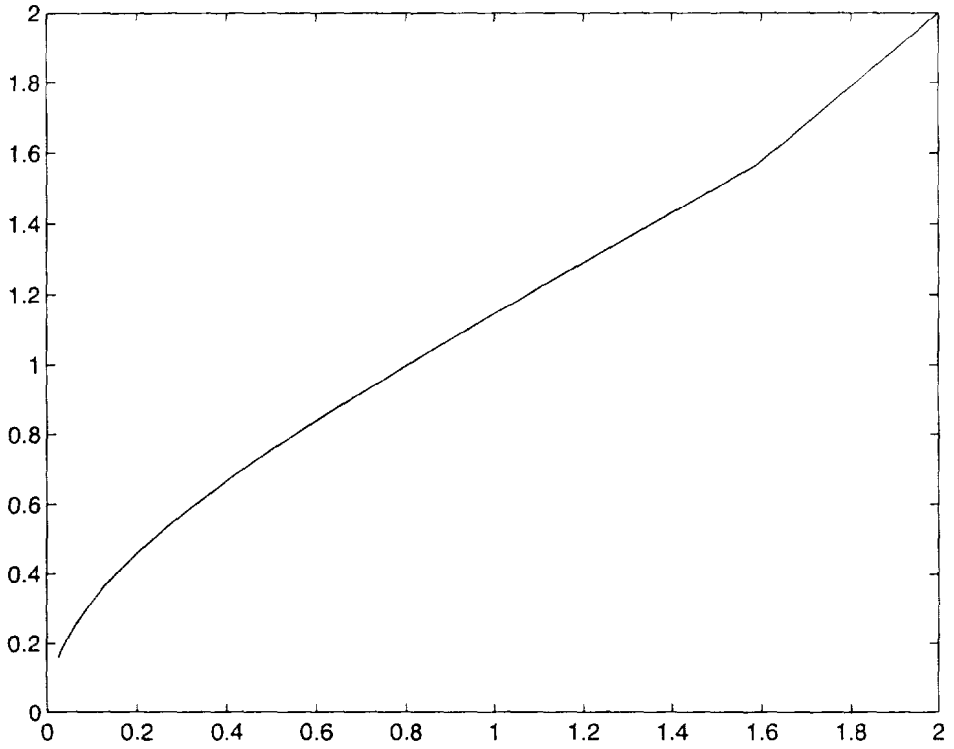
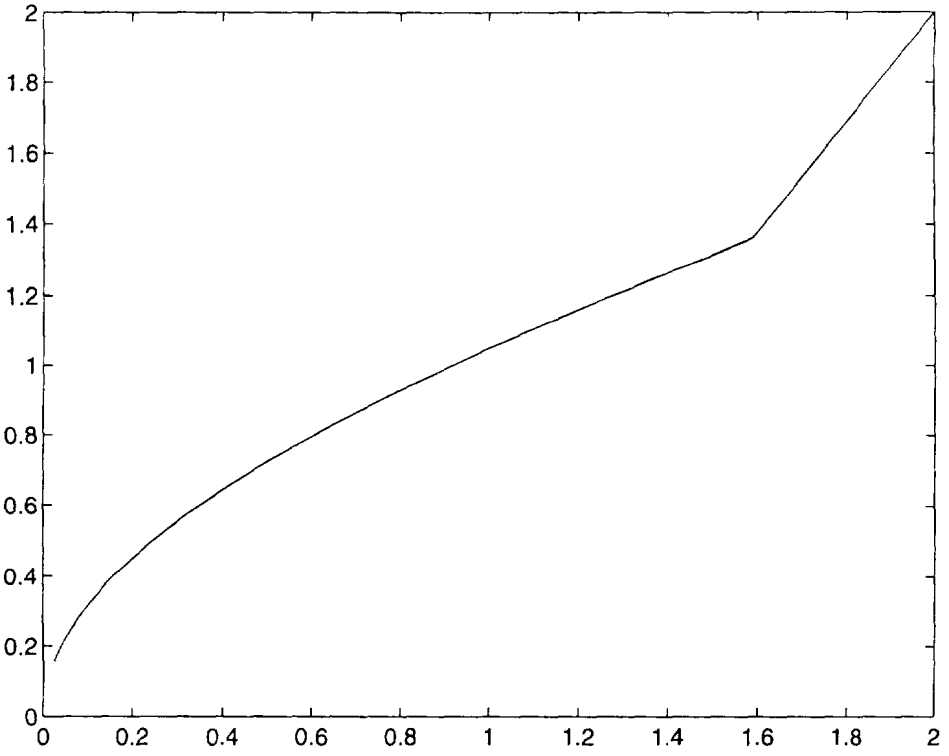


Figure 8. Estimated Twenty-year Return ($\sigma = \sqrt{20-t}$, $\mu = 15\% + 0.001t$.)



5. Concluding Remarks

This study has developed a stochastic model for forecasting the long-term return of interest rate process. The paper derived a dynamic model for average short rate over time to maturity period. Both closed-form and numerical solution are presented is presented together with numerical examples. The method developed in this paper is suitable for any interest rate process including multi-factor models. For example, to estimate the expected long term return with short rate modeled by the two factor CIR model, one could solve PDE (2.4) in two dimensional space. The method could be used for hedging interest rate risk, pricing and managing interest rate derivatives and interest rate sensitive insurance products.

References

- [1] Albizzati, M. and H. Geman (1994), Interest Rate Risk Management and valuation of the Surrender Option in Life Insurance Policies, *Journal of Risk and Insurance*, 61, 616-637.
- [2] Beekman, J. A. and Shiu, E. S. W. (1988), Stochastic models for bond prices, function space integrals and immunization theory, *Insurance: Mathematics and Economics*, 7, 163-173.
- [3] Boyle, P. P. (1978), Immunization under stochastic models of the term structure, *Journal of the Institute of Actuaries*, 105, 177-187.
- [4] Deelstra, G. D. and Delbaen, F., (1995), Long-term returns in stochastic interest rate models, it *Insurance: Mathematics and Economics* 17, 163-169.
- [5] Heath, D.; Jarrow, R. and Morton, A. (1992), Bond pricing and the term structure of interest rates: a new methodology for contingent claims evaluation, *Econometrica*, 77-105.
- [6] Karatzas, L. and Shreve, S. E. (1991), *Brownian Motion and Stochastic Calculus*, Second Edition, Springer-Verlag, New York.