ACTUARIAL RESEARCH CLEARING HOUSE
1996 VOL. 1

Premium Calculations by Transformed Distributions

A. Hoque Sharif
Department of Statistics and Actuarial Science
University of Waterloo, Waterloo, Ontario, Canada

ABSTRACT

Venter (1991) showed that the only premium calculation principles that preserve layer additivity are those that can be generated from transformed distributions, where the price for any layer is the expected loss for that layer under the transformed distribution. Stimulated by this results, Wang (1995) introduced the concept of PH-transform of a random risk X and hence calculated risk adjusted premium by using transformed distributions. The concept of transformed distributions is generalized in this paper. First the concepts of net premium intensity, loaded premium intensity and load generators are introduced. Then transformed distributions are identified with premium intensity and hence the loaded premium is calculated from transformed distributions. Finally it is shown that this method of premium calculation is arbitrage free and it incorporates the strengths of the utility approach.

Contents

Key words: *Premium, loading, layering, intensity*

1 Introduction

An insurance risk is a contingent claim X, having a probability distribution function. In other words, an insurance risk is a random loss. An insurance in its technical legal meaning, is a legal mutual agreement between two parties exchanging an insurance risk for a fixed payment called premium. The principle of assigning a premium to a risk is an essential issue to pricing an insurance risk. The calculation of an insurance premium is one of the most important function of a practicing actuary.

Risk loadings are required by insurers/reinsurers as a source of solvency margin and potential profit. But the question is : how to decide on risk loading to different risks? Extensive research has ended up with numerous principles of premium calculations (see Goovaerts *et al* (1984)). Each one of them has its pros and cons. None of them totally satisfy all the ideal premium principle. None of them satisfy layer additivity principle. Recently Wang (1995a) proposed a PH transform principal which satisfy several ideal principles of premium calculations, namely layer additivity, scale invariant, translation invariant. In this chapter we will propose two new principles which will be a generalization of Wang's results. First let us describe various principles suggested by different researchers as found in the literature. Our discussion will follow along the lines of Goovaerts, De Vylder and Haezendonck (1984).

2 Premium Calculation Principles and its Properties

A premium principle is a well-defined rule for calculating the premium for a given risk which is a random variable. The term premium usually means the risk premium which incorporate both process risk and parameter risk; commission and expenses are always excluded from premium principle and handled separately. For a class $\mathcal{R}$ of all risks, a premium principle π is a mapping

$$\pi : \mathcal{R} \longrightarrow \boldsymbol{R},$$

which means that for any risk $X \in \mathcal{R}$ a premium $P = \pi(X)$ is well-defined. One can consider $\mathcal{R}$ as a class of distribution functions of all risks. If F_X be the distribution function of any risk X, then the premium $P = \pi(F_X)$ is the unique value assigned to the risk X. Note that the actual real premium charged for a risk X will have an additional component for commission and expenses. Let us now briefly describe the twelve different principle of calculating risk premium. For the first ten, detail descriptions are given by Goovaerts, De Vylder and Haezendonck (1984). For the last two see Kaas, van Heerwaarden and Goovaerts (1994) and Wang (1995b).

All the symbols used in this section have the standard meaning, namely E for expectation, σ^2 for variance, F_X for distribution function of random risk X.

P.1 The expected value principle :

Definition 1 *The premium calculated according to the expected value principle is given by*

$$\pi(X) = (1 + \lambda)E(X)$$

where $\lambda \in \boldsymbol{R}^+$ is the premium loading. For $\lambda = 0$, this principle is known as **Net Premium** *principle.*

P.2 The maximal loss principle :

Definition 2 *The premium $\pi(X)$ calculated according to the maximal loss principle is given by*

$$\pi(X) = pE(X) + qMax(X)$$

where $q = 1 - p$, and $Max(X)$ denotes the right end point of the range of X.

P.3 The variance principle :

Definition 3 *The premium $\pi(X)$ determined by the variance principle for a given risk X is given by*

$$\pi(X) = E(X) + \beta\sigma^2(X)$$

where $\beta \in \boldsymbol{R}^+$. In this case the safety loading is proportional to the variance.

P.4 The Standard deviation principle :

Definition 4 *The premium $\pi(X)$ determined by the standard deviation principle for a given risk X is given by*

$$\pi(X) = E(X) + \alpha\sigma(X)$$

where $\alpha \in \boldsymbol{R}^+$. In this case the safety loading is proportional to the standard deviation.

P.5 The semi-variance principle :

Definition 5 *The premium $\pi(X)$ determined by the semi-variance principle for a given risk X is given by*

$$\pi(X) = E(X) + \beta\sigma_+^2(X)$$

where $\beta \in \boldsymbol{R}^+$ and

$$\sigma_+^2(X) = \int_{E(X)}^{\infty} (x - E(X))^2 dF_X(x).$$

P.6 The Mean value principle :

Definition 6 *Let $f(.)$ be a continuous and strictly monotonic function on a domain D (the domain of the random risk X). The premium calculated according to the mean value principle, denoted by $\pi(X, f)$, is the unique root of the following equation*

$$f(\pi) = E(f(X)).$$

For $f(x) = e^{\alpha x}$, this principle is known as **Exponential Principle**, *and the corresponding premium is given by*

$$\pi(X) = \frac{1}{\alpha} log E[e^{\alpha X}].$$

P.7 The zero utility principle :

Definition 7 *Let $u(.)$ be a utility function. The $\pi(X, f)$ calculated according to the principle of zero utility is the root π of the equation*

$$E[u(\pi - X)] = 0$$

where X is the random risk. Note that $u(0) = 0$ for any utility function.

P.8 The Swiss premium principle :

Definition 8 *Let $f(.)$ be a continuous strictly monotonic real function defined on $\boldsymbol{R}$. Let $z \in [0, 1]$. Let X be a real random variable (risk). The Swiss premium associated to the risk X is the root of the equation in p*

$$E(f(X - zp)) = f((1 - z)p)$$

The Swiss premium is denoted by $\pi(X, f, z)$, since it is dependent on the choice of f and z. Note that $z = 0$ implies mean value principle while $z = 1$ implies zero utility principle.

P.9 The Orlicz principle :

Definition 9 *The premium $\pi(X)$ calculated according to the Orlicz principle is given as the root of the equation*

$$E[\phi(\frac{X}{p})] = \phi(1) \quad in \quad p$$

where $\phi(x)$, $x \geq 0$ is a function with the following properties

- *$\phi(x)$ is continuous and increasing in $\boldsymbol{R}$*

- *$\phi'(x)$ is nondecreasing in $\boldsymbol{R}$.*

P.10 The Esscher principle :

Definition 10 *The premium $\pi(X)$ calculated according to the Esscher principle is given by*

$$\pi(X) = \frac{E(Xe^{\alpha X})}{E(e^{\alpha X})}.$$

P.11 The Dutch premium principle : This was introduced by Van Heerwaarden and Kass (1992). The motivation was to incorporate some of the basic properties of premium principle, namely unjustified loading, no rip-off, preservation of stochastic order and of stop-loss order etc. For detail see Kaas, van Heerwaarden, and Goovaerts (1994).

Definition 11 *The Dutch premium for a risk X is given by*

$$\pi(X; \theta_r, \alpha] = E[X] + \theta_r E[(X - \alpha E[X])_+], \quad \alpha \geq 1,\ 0 \leq \theta_r \leq 1.$$

P.12 The PH transform principle : In a recent paper, Wang (1995a) proposed a new principle to calculate the risk-adjusted premium by using proportional hazard transform(PH) to a random risk. For a insurance risk X, we define the survivor function $S_X(t) = 1 - F_X(t)$, where $F_X(t)$ is the left hand tail probability. This tail probability plays a crucial role to define the new premium principle. The appropriate definition of this premium principle as given by Wang (1995b) is as follows.

Definition 12 *The PH transform is a mapping of one random variable X into another random variable Y*

$$\Pi_\rho : X \mapsto Y$$

such that

$$S_Y(t) = S_X(t)^{\frac{1}{\rho}} \quad (\rho \geq 1).$$

Now for a risk X, the risk-adjusted premium is the mean of the transformed variable $Y = \Pi_\rho(X)$ and is given by

$$\pi_\rho(X) = E[\Pi_\rho(X)] = \int_0^\infty S_X(t)^{\frac{1}{\rho}},$$

where $\rho \geq 1$ is called the (risk-averse) index.

2.1 Basic requirements for a consistent premium principle

Let X be an insurance risk with a distribution function $F_X(t) = \Pr(X \leq t)$. A premium principle is a rule that assigns a premium value to a given risk. To be consistent, a premium principle must satisfy some basic requirements. Goovaerts, De Vylder, and Haezendonck (1984) has extensive discussion of basic requirements. Boyle and Nye (1991) pointed out some constraints on a premium principle required for arriving at premiums for stop-los contracts. Wang (1995c) has compiled some most common requirements for a consistent premium principle which are as follows.

R.1 Positive loading and no ripoff : $E(X) \leq \pi(X) \leq max(X)$.

R.2 Linearity : $\pi(aX + b) = a\pi(X) + b, \quad a \geq 0$.

- $\pi(aX) = a\pi(X)$ is called scale invariant (homogeneous);
- $\pi(X + b) = \pi(X) + b$ is called translation invariant;
- $\pi(b) = b$ is called no unjustified loading.

R.3 Sub-additivity : For any two risks U and V regardless of dependence,

$$\pi(U + V) \leq \pi(U) + \pi(V).$$

R.4 Higher loading for a higher risk : If U is less risky than V (notation $U \prec V$ in some sense) then $\pi(U)$ should be less than $\pi(V)$.

R.5 Layer additivity : If a risk X is divided into countable stop-loss layers, then the layer price should be additive.

R.6 Decreasing stop-loss layer premiums : For any two stop-loss layer of the same length, higher layer should have lower premium than that of lower layer.

R.7 Increasing relative risk-loading.

3 Premium Calculation by spreading load over stop-loss layers.

Venter (1991, p.228) showed that "the only premium calculation principles that preserve additivity are those generated by transformed distributions." Similar results were pointed out by Harrison and Kreps (1979), Harrison and Pliska (1981), Delbaen and Haezendonck (1989), and Sondermann (1991). Their conclusive comments were that in an arbitrage free market, pricing of a risk should take place according to the expectation under a risk adjusted probability distribution. In the finance literature, this risk adjusted pdf is sometimes called risk-neutral distribution. In the same principle, Wang (1995a) used the proportional hazard transform measure to find a risk load adjusted distribution. The risk adjusted premium is then calculated by taking the expectation of the risk X, with respect to the adjusted distribution. The amount of risk load for a risk is determined by the market. Once the amount of risk load is determined by the market, does it uniquely determine the corresponding adjusted distribution or could it create infinitely many adjusted random variable whose expected value equals the market premium? So the question arises, how to choose the adjusted distribution? An insurer's assessment of a risk will reflect the subjective attitude of the individual company towards different layer of uncertain outcomes. As an example, let X be a risk ranges over $[0, 1,000,000)$ with expected net premium \$1,000, and risk load \$100. The spread of the risk load \$100 over the range (virtually layers) of X is subject to the attitude of the insurer's view to different layers of the risk. An insurer with theoretically very large wealth will spread the load almost uniformly over the layer. On the other hand, a small insurer will spread this risk load over different layers in a very skewed manner, very high load for the upper layer and smaller load for lower layers. The net premium for a risk is fixed by the nature of the risk and is given by the expected value of the risk, and its spread over the layer is determined by the inherent nature of the risk. The amount of load by each insurer and its spread are determined by the individual insurer whose reactions totally depends on its financial health. The market premium is determined by the market equilibrium which is caused by the joint effect of all individual insurers that constitute a market. So the amount of market load settled by the market may not reflect the actual behaviour of individual company but the joint behaviour of all insurers. Detail mathematical interpretation is deferred to section 9.5. Here our aim is to model the market behaviour in loading a gross premium. Before going in detail, let us put forward few definitions needed later for our pathological treatments to premium calculation for an insurable risk.

3.1 Some basic definitions

Definition 13 *A* **layer** $I_{(a,b]}$ *of a given risk* X *is defined by a stop-loss cover:*

$$I_{(a,b]} = \begin{cases} 0, & 0 \leq X < a; \\ (X-a), & a \leq X < b; \\ (b-a), & b \leq X. \end{cases}$$

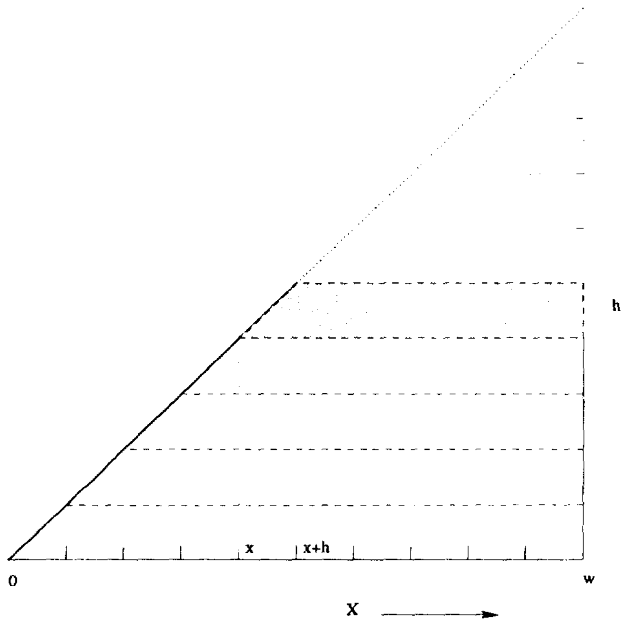

Figure 1: Stop-loss slicing of risk (cross-sectional slicing)

which has a distribution function

$$F_{I_{(a,b]}}(t) = \begin{cases} F_X(a+t), & 0 \le t < b-a, \\ 1, & b-a \le t. \end{cases}$$

where $F_X(.)$ is the cumulative distribution function of the risk X.

Obviously the amount of coverage defined by the definition of the layer above is a random quantity. So its expected value is meaningful.

Definition 14 *Net premium for a layer $I_{(a,b]}$ is defined to be the expected value of the layer and is denoted by $\pi(I_{(a,b]})$.*

Remark : The net premium for a risk X is denoted by $\pi(X)$ and is defined as the expected value of the layer $I_{(0,M]}$ where M is the largest value X can take.

Now we are going to define a infinitesimal concept of premium that will help to define the net premium of an arbitrary layer.

Definition 15 *The* **Net premium intensity** *of a random risk X at a point $X = x$, denotes by $\phi(x)$ is defined to be the derivative of the net premium for a layer $I_{(0,x]}$ and is given by*

$$\phi(x) = \frac{d}{dx}\pi(I_{(0,x]}).$$

In other words,

$$\phi(x) = lim_{h \to 0}\frac{\pi(I_{(x,x+h]})}{h}.$$

Literally, $\phi(x)$ is the expected loss per first dollar claim in an infinitesimal layer at stop loss level $X = x$. In other words, it is the marginal net premium at x.

Remark : Having defined net premium intensity, the net premium for any layer is easily given by

$$\pi(I_{(a,b]}) = \int_a^b \phi(x)dx.$$

Similarly,

$$\pi(X) = \int_0^\infty \phi(x)dx.$$

Note that if we slice the risk into equal layers (in fact it works for any arbitrary partition of the range of the risk) of width h, we have

$$X = \sum_{i=0}^{\infty} I_{(ih,(i+1)h]}$$

and

$$\pi(X) = \sum_{i=0}^{\infty} \pi(I_{(ih,(i+1)h]}).$$

• With the help of the concept of net premium intensity, layer additivity of net premium is easily demonstrated above. Until now, we have no idea what the function $\phi(x)$ will look like. But since the net premium principle satisfies all the basic requirements listed in R.1 to R.6 in the previous section, we can list the following characteristic of $\phi(x)$ whose proofs are obvious from previous section.

1. $0 \leq \phi(x) \leq 1$
2. $\phi(0) = 1$ and $\phi(M) = 0$ where M is the largest value X can take
3. $\phi(x)$ must be function of $F_X(x)$ (because of R.2).
4. $\phi(x)$ must be decreasing in x (on account of R.6).
5. $\phi(x)$ is independent of market and uniquely determined by the nature of the risk X.

Examples :

Let X be a risk with cdf $F_X(x)$, then by definition
Net premium for $I_{(0,x]} : \pi(I_{(0,x]}) = \int_0^x (1 - F_X(t))dt$
Net premium intensity : $\phi(x) = 1 - F_X(x)$
Net premium for $I_{(a,b]} : \pi(I_{(a,b]}) = \int_a^b \phi(t)dt$
Net premium for $X : \pi(X) = \int_0^\infty \phi(t)dt$

Using the concept of net premium intensity, our next task is to introduce the concept of loaded premium intensity which will be used to calculate the loaded premium for a risk and that of its arbitrary stop-loss layers.

3.2 Loaded premium calculation

In this section, our first task is to define a loaded premium intensity along the line of net premium intensity defined in the last section.

Definition 16 *Let ρ be the index of loading, and $\pi_\rho(I_{(0,x]})$ be the loaded premium for the layer $I_{(0,x]}$, then the* **loaded premium intensity** *at a point $X = x$, denoted by $\phi_\rho(x)$ is defined to be the derivative of the loaded premium for the stop-loss layer $I_{(0,x]}$ and is given by*

$$\phi_\rho(x) = \frac{d}{dx}\pi_\rho(I_{(0,x]}).$$

In other words,

$$\phi_\rho(x) = lim_{h\to 0}\frac{\pi_\rho(I_{(x,x+h]})}{h}.$$

Literally, $\phi_\rho(x)$ is the expected premium per first dollar claim in an infinitesimal layer at stop loss level $X = x$.

Remark : Having defined loaded premium intensity, the loaded premium for any layer is easily given by

$$\pi_\rho(I_{(a,b]}) = \int_a^b \phi_\rho(x)dx.$$

Similarly the loaded premium for the risk X is given by,

$$\pi_\rho(X) = \int_0^\infty \phi_\rho(x)dx.$$

Note that if we slice the risk into equal layers of width h, we have

$$\pi_\rho(X) = \sum_{i=0}^{\infty} \pi_\rho(I_{(ih,(i+1)h]}).$$

With the help of the concept of loaded premium intensity, layer additivity of loaded premium is easily demonstrated above. Until now, we have no idea what the function $\phi_\rho(x)$ will look like. But since the loaded premium principle should satisfy all the basic requirements listed in R.1 to R.7 in the previous section, we can list the following characteristic of $\phi_\rho(x)$ whose proofs are obvious from previous section.

1. For given $x, \phi_\rho(x)$ must be monotonic in ρ and there should be a unique value of ρ for which $\phi_\rho(x)$ is identical with net premium intensity $\phi(x)$ signifying zero loadings.

2. $0 \le \phi_\rho(x) \le 1$

3. $\phi_\rho(0) = 1$ and $\phi_\rho(M) = 0$ where M is the largest value X can take

4. $\phi_\rho(x)$ must be function of $F_X(x)$ (because of R.2).

5. $\phi_\rho(x)$ must be decreasing in x on account of R.6

6. $\phi_\rho(x)$ is not independent of market and is uniquely determined by the market and the nature of the risk X.

So far we have defined loaded premium intensity in terms of loaded layer premium $\pi_\rho(I_{(0,x]})$ which in turn depends on loaded premium intensity. Therefore we need to explore some other way how to create or model loaded premium intensity. With that view in mind let us define the concept of relative loading.

Definition 17 *The* **relative loading factor** *of a random risk X at a point $X = x$, denoted by $\psi_\rho(x)$ is defined to be the ratio of loaded premium intensity to the net premium intensity and is given by*

$$\psi_\rho(x) = \frac{\phi_\rho(x)}{\phi(x)}$$

where $\phi(x)$ is non-zero. Literally, $\phi_\rho(x)$ is the market loading on expected premium per first dollar possible claim in an infinitesimal layer at stop loss level $X = x$.

Until now, we have no idea what the function $\psi_\rho(x)$ will look like. But since the loaded premium principle should satisfy all the basic requirements listed in R.1 to R.7 in the previous section, we can list the following characteristic of $\psi(x)$ whose proofs are obvious from previous section.

1. For given x, $\psi_\rho(x)$ must be monotonic in ρ and there should be a unique value of ρ for which $\psi_\rho(x)$ is identically unity signifying zero loadings.

2. $1 \leq \psi_\rho(x)$

3. $\psi_\rho(0) = 1$ and $\psi_\rho(x) \geq 1$

4. $\psi_\rho(x)$ must be function of $F_X(x)$ (because of R.2).

5. $\psi_\rho(x)$ must be increasing in x on account of R.7

6. $\psi_\rho(x) \times \phi(x)$ must be decreasing in x on account of R.6

7. $\psi_\rho(x)$ is not independent of market and is uniquely determined by the market and the nature of the risk X.

On account of R.2, $\psi_\rho(x)$ must be a function of $F_X(x)$. In order to facilitate the modeling of relative loading factor, let us introduce the following definition of load generator.

Definition 18 *The* **load generator** $g_\rho(.)$, *indexed by* ρ, *is a mapping* $g_\rho(.): [0,1] \mapsto [1,\infty)$ *such that* $g_\rho(F_X(x))$ *is a relative loading factor and* $g_\rho(F_X(x))(1-F_X(x))$ *is a loaded premium intensity.*

Having introduced the concept of load generator, we have the following theorem about the intrinsic behaviour of the load generator.

Theorem 3.1 *Let* $g_\rho(t)$ *be a mapping* : $[0,1] \mapsto [1,\infty)$. *If* $g_\rho(t)$ *is continuous and differentiable then it is a load generator iff, (a)* $g_\rho(0) = 1$ *(b)* $g_\rho'(t) \geq 0$ *and (c)* $\frac{g_\rho'(t)}{g_\rho(t)} \leq \frac{1}{1-t}$

Proof : The results immediately follows because of R.6 and R.7. We will prove the "if" part (only forward direction). The "only if" (the backward direction) part follows immediately. Now let $F_X(x)$ be the cdf of the risk X. Condition (a) follows by definition. Since the relative loading factor $g_\rho(F_X(x))$ must be increasing in x (ref R.7), we have by differentiation $g_\rho'(t)F_X'(x) \geq 0$ for all x where $t = F_X(x)$. Since $F_X'(x) \geq 0$, $g_\rho'(t)$ must be non-negative. To prove (c), recall that the loaded premium intensity given by

$$\phi_\rho(x) = g_\rho(F_X(x))(1 - F_X(x))$$

must be decreasing in x. Differentiating both sides with respect to x we have

$$\phi_\rho'(x) = \phi_\rho(x)\{\frac{g_\rho'(t)}{g_\rho(t)} - \frac{1}{1-t}\}F_X'(x)$$

Since $F_X'(x) \geq 0$, (c) follows immediately. The "only if" part follows as a consequence of the above definition of load generator. (QED)

What we have achieved so far is that first we defined a load generator which can be easily built up depending on the market (see examples below). A load generator gives a loading factor for all values of x. When multiplied with the net premium intensity, it gives the loaded premium intensity. Then premium can be calculate for any stop-loss layer just simply by integration. The premium calculated using load generator satisfies all the basic requirements given in R.1 to R.7.

Examples 1. $g_\rho(t) = e^{\rho t}$ for $0 \leq \rho < 1$.

Note that $g_0(t) = 1$, hence $\rho = 0$ gives the case of no loading. $g_\rho(0) = 1$ and $g_\rho'(t) > 0$ and $\frac{g_\rho'(t)}{g_\rho(t)} = \rho \leq \frac{1}{1-t}$.

Examples 2. $g_\rho(t) = (1-t)^{-\rho}$ for $0 \leq \rho < 1$.
One can easily check that $g_\rho(t)$ is a load generator. $\rho = 0$ gives the case of no loading.

Examples 3. $g_\rho(t) = (1+\rho t)$ for $0 \leq \rho < 1$.
One can easily check that $g_\rho(t)$ is a load generator. $\rho = 0$ gives the case of no loading.

Examples 4. $g_\rho(t) = (Sec(\frac{\pi}{2}t)^\rho$ for $0 \leq \rho < 1$.
One can easily check that $g_\rho(t)$ is a load generator. $\rho = 0$ gives the case of no loading.

Examples 5. $g_\rho(t) = (1-t)^{-\frac{\rho-1}{\rho}}$ for $1 \leq \rho < \infty$.
One can easily check that $g_\rho(t)$ is a load generator. $\rho = 1$ gives the case of no loading.

Examples 6. Any convex combination of load generators given in example 1, 2, and 3.
In fact there are infinitely many load generator one could create. The above examples are only a few. For a pricing actuary, the first prudent job is to choose a suitable load generator that closely reflects the market. Once the load generator is chosen, the adjusted distribution could be found by routine operation mentioned above. The calculation of of premium of any coverage of the risk X is the appropriate expected value with respect to the adjusted distribution. So the final definition of the loaded premium is given by the the following definition.

Definition 19 *Let X be a risk with cdf $F_X(x)$. Let $g_\rho(t)$ be the load generator. Then the loaded premium for any stop-loss layer $I_{(a,b]}$ is given by*

$$\pi(I_{(a,b]}) = \int_a^b g_\rho(F_X(x))(1 - F_X(x))dx$$

and the loaded premium for the risk X is given by

$$\pi(X) = \int_0^\infty g_\rho(F_X(x))(1 - F_X(x))dx.$$

3.3 Link with PH transform premium

Wang (1995a) recently used PH-transform to define a premium principle which satisfy all the properties in R.1 to R.7. Let us see how PH-transform fits into our load generator. In order to do that let us start with our load generator given in example 4. The loading factor $\psi_\rho(x)$ is given by

$$g_\rho(F_X(x)) = (1 - F_X(x))^{\frac{1}{\rho}-1} \quad \text{for} \quad \rho \geq 1$$

and the loaded premium intensity is given by

$$\phi_\rho(x) = (1 - F_X(x))^{\frac{1}{\rho}}.$$

Hence the loaded premium for a stop-loss layer $I_{(a,b]}$ is given by

$$\pi_\rho(I_{(a,b]}) = \int_a^b (1 - F_X(x))^{\frac{1}{\rho}} dx$$

and the total loaded premium for a risk X is given by

$$\pi_\rho(X) = \int_0^\infty (1 - F_X(x))^{\frac{1}{\rho}} dx \quad \text{for} \quad \rho \geq 1$$

which is same as the premium calculated by PH-transform as was introduced by Wang (1995a). So our method is a generalization of PH-transforms to calculate the loaded premium.

3.4 Hazard rate of adjusted distributions

The loaded premium intensity $\phi_\rho(u)$ is a non-increasing function with $\phi_\rho(0) = 1$ and decreases down to zero. It can be considered as the survival function of a random variable. Let Y be such a random variable whose right hand tail probability at u matches with the loaded premium intensity $\phi_\rho(u)$ at $X = u$. Y is called risk neutral adjusted random variable whose survival function is given by

$$S_Y(u) = g_\rho(F_X(u))(1 - F_X(u)).$$

We can easily find a relation between the hazard rates of X and Y. Let $\lambda_X(u)$ be the hazard rate of the risk X and $\lambda_Y(u)$ be the hazard rate of the adjusted random variable Y. Then we have by definition

$$\lambda_Y(u) = -\frac{d}{du}\log S_Y(u) = \lambda_X(u) - \frac{g_\rho'(t)}{g_\rho(t)} F_X'(u)$$

where $t = F_X(u)$. Hence we have the following theorem.

Theorem 3.2 *(a) The hazard rate of the risk adjusted random variable Y at $Y = u$ is always less than or equal to the hazard rate of the risk X at $X = u$ for all u.*

(b) The adjusted random variable Y is stochastically larger than the risk X, i.e. $X_{st} \leq Y$.

Proof : (a) From above, we have

$$\lambda_Y(u) = \lambda_X(u) - \frac{g_\rho'(t)}{g_\rho(t)} F_X{}'(u)$$

where $t = F_X(u)$. Since $g_\rho'(t) \geq 0$ and $F_X{}'(u) \geq 0$, results immediately follow.
(b) By definition, we have

$$1 - F_Y(u) = g_\rho(F_X(u))(1 - F_X(u))$$

and $g_\rho(F_X(u)) \geq 1$ for all u. Hence $1 - F_Y(u) \geq 1 - F_X(u)$. Or $F_Y(u) \leq F_X(u)$ for all u and that completes the proof.

Examples 1 (revisited) : $g_\rho(t) = e^{\rho t}$ for $0 \leq \rho < 1$. So,

$$1 - F_Y(u) = e^{\rho F_X(u)}(1 - F_X(u)) \text{ and } \lambda_Y(u) = \lambda_X(u) - \rho F_X{}'(u).$$

The hazard rate at u is decreased by the amount $\rho F_X{}'(u)$.

Examples 5 (revisited) : $g_\rho(t) = (1-t)^{-\frac{\rho-1}{\rho}}$ for $1 \leq \rho < \infty$. So,

$$1 - F_Y(u) = (1 - F_X(u))^{\frac{1}{\rho}}, \text{ and } \lambda_Y(u) = \frac{1}{\rho}\lambda_X(u).$$

Obviously, the hazard rate is proportionately deflated in this case. In fact in each of those examples 1 to 5 mentioned earlier, the adjusted distribution is created by deflating the hazard rate of X.

4 Premium Calculation by spreading load over franchise layers

Since Venter (1991) proposed to calculate premium using transformed distribution, Albrecht (1992) created an strange coverage (could be called franchise coverage) and argued that the premium principle using transformed distribution fails to calculate a consistent premium for the franchise coverage. In this section we will attempt to use our load spreading technique on the franchise layers of the risk. In the earlier section, we have sliced the risk X in stop-loss layer and then spread the load over the stop-loss layer in a consistent fashion. In this section we will attempt to use our load spreading technique on the franchise layers of the risk. First we will split our risk X into franchise layers (to be called longitudinal slicing) and then spread the load over the franchise layers. So the approach will be exactly same as in the previous subsection but applied on a longitudinal slicing as opposed to cross section slicing used in stop-loss layerings.

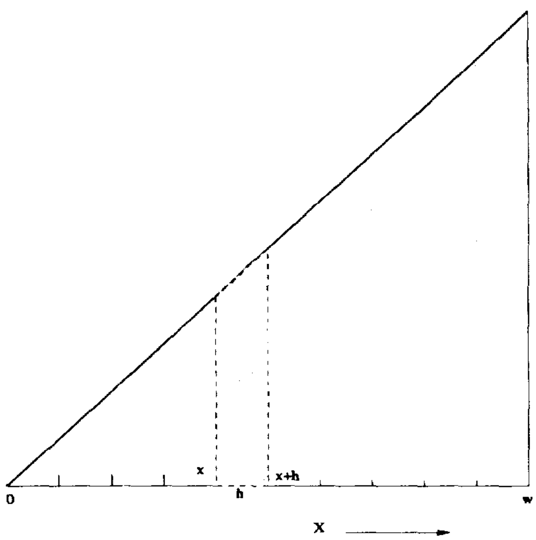

Figure 2: longitudinal slicing of risk (franchise-sectional slicing)

4.1 Some basic definitions related to longitudinal slicing

Definition 20 *A* **longitudinal layer** $L_{(a,b]}$ *of a given risk* X *is defined by a franchise cover:*

$$L_{(a,b]} = \begin{cases} 0, & 0 \le X < a; \\ X, & a \le X < b; \\ 0, & b \le X. \end{cases}$$

which has a distribution function

$$F_{L_{(a,b]}}(t) = \begin{cases} F_X(a) + 1 - F_X(b), & 0 \le t < a, \\ F_X(t) + 1 - F_X(b), & a \le t < b, \\ 1, & b \le t. \end{cases}$$

where $F_X(.)$ *is the cumulative distribution function of the risk* X.

Obviously the amount of coverage defined by the definition of the layer above is a non-trivial random quantity. So its expected value is meaningful.

Definition 21 *Net premium for a layer* $L_{(a,b]}$ *is defined to be the expected value of the layer and is denoted by* $\pi(L_{(a,b]})$.

Remark : The net premium for a risk X is denoted by $\pi(X)$ and is defined as the expected value of the layer $L_{(0,M]}$ where M is the largest value X can take.

Now we are going to define a infinitesimal concept of premium that will help to define the net premium of an arbitrary layer.

Definition 22 *The* **Net premium intensity** *of a random risk* X *at a point* $X = x$ *under longitudinal slicing, denotes by* $\phi^l(x)$ *is defined to be the derivative of the net premium for a layer* $L_{(0,x]}$ *and is given by*

$$\phi^l(x) = \frac{d}{dx}\pi(L_{(0,x]}).$$

In other words,

$$\phi^l(x) = lim_{h \to 0} \frac{\pi(L_{(x,x+h]})}{h}.$$

Literally, $\phi^l(x)$ *is the expected loss per first dollar claim in an infinitesimal layer at franchise cover level* $X = x$.

Remark : Having defined net premium intensity, the net premium for any layer is easily given by

$$\pi(L_{(a,b]}) = \int_a^b \phi^l(x)dx.$$

Similarly,

$$\pi(X) = \int_0^\infty \phi^l(x)dx.$$

Note that if we slice the risk into equal longitudinal layers (in fact it works for any arbitrary partition of the range of the risk) of width h, we have

$$X = \sum_{i=0}^{\infty} L_{(ih,(i+1)h]}$$

and

$$\pi(X) = \sum_{i=0}^{\infty} \pi(L_{(ih,(i+1)h]}).$$

With the help of the concept of net premium intensity, layer additivity of net premium is easily demonstrated above. Until now, we have no idea what the function $\phi^l(x)$ will look like. But since the net premium principle satisfies all the basic requirements listed in R.1 to R.6 in the previous section, we can list the following characteristic of $\phi^l(x)$ whose proofs are obvious from previous section.

1. $0 \leq \phi^l(x)$
2. $\phi^l(x)$ is independent of market and uniquely determined by the nature of the risk X.

Note that $\phi^l(x)$ is much simpler than that in case of cross sectional slicing.

Examples :

Let X be a risk with cdf $F_X(x)$, then by definition
Net premium for $L_{(0,x]}$: $\pi(L_{(0,x]}) = \int_0^x (tf_X(t))dt$
Net premium intensity : $\phi^l(x) = xf_X(x)$
Net premium for $L_{(a,b]}$: $\pi(L_{(a,b]}) = \int_a^b \phi^l(t)dt$
Net premium for X : $\pi(X) = \int_0^\infty \phi^l(t)dt$

Using the concept of net premium intensity, our next task is to introduce the concept of loaded premium intensity which will be used to calculate the loaded premium for a risk and that of its arbitrary franchise cover layers.

4.2 Loaded premium calculation under longitudinal slicing

In this section, our first task is to define a loaded premium intensity along the line of net premium intensity defined in the last section.

Definition 23 *Let ρ be the index of loading, and $\pi_\rho(L_{(0,x]})$ be the loaded premium for the layer $L_{(0,x]}$, then the* **loaded premium intensity** *at a point $X = x$, denoted by $\phi^l_\rho(x)$ is defined to be the derivative of the loaded premium for the franchise cover layer $L_{(0,x]}$ and is given by*

$$\phi^l_\rho(x) = \frac{d}{dx}\pi_\rho(L_{(0,x]}).$$

In other words,

$$\phi^l_\rho(x) = lim_{h\to 0}\frac{\pi_\rho(L_{(x,x+h]})}{h}.$$

Literally, $\phi^l_\rho(x)$ is the expected premium per first doller claim in an infinitesimal layer at franchise coverage $X = x$.

Remark : Having dubiously defined loaded premium intensity, the loaded premium for any longitudinal layer is easily given by

$$\pi_\rho(L_{(a,b]}) = \int_a^b \phi^l_\rho(x)dx.$$

Similarly the loaded premium for the risk X is given by,

$$\pi_\rho(X) = \int_0^\infty \phi^l_\rho(x)dx.$$

Note that if we slice the risk into equal layers of width h, we have

$$\pi_\rho(X) = \sum_{i=0}^{\infty} \pi_\rho(L_{(ih,(i+1)h]}).$$

With the help of the concept of loaded premium intensity, longitudinal layer additivity of loaded premium is easily demonstrated above. Until now, we have no idea what the function $\phi^l_\rho(x)$ will look like. But since the loaded premium principle should satisfy all the basic requirements listed in R.1 to R.7 in an earlier section, we can list the following characteristic of $\phi^l_\rho(x)$ whose proofs are obvious from previous section.

1. For given $x, \phi^l_\rho(x)$ must be monotonic in ρ and there should be a unique value of ρ for which $\phi^l_\rho(x)$ is identical with net premium intensity $\phi^l(x)$ signifying zero loadings.

2. $0 \leq \phi^l_\rho(x)$

3. $\phi^l_\rho(x)$ is not independent of market and is uniquely determined by the market and the nature of the risk X.

So far we have defined loaded premium intensity in terms of loaded layer premium $\pi_\rho(L_{(0,x]})$ which in turn depends on loaded premium intensity. Therefore we need to explore some other way how to create or model loaded premium intensity. With that view in mind let us define the concept of relative loading.

Definition 24 *The* **relative loading factor** *of a random risk X at a point $X = x$, denoted by $\psi_\rho(x)$ is defined to be the ratio of loaded premium intensity to the net premium intensity and is given by*

$$\psi^l_\rho(x) = \frac{\phi^l_\rho(x)}{\phi^l(x)}$$

where $\phi^l(x)$ is non-zero. Literally, $\phi^l_\rho(x)$ is the market loading on expected premium per first doller possible claim in an infinitesimal layer at franchise cover level $X = x$.

Until now, we have no idea what the function $\psi^l_\rho(x)$ will look like. But since the loaded premium principle should satisfy all the basic requirements listed in R.1 to R.7 in the previous section, we can list the following characteristic of $\psi^l_\rho(x)$ whose proofs are obvious from previous section.

1. For given x, $\psi^l_\rho(x)$ must be monotonic in ρ and there should be a unique value of ρ for which $\psi^l_\rho(x)$ is identically unity signifying zero loadings.

2. $1 \leq \psi^l_\rho(x)$

3. $\psi^l_\rho(0) = 1$ and $\psi^l_\rho(x) \geq 1$

4. $\psi^l_\rho(x)$ must be function of $F_X(x)$ (because of R.2).

5. $\psi^l_\rho(x)$ must be increasing in x on account of R.7

6. $\psi^l_\rho(x)$ is not independent of market and is uniquely determined by the market and the nature of the risk X.

On account of R.2, $\psi_\rho^l(x)$ must be a function of $F_X(x)$. In order to facilitate the modeling of relative loading factor, let us introduce the following definition of load generator.

Definition 25 *The* **load generator** $g_\rho(.)$, *indexed by* ρ, *is a mapping* $g_\rho(.)$: $[0,1] \mapsto [1,\infty)$ *such that* $g_\rho(F_X(x))$ *is a relative loading factor and* $g_\rho(F_X(x))\phi^l\rho(x)$ *is a loaded premium intensity.*

Having introduced the concept of load generator, it is important to note that the load generator must be increasing function because of empirical restriction on premium as pointed in Venter (1991) One can consider the load generator as a increasing function of x, in which case it maps $[0,\infty)$ to $[1,\infty)$ and the principle looses its homogeneity property.

What we have achieved so far is that first we defined a load generator which can be easily built up depending on the market (see examples below). A load generator gives a loading factor for all values of x. When multiplied with the net premium intensity, it gives the loaded premium intensity. Then premium can be calculate for any franchise layers just simply by integration. The premium calculated using load generator satisfies all the basic requirements given in R.1 to R.7.

Examples 1. $g_\rho(t) = e^{\rho t}$ for $0 \leq \rho < 1$.

Note that $g_0(t) = 1$, hence $\rho = 0$ gives the case of no loading. $g_\rho(0) = 1$ and $g_\rho{}'(t) > 0$.

Examples 2. $g_\rho(t) = (1-t)^{-\rho}$ for $0 \leq \rho < 1$.

One can easily check that $g_\rho(t)$ is a load generator. $\rho = 0$ gives the case of no loading.

Examples 3. $g_\rho(t) = (1-t)^{-\frac{\rho-1}{\rho}}$ for $1 \leq \rho < \infty$.

One can easily check that $g_\rho(t)$ is a load generator. $\rho = 1$ gives the case of no loading.

Examples 4. Any convex combination of load generators given in example 1, and 2.

In fact there are infinitely many load generator one could create. The above examples are only a few. For a pricing actuary, the first prudent job is to choose a suitable load generator that closely reflects the market. Once the load generator is chosen, the adjusted distribution could be found by routine operation mentioned above. The calculation of premium of any coverage of the risk X is the appropriate expected value with respect to the adjusted distribution. So the final definition of the loaded premium using longitudinal slicing is given by the the following definition.

Definition 26 *Let X be a risk with cdf $F_X(x)$. Let $g_\rho(t)$ be the load generator. Then the loaded premium for any franchise layer $L_{(a,b]}$ is given by*

$$\pi(L_{(a,b]}) = \int_a^b g_\rho(F_X(x))\phi^l(x)dx$$

and the loaded premium for the risk X is given by

$$\pi(X) = \int_0^\infty g_\rho(F_X(x))\phi^l(x)dx.$$

4.3 Link with net premium principle

In case of a net premium principle, total premium is given by $(1+\theta)E(X)$ for the risk X where θ is the constant load which is same as taking our loading function $(1+\theta)$ independent of x but uniquely determined by ρ. On the other hand if θ becomes function of ρ and x, we get the above premium principle based on longitudinal slicing and is given by $\pi(X) = E(1+\theta(\rho, X))X$. If we take $\theta(\rho, X) = (1 - F_X(x))^{\frac{1}{\rho}-1} - 1$, then $\pi(X) = \int_0^\infty x(1 - F_X(x))^{\frac{1}{\rho}-1}dF_X(x)$ which is similar (not equal) to what we have found in example 5 under stop loss slicing. Note that under PH-transform, Wang (1995a) showed that $\pi_\rho(X) = \int_0^\infty (1 - F_X(x))^{\frac{1}{\rho}}dx$, which is equal to $\int_0^\infty \frac{x}{\rho}(1 - F_X(x))^{\frac{1}{\rho}-1}dF_X(x)$, which is similar (but not equal) to what we have derived above. By choosing suitable function for $\theta(\rho, X)$, we can easily show that our longitudinal slicing principle leads to the total premium equal to the sum of $E(X)$ and a risk premium $R(X) = E(X\theta(\rho, X))$ as was done in Ramsay (1994) or as formulated in Carrier (1994). So our method is a generalized result to calculate the loaded premium.

5 Link with utility theory and arbitrage free market

Venter (1991) studied premium calculation principles under one aspect of competitive market theory : the impossibility of systematic arbitrage. He showed that the principles based on second moments or utility theory lead to arbitrage possibilities some other principles, namely adjusted distribution, do not.

Albrecht (1992), in his discussion paper to Venter contributions, argued that -

> *... in contrast to the theory of financial markets–it is not reasonable to demand that insurance markets are arbitrage free.*

In addition he claimed that the adjusted distribution principles put forward by Venter are invalid. Both Venter (1991) and Albrecht (1992) had discussed some important issues but failed to justify (i) no arbitrage (ii) utility theory and (iii) adjusted distribution principle in premium calculation. Our load spreading principle really accommodate and justify all of the above three issues.

In earlier sections, we have derived the load spreading technique and hence premium calculation. It is virtually nothing but creating an adjusted distribution with a careful attention to the empirical restrictions imposed on premium calculation. Now we are going to show how no-arbitrage principle and utility theory are taken care in our premium calculation.

Let $U(t)$ be the utility function followed by all the insurance/reinsurance company. Let $w \in [0, \infty)$ be the amount of wealth owned by a particular company. Note that w could be any positive but finite number. Let P_w be the premium (under no competition) assumed to be charged by a company having wealth w. Under utility principle P_w is given by the equation

$$E(U(w + P_w - X)) = U(w)$$

where X is the random risk to be insured and E stands for the expected value. Obviously the premium P_w will be a decreasing function of w in accordance with the fact that large insurers can maintain the same level of security at a lower price. Venter (1992) argued that

> *Our risk theory training leads actuaries to believe that the smaller needed security premium for large insurers will induce them [all insurers] to charge lower prices. This is not necessarily true in the market, however. Larger insurers may in fact charge the market price and make more profit.*

In our view, none of the above arguments is precise and the market price is not properly defined. In what follows, we will show that the market price is determined by load spreading technique and the heterogeneity of premium P_w and its spread induce reinsurances and risk sharing.

Let $\phi_{P_w}(x)$ be the intensity of loaded premium P_w. For two arbitrary level of wealth say u and v, ,where $u < v$, P_v must be less than P_u. The intensity of loaded premium must be dependent on wealth level such that for every pair u and v where $u < v$, there should be at least one real number x_* such that $\phi_{P_u}(x_*) = \phi_{P_v}(x_*)$, and for all $x < x_*$, $\phi_{P_u}(x)$ must be less than or equal to $\phi_{P_v}(x)$, and for all $x > x_*$, $\phi_{P_u}(x)$ must be greater than or equal to $\phi_{P_v}(x)$. In other words, the intensity of loaded premium is less variable for a large insurer than that of a small insurer. Let $\phi_m(x)$ be the intensity of market premium, where m stands for market (an abuse of indexing notation). Obviously

$$\phi_m(x) = Min_{w\in[0,\infty)}\{\phi_{P_w}(x)\}.$$

Hence the market premium for the risk X is given by

$$\pi(X) = \int_0^\infty \phi_m(s)ds. \tag{5.1}$$

Note that

$$\pi(X) \leq P_w \quad \text{for all} \quad w \in [0,\infty).$$

It is interesting to note that the market premium has an upper bound which is the minimum of all P_w for $w \in [0,\infty)$. It is the utility function and various nature of spreading of load by different insurer which induce risk sharings in layers by different insurer. Let

$$\begin{aligned} \Gamma_w &= \text{set of real numbers } x \text{ where intensity for market premium equals intensity for premium} \\ &= \{x : \ \phi_m(x) = \phi_{P_w}(x), \ x \in R^+\} \end{aligned}$$

and $\Gamma_w = \cup_{i=1}^k [a_i, b_i]$ be the k disjoint union of the intervals $[a_i, b_i]$ where k is some positive integer. Then a insurer with wealth w will get the business of stop-loss coverage $I_{\Gamma_w} = \sum_{i=1}^k I_{[a_i,b_i]}$ at a premium

$$\begin{aligned} \pi(I_{\Gamma_w}) &= \int_{\Gamma_w} \phi_m(x)dx \\ &= \sum_{i=1}^k \int_{a_i}^{b_i} \phi_{P_w}(x)dx \end{aligned}$$

and get reinsurance coverage from the market for the coverage $X - I_{\Gamma_w}$ at market price. Thus the risk X is sold in the market at a uniform price $\pi(X)$ creating no arbitrage opportunity and the price does take utility theory into considerations.

6 Conclusion

First we have introduced the concept of net and loaded premium intensity. The concept of load generator is introduced and it culminates into an adjusted distribution which is used to calculate a loaded premium for a risk X. The premium calculation method under the above load generator technique satisfies all of the basic requirements $R.1$ to $R.7$ for a consistent premium principle as illustrated in section 9.2.1. Utility theory and arbitrage free market concept are for the first time in the literature properly accommodated in this method of premium calculation.

In this paper we have simply put forward the definition of the new method of premium calculation. One can easily use this method in a simple and consistent way to calculate the increased limit factors (ILF) for casualty actuaries.

Further research are needed to investigate the properties of the load generator methods of premium calculation, in particular order preserving properties.

References

Albrecht, P. (1992). Premium calculation without arbitrage– a note on a contribution by g. venter. *ASTIN Bulletin*, 22:247–254.

Borch, K. H. (1961). The utility concept applied to the theory of insurance. *ASTIN Bulletin*, 1:170–191.

Borch, K. H. (1990). *Economics of Insurance*. Advanced Textbook in Economics, vol. 29, Amsterdam: Elsevier.

Boyle, P. P. and Nye, D. J. (1991). A note on stop loss premiums. *Journal of Risk and Insurance*, 58:536–547.

Buhlmann, H. (1980). An economic premium principle. *ASTIN Bulletin*, 11:52–60.

Buhlmann, H. (1985). Premium calculation from top down. *ASTIN Bulletin*, 15:89–101.

Carriere, J. F. (1993). Discussion of ramsay's paper – loading gross premiums for risk without using utility theory. *Transactions of the Society of Actuaries*, XLV:337.

Cummins, J. D. (1990). Asset pricing models and insurance ratemaking. *ASTIN Bulletin*, 20:125–166.

Cummins, J. D. (1991). Statistical and financial models of insurance pricing. *Journal of Risk and Insurance*, 58:261–302.

Delbaen, F. and Haezendonck, J. (1989). A martingale approach to premium calculation principles in an arbitrage free market. *Insurance: Mathematics and Economics*, 8:269–277.

Freifelder, L. R. (1976). *A Decision Theoretic Approach to Insurance Ratemaking*. The S.S. Huebner Foundation, Pa, distributed by Irwin Inc., Homewood, Ill.

Gerber, H. U. (1974). On additive premium calculation principles. *ASTIN Bulletin*, 7:215–222.

Goovaerts, M. J., De Vylder, F., and Haezendonck, J. (1984). *Insurance Premiums: Theoy and Applications*. North-Holland, Amsterdam.

Hogg, R. and Klugman, S. (1984). *Loss Distributions*. John Wiley, New York.

Kaas, R., van Heerwaarden, A., and Goovaerts, M. (1994). *Ordering of Actuarialal Risks*. CAIRE, Brussels (Education Series).

Meyers, G. G. (1991). The competitive market equilibrium risk load formula for increased limits ratemaking. *Proceedings of the Casualty Actuarial Society*, LXXVIII:163–200.

Ramsay, C. M. (1993b). Loading gross premiums for risk without using utility theory; and discussions. *Transactions of the Society of Actuaries*, XLV:305–349.

Reich, A. (1986). Properties of premium calculation principles. *Insurance: Mathematic and Economics*, 5:97–101.

Robbin, I. (1992). Discussion of meyers' paper – the competitive market equilibrium risk load formula for increased limits ratemaking. *Proceedings of the Casualty Actuarial Society*, LXXIX:367–384.

Rothschild, M. and Stiglitz, J. E. (1970). Increasing risk i: a definition. *Journal of Economic Theory*, 2:225–243.

Sondermann, D. (1991). Reinsurance in arbitrage-free markets. *Insurance: Mathematics and Economics*, 10:191–202.

Van Heerwaarden, A. E. and Kass, R. (1992). The dutch premium principle. *Insurance: Mathematics and Economics*, 11:129–133.

Van Heerwaarden, A. E., Kass, R., and Goovaerts, M. J. (1989). Optimal reinsurance in relation to ordering of risks. *Insurance: Mathematics and Economics*, 8(1):11–17.

Venter, G. G. (1991). Premium calculation implications of reinsurance without arbitrage. *ASTIN Bulletin*, 21:223-230.

Venter, G. G. (1992). Premium calculation implications of reinsurance without arbitrage– author's reply on the note by p. albrecht. *ASTIN Bulletin*, 22:225–226.

Wang, S. (1995a). Insurance pricing and increased limits ratemaking by proportional hazards transforms. Research report 94-22, The Institute of Insurance and Pension Research, University of Waterloo. (Also to appear on Insurance: Mathematics and Economics).

Wang, S. (1995b). Ordering of risks under ph transforms. Research report 95-05, The Institute of Insurance and Pension Research, University of Waterloo.

Wang, S. (1995c). The price of risk : an actuarial/economic model. Research report 95-06, The Institute of Insurance and Pension Research, University of Waterloo.