ACTUARIAL RESEARCH CLEARING HOUSE
1999 VOL. 1

Insurance and Annuity Calculations

in the Presents of Stochastic Interest Rates

by

Dale Borowiak

Department of Statistics

The University of Akron

Akron, Ohio 44325

Abstract

In computing actuarial measurements, such as the moments of present value functions for insurance and annuities, the stochastic nature of the interest rate is commonly ignored. In this paper we treat the interest rate as a normal random variable and present general formulas for the mean and variance of the present value function for insurance and annuities. An example demonstrating these formulas is presented. It is found that taking the force of interest to be fixed at a mean value results in slightly under estimating the actuarial present value along with the variance for both insurance and annuities.

1. Introduction

The effect of changing interest rates on actuarial calculations has attracted some attention. Jordan (1967) observed the change on annuity and reserve computations when the interest rate was varied from one value to another. Recently, the effect of interest rates on surrender features of insurances have been explored by Grosen and Jorgensen (1997). More specifically, the force of interest has been modeled using a stochastic process component. Examples of these approaches are given by Nielsen and Sandmann (1995) and Panjer and Bellhouse (1980).

In this paper we first present a general present value function based on a stochastic interest rate in the discrete future lifetime setting. To compute actuarial calculations the direct approach of Panjer and Bellhouse (1980) is used. As in the approach of Kellison (1991) the force of interest is asumed to be normal. The mean and variance for both life insurance and life annuities are then discussed and an example is given. The paper ends with a consideration of the extention of these computations to the continuous future time structure.

2. Present Value Function

There are two basic modeling frameworks in actuarial computations. The first is a discrete system where the future time period is partitioned into equal length intervals corresponding to (1/m) th of a year. These time periods are denoted $E_j = [(j - 1)/m, j/m)$ for $j = 1, 2, \ldots$ The financial action, such as a premium payment or benefit payment occurs at some predefined point in these intervals. The second, referred to as the continuous system, can be considered as the limiting case of the discrete system as m approaches infinity. In this section only the discrete system is considered.

The analysis of a financial action occurring at some future time involves the computation of the present value function. In the discrete model let the force of interest in E_j be δ_j while the benefit amount after n periods or at future time n/m is denoted b(n). The corresponding present value function is

$$(2.1) \qquad PV(n) = b(n) \exp(-\psi_n) \quad \text{where} \quad \psi_n = \sum_{j=1}^{n} \delta_j$$

The present value function (2.1) forms the basis of the analysis of life insurance models.

The common discrete annuity setting in which a payment, denoted π_j is made at the beginning of each period. The present value of the annuity at time n/m is the sum of the present value functions corresponding to each payment up to time n/m and is given by

$$(2.2) \qquad PVA(n) = \sum_{j=o}^{n} \pi_j \exp(-\psi_j)$$

where $\psi_o = 1$.

3. Moments of the Present Value Function

In this section the force of interest is considered a random variable. In general, many distributions may be utilized to model the force of interest but we consider only the normal distribution. A dependent structure may be imposed by applying an autoregressive model (see Box and Jenkins (1976)) on the interest rates over the time periods. In this paper we consider independent interest rates..

Let the discrete setting hold where in E_j the force of interest, δ_j, is considered a independent normal random variable with μ_j and variances σ_j^2 for $j \geq 1$. For integers n and $i < j$ let

$$(3.1) \qquad \alpha_n = \sum_{j=1}^{n} \mu_j \qquad \alpha_{j,i} = \sum_{r=i+1}^{j} \mu_\rho \qquad \beta_n = \sum_{j=1}^{n} \sigma_j^2 \qquad \text{and} \qquad \beta_{j,i} = \sum_{r=i+1}^{j} \sigma_r^2.$$

Central moments, with respect to the force of interest, of (2.1) and (2.2) are now computed. The form of the normal moment generating function along with the notations given by (3.1) are utilized.

For positive integer s taking the expectation of (2.1) yields

$$(3.2) \qquad E\{PV(n)^s\} = b(n)^s \exp(-s\,\alpha_n + (1/2)s^2\beta_n)$$

For discrete annuities taking the expectation of (2.2) gives

$$(3.3) \qquad E\{PVA(n)\} = \sum_{j=o}^{n} \pi_j \exp(-\alpha_j + (1/2)\,\beta_j)$$

The second moment, obtained by squaring (2.2) and then taking the expectation, is

$$(3.4) \qquad E\{PVA(n)^2\} = \sum_{j=1}^{n} \pi_j^2 \exp(-2\alpha_j + 2\,\beta_j) + 2 \sum_{i<j}^{n}\sum \pi_i \pi_j \exp(-2\,\alpha_i + 2\,\beta_j - \alpha_{j,i} + (1/2)\,\beta_{j,i})$$

In the stochastic force of interest setting to compute the actuarial present value, APV, and higher order moments for insurance and annuities the expectation of (3.2) - (3.4), over the appropriate range, with respect to the curtate future lifetime K is calculated.

4. Homogeneous Moments

In this section the homogenous situation where the period interest rates are independent and normal with $\mu_j = \mu$ and $\sigma_j^2 = \sigma^2$ for $j \geq 1$ is considered. For discrete life insurance let K be the curtate future lifetime and the benefit be the constant b which is paid at the end of the period. For positive integer s if

(4.1) $$\delta(s) = s\,\mu + (1/2)s^2\sigma^2$$

then (3.2) becomes

(4.2) $$E\{PV(k)^s\} = b \exp(-(k+1)\delta(s))$$

Also, for a discrete life annuity where the payments are all π and occur at the start of each period the expected value (3.3) is

(4.3) $$E\{PVA(k)\} = \pi \sum_{k=0}^{k} \exp(-k\,\delta(1))$$

The higher moments for annuities can be computed using (3.4).

In the homogeneous stochastic interest case the expected present value functions, given by (4.2) and (4.3), are the standard present value functions for discrete life insurance and life annuities. For a complete reference see Bowers, et. al. (1986). For $s = 1$, (4.1) becomes $\delta(1) = \mu + (1/2)\sigma^2$. We note that for most values of σ, $\delta(1)$ is close to μ and the effect of the stochastic nature of the interest rate is small. This is demonstrated in the following example.

Ex. In this example we utilize the life tables given in Bowers et. al. (1986, pg. 560). For a person age 30 consider both whole life insurance and a whole life annuity. Benefits are paid at the end of the year and annuity payments are made at the start of each year surviving year. Here $\mu = .08$ and for chosen values of σ the actuarial present value, APV, and the variances of both the insurance and the annuities are computed. These values are listed in Table I.

Table I : Whole Life Insurance and Annuity Computations

	Whole Life Insurance		Whole Life Annuity	
σ	APV	Variance	APV	Variance
.00	.051037	.009402	12.3428	1.59071
.01	.051105	.009423	12.3494	1.67665
.02	.051310	.009487	12.3690	1.93737
.03	.051654	.009596	12.4018	2.37937
.04	.052142	.009752	12.4481	3.01484

From Table I we observe that the effect of stochastic interest rate is minimal. There is, however, more of an impact on annuity calculations as contrasted to the insurance computations.

5. Continuous Models

The continuous setting can be viewed as the limiting case of the discrete setting where m approaches infinity. In this section the continuous counterparts of the previous formulas are presented.

Let the future lifetime be the continuous random variable T where the continuous payment is given by π_t for $t \geq 0$.. Further, for the force of interest the mean and variance functions are assumed to be integrable over the future lifetime and are denoted by μ_u and σ_u^2 for $u \geq 0$. For continuous insurance the s moment is given by (3.2) where for t replacing n

(5.1) $$\alpha_t = \int_o^t \mu_u \, du \qquad \text{and} \qquad \beta_t = \int_o^t \sigma_u^2 \, du$$

For continuous annuities the expected present value function, from (3.3) is

(5.2) $$E\{PVA(t)\} = \int_o^t \pi_u \exp(-\alpha_u + (1/2)\beta_u) du$$

Also, the second moment, based on (3.4), is

(5.3) $$E\{PVA(t)\} = \int_o^t \pi_u^2 \exp(-2\alpha_u + 2\beta_u) du$$

$$+ \int_{o \leq s \leq v \leq t}^{t} \int^{t} \pi_s \pi_v \exp(-(\alpha_s + \alpha_v) + (1/2)(3\beta_s + \beta_v)) ds \, dv$$

The homogeneous case for continuous models is analogus to that of the discrete setting. Here $\mu_u = \mu$ and $\sigma_u^2 = \sigma^2$ for $u \geq 0$ produces $\alpha_t = t\mu$ and $\beta_t = t\sigma^2$. As in the discrete setting the standard continuous moment formulas hold with fixed force of interest $\delta(1)$.

References

Bowers, N., Gerber, H., Hickman, J., Jones, D. and Nesbit, C. (1986) **Actuarial Mathematics**, Published by The Society of Actuaries.

Box, G. E. P. and Jenkins, G. M. (1976) **Time Series Analysis, Second Edition.** Holden-Day, San Francisco.

Jordan, C. (1991) **Society of Actuaries Textbook on Life Contingencies**, Published by The Society of Actuaries.

Grosen, A. and Jorgensen, P. L. (1997) "Valuation of Early Exercisable Interest Rate Guarantees", **The Journal of Risk and Insurance**, Vol.64, No.3, pp. 481-503.

Kellison, S , G. (1991) **The Theory of Interest**, Published br Irwin, Burr Ridge Ill.

Nielsen, J. and Sandmann, K. (1995) "Equity - Linked Life Insurance" **Insurance : Mathematics and Economics**, No.16, pp. 225-253.

Panjer, H. H. and Bellhouse, D. R. (1980) "Stochastic Modelling of Interest Rates with Applications to Life Contingencies, **The Journal of Risk and Insurance**, Vol. 27, pp. 91-110.

ACTUARIAL RESEARCH CLEARING HOUSE
1999.1

Production supported by the William-Elliott Chair of Insurance
and Resource Technologies, Inc.
Distribution sponsored by the SOA:
ARCH, Education and Research Section, and Committee on Knowledge Extension Research

THIS MODULE WAS PRODUCED BY:
Arnold F. Shapiro
Technical Support: support@afShapiro.com (E-mail) or 814-234-0995 (FAX)

SYSTEM REQUIREMENTS:

The minimum system requirements to run this software are:

Microsoft Windows© version 3.1 or higher
IBM-compatible (Pentium 100 MHz or better)
CD-ROM player (4X or better)
Sound card (SoundBlaster Pro or compatible)
640K plus 8MB extended memory
Hard disk with at least 7MB of free disk space
SVGA Windows©-supported graphics monitor
Windows©-compatible mouse or other pointing device

INSTALLATION (Windows© 3.1x)

Open Microsoft Windows© 3.1x
Insert CD into your CD-ROM drive
Open the Program Manager
Bring up the run dialogue box [click on "File"; click on "Run"]

Type the CD-ROM drive letter followed by a colon and a backslash (:\)
Type: SETUP and press ENTER
Follow the directions on the screen

The following things will happen when you install this software:
The program is installed in a directory on your hard disk
The default directory is "C:\arc_33"
ARC = Actuarial Research Conference
A group is created in Program Manager
The title of the group is "arc33_10"
The group name is "arc33_10.GRP"

RUNNING THE PROGRAM

Restore the Program Manager, if necessary
Double click the "Program Manager" icon
Alternately, click the "Program Manager" icon and click "Restore"
Restore the "arc33_10" group, if necessary
Double click the "arc33_10" icon
Alternately, click the "arc33_10" icon and click "Restore"
Double click the "ARC 33 1.0" icon

INSTALLATION (Windows© 95)

Start the computer, Windows© 95 appears
Insert CD into your CD-ROM drive
Click the start button at the bottom (left) of the screen
If the start button does not show, move cursor to screen bottom
Clicking start reveals a menu

Select "Run" on the menu
In "run" dialog box type CD-ROM Drive letter, a colon, and a backslash
Example: If Drive = D, type "D:\" (no quotes)
Next, type "Setup.exe" and press Enter
Directions appear. Follow them
The program installs
The following things will happen when you install this software:

The program is installed in a directory on your hard disk
The default directory is "C:\arc_33"
ARC = Actuarial Research Conference

A Group program is created in "Program" or the menu that "Start" reveals.
The title of the group is "arc33_10"
The group name is "arc33_10.GRP"