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Stochastic Optimization Techniques for Pricing Callable Bonds: Continuous time approach

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Paper presents a new methodology for obtaining fast algorithms and closed form solutions for pricing callable bonds in continuous time models.. The methodology uses stochastic optimization techniques where an issuer of a bond is trying to minimize the price of a callable bond in a game against the bondholder. Some flexibility to the model is added by allowing partially irrational calls. We assume that the term structure of interest rates is described by a set of stochastic differential equations with respect to Brownian Motion, which satisfy the Markovian Property. Theoretical results obtained in the paper allow for quick estimation of different market value characteristics such as duration and convexity for portfolios consisting of huge numbers of callable bonds. Results are directly applicable to regulatory scenario testing, immunization and market value accounting.

Recent developments in the Fixed Income Securities Market generated high demand for pricing valuation models. As market becomes more complex and bond prices become more volatile those models produce dramatic impact on trading strategies, portfolio management and other types of financial analysis. This process has significant impact back on the market creating new demand for new type of securities. In particular the idea of replicating portfolios creates hedging strategies which in turn give rise to numerous financial instruments. Some of them are so complex that any visible analytics with respect to price sensitivity analysis are basically reduced to cash-flow testing under Monte-Carlo stimulation technique. In this spiral evolution of financial market and financial science we may observe two types of requirements for market price valuation models. On one hand we have sophisticated multidimensional models designed to price individual securities, on the other we have less time-consuming models designed to evaluate pricing characteristics of huge portfolios. In confronting the problem of determining which model to use a portfolio analyst may have to consider many issues. Time limitation may be one of them. Consider market value accounting in determining present value of future surplus in an asset/liability management game in a large insurance company. Assume that portfolio analyst wants to calculate the distribution of such a surplus. He faces a problem of pricing thousands of callable bonds along different scenarios under a dynamically evolving structure of interest rates. In such a case the user may well prefer the crude *Price-to-worst* formula to the generally accepted binomial lattice backward substitution. Similar situation will happen with passthroughs. And again harsh reality inclines us in favor of simplistic crude formulas. We observe here the famous Heisenberg principle from physics which when applied here says that one can only get accuracy by sacrificing the speed of calculations. In this article we will try to show that the gap between slow accurate and fast crude algorithms can be reduced by looking at the matter from a different angle.

It is well known that any contingent cash flow may be considered as combinations of generic options. Among them the American Call option is considered to be the most

/complicated for calculations. Two major reasons for this fact are an added dimension (user may exercise call at any point in time) and embedded assumptions of the effectiveness of an issuer. The last assumption converts the pricing problem into a stochastic optimization problem with stopping time. To make this statement clear, consider the callable bond CB and its counterpart regular Bond B_S with sinking fund S. Assume that the schedule of sinking fund payment depends on interest rate movement, but stipulated up front. Denote P_b -price of bond CB. and P_s -price of bond B_S. Then

Optimization Problem 1

$$P_b = \min_{s \in S} P_s$$

This fact explains why Dynamic Programming is the main tool in dealing with callable bonds. In general practice the price of an American call option is calculated by backward substitution using a version of a discrete or discretized arbitrage free model for interest rate movement. In such an algorithm it is assumed that the issuer(caller) is absolutely efficient, meaning that he knows the exact solution of the optimization problem 1. This 100% efficiency assumption is obviously not true. There are plenty of factors involved. One of them is a wide bid-ask spread which reflects the low liquidity of the corporate bond market. Another is inefficiency in the process of making a decision. A bond may be called if the treasurer does not have enough information or the information he has is not exactly correct. Bond may be called as a result of debt restructuring, refinancing due to reorganization, merger, acquisitions, etc.

Here we consider a continuous time model justified in addition to other reasons by the fact that in the Bond Market the caller has to pay accrued interest at the exercise date.
In the model we introduce two causes of calls. First is the rational cause, when a caller exercises his option based on his idea of what the fair market price of the bond is at the moment. Second is the irrational cause. The irrational cause forces the issuer to call a bond regardless of financial market analysis and therefore completely randomly from the issuer's point of view. We do not deny rationality to irrational cause. Irrationality in this case means independence from issuer's will. Necessity of financial restructuring may be one of examples of irrational cause. We further assume that the caller is subject to error while making a decision and therefore his solution is quasi-optimal. The methodology we use to implement these aberrations is discussed later in the more technical part of the paper. As it happens quite often in stochastic processes theory the added stochastic complexity simplifies the final result. Consequentially we are able to deliver methodology which bears significant potential in obtaining fast algorithms and closed form solutions for problems of pricing callable bond. Finally we will discuss how this methodology may be developed for pricing mortgage backed securities.
In the paper we demonstrate

a) how irrational cause and an imperfection in a rational decision may be taken into account.

b) how model built on top of the imperfection assumption allows for different realizations dependent on user's preferences.

c) methodology which bears potential for a wide range pricing model including MBS.

d) that under certain assumptions the price of a callable bond satisfies second order *Ordinary Differential Equation* (Equations 8,9,10) with general solution available from the traditional sources [5][6] .

Underlying Stochastic Interest Rate Model

As we have mentioned before, we build our theory under continuous time assumptions. We assume that the term structure of interest rates is described by a stochastic differential equation with respect to a Wiener process w_t. For our purposes we do not have to specify the underlying model. However we are going to be more specific while demonstrating examples. We assume that the underlying model allows no arbitrage. To eliminate market price of risk and to be able to use the discounting technique to obtain prices we assume that stochastic measure is already risk neutral. According to [7] this assumption would not restrict the generality of the model. We define interest rates dynamics as follows

Equation 1

$$dx_t = b_t(x_t) \bullet dt + \sigma_t(x_t) dw_t$$

We assume that x_t describes term structure of interest rates, b and σ are the so called drift and diffusion coefficients. Parameter σ reflects volatility of x, and b describes deterministic characteristics of the movement. For illustrative purposes we assume single dimension for all the parameters. As an example consider Cox, Ingersoll, Ross (CIR) [2] model

Equation 2

$$dx_t = k*(\theta - x_t) \bullet dt + \sigma \cdot \sqrt{x_t} dw_t$$

Here k is the so called elasticity parameter, θ -meanreversion and σ is the volatility parameter for the model. This one dimensional model has significant accumulation of analytical tools [2], [3] and therefore is very attractive as a basis for building applications. To understand the meaning of θ, one may set σ to zero. It is easy to see that in the resulting deterministic equation, x_t converges to θ, with $t \to \infty$. The same is true for the mathematical expectations Ex_t. Parameter k controls the speed of convergence.

Introducing Irrational Cause (IC)

Assume existence of intensity function $r(t,x)$ which depends on time t and interest rate x. This function describes intensity of irrational cause. It says that conditional probability that the bond will be called due to IC on the interval [t, t+Δt] is $r \cdot \Delta t + o(\Delta t)$. Naturally, $r(t,x)$ has to be a non-decreasing function of x. Note that the intensity function may be used to control mandatory maturity. Any fixed maturity may be approximated by IC with intensity unrestrictedly high in the neighborhood of the maturity. The simplest example of an IC function is a constant function $r(t,x) = 1/T$ where T is a constant number which we choose to equal maturity. Another example of $r(x,t)$ is $r(x,t) = a_t + b_t \cdot f(d_t \cdot x + c_t)$ where f may be any S-shaped decreasing function. The coefficients are part of tuning up a model and may also depend on time to maturity T and type of stochastic interest rate model used as a basis.

Introducing quasi-optimality in a caller decision.

We introduce quasi-optimality in caller decision by allowing the caller to deviate somewhat from optimality. How far a caller may deviate from the actual optimal solution is controlled by quasi-optimality parameter ε. We utilize here Prof. N. Krylov's [1] idea of controlling optimal call with non-negative intensity function. Roughly speaking this means the following. Assume that T_O is the optimal stopping (calling) time. If the issuer has complete control over the situation, he will call exactly at T_O minimizing the price of the security. If he deviates from T_O and calls the bond at times $T_O-\varepsilon$, T_O, $T_O+\varepsilon$ with equal probabilities of 1/3, the resulting price will be higher than optimal, though the difference will be negligibly small for small ε. This means that the caller has a positive intensity function in the vicinity of the optimal calling time. Lower ε causes higher value of the intensity function in the vicinity of T_O and therefore better approximation of the optimal stopping time.

Notations and Preliminary Information

Consider a callable bond maturing at time T and paying coupon f_t with continuously compounded interest. Assume that the current level of interest rate is equal to x. Assume also that the bondholder is entitled to a premium g_t at time t if the bond is called. Assume that the bond has par value of 1 and that g_t is a decreasing continuous function such that $g_T = 1$. Denote τ as a fixed random moment (not necessarily optimal) when the bond is called. In accordance with our assumption of no-arbitrage and risk-neutrality the price $v_\tau(x,t)$ of such a bond is a mathematical expectation of discounted future cashflow.

$$v_\tau(x,t) = E\{ \int_t^{\tau \wedge T} \exp\left(-\int_t^s x_u \cdot du\right) \cdot f_s ds + \exp\left(- \int_t^{\tau \wedge T} x_u \cdot du \right) \cdot g(\tau \wedge T)$$

This is not a price of the callable bond CB yet. It is rather the price of a non-callable bond with a scheduled circumstantial call τ. It is possible to show that price $v(x,t)$ of a callable issue is a minimum price that the issuer of the bond may obtain given that he is allowed to chose the most beneficial for him circumstantial call. Therefore

Optimization Problem 2

$$v(x,t) = \min_{\tau} E[\{ \int_{t}^{\tau \wedge T} \exp\left(-\int_t^s x_u \cdot du\right) \cdot f_s ds + \exp\left(- \int_{t}^{\tau \wedge T} x_u du \right) \cdot g(\tau \wedge T) \}]$$

$$dx_s = b ds + \sigma dw_s \qquad x_0 = \mathrm{x}$$

It is shown in [1] that $v(x,t)$ satisfies the Non-Linear Partial Differential Equation.

Equation 3

$$g - v(t,x) + [\frac{1}{2} \cdot \sigma^2 \frac{\partial^2 v}{\partial x^2} + b \cdot \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} - x \cdot v + f_t + v - g]_- = 0$$

$$[a]_- = \min(a,0)$$

This in turn is equivalent to the following three conditions:

Equation 4

$$g - V \geq 0$$

$$g - v > 0 \Rightarrow Lv + f = 0$$

$$g = v \Rightarrow Lv + f \geq 0$$

$$\text{where } Lv = \frac{1}{2} \sigma \frac{\partial^2 v}{\partial x^2} + b \cdot \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} - x \cdot v$$

Effective numerical procedures have been developed to solve equation *4*, but a description of them is beyond the scope of this paper.

Pricing with irrational cause

As we have discussed earlier, we define the irrational cause through intensity function $r(x,t)$. This means that probability at time t_0 that the bond will be called due to the irrational cause on the interval $(t,t+dt)$ is

$$r(x_t,t) \cdot \exp(-\int_{t_0}^{t} r(x_u,u) \cdot du) \cdot dt$$

For the sake of simplicity assume that $t_0 = 0$. We reformulate the optimization problem 2 by introducing additional variable y_t, and assuming perpetuity in payment T=∞. It is not difficult to see that the last assumption does not cause a loss in generality. To achieve the actual maturity at T one has to choose unrestrictedly high intensity function in the small vicinity of T. The optimization problem 2 may be rewritten as

Optimization Problem 3

$$v(x,y) = \min_{\tau} E[\int_0^{\tau} \exp(-y_s)\cdot f_s\, ds + \exp(-y_\tau)\cdot g(y_\tau)]$$

$$dx_s = b ds + \sigma dw_s \qquad x_0 = x$$

$$dy_s = x_s ds \qquad y_0 = y$$

Now consider an individual trajectory ω where the bond is scheduled to be called at time τ(ω). The conditional contribution of this trajectory to the criteria of the optimization problem 3 is

$$\int_0^{\tau} \exp(-y_s)\cdot f_s\, ds + \exp(-y_\tau)\cdot g(y_\tau)]$$

Assume now that together with circumstantial call τ the bond may be called due to the irrational cause defined by intensity function ***r(x,t)***. Therefore the expected contribution from the individual trajectory is

$$\exp(-\int_0^{\tau} r(x_t,t)dt)\cdot\{\int_0^{\tau} \exp(-y_s)\cdot f_s\, ds + \exp(-y_\tau)\cdot g(y_\tau)\} +$$

$$\int_0^{\tau}[r(x_t,t)\cdot \exp(-\int_0^{t} r(x_s,s)ds)\cdot\{\int_0^{t} \exp(-y_s)\cdot f_s\, ds + \exp(-y_t)\cdot g(y_t)\}]dt =$$

$$= \mathrm{I} + \mathrm{II}$$

If the issuer decides to call the bond at τ, two different events may happen . First takes place when the bond indeed called at τ and the first part (**I**) of the expression above evaluates the expected contribution from this event. In this case $\exp(-\int_0^{\tau} r(x_t,t)dt)$ is the probability that the bond will be called by the issuer without interference of the irrational cause. The second event takes place when the irrational call happens before τ. Accordingly the second part (**II**) is a contribution from such an event. After some transformations we have

$$\mathrm{I}+\mathrm{II} = \exp(-\int_0^{\tau} r(x_t,t)dt)\cdot \exp(-y_\tau)g(\tau) + \{\int_0^{\tau} \exp\left(-y_t - \int_0^{t} r(x_s,s)ds\right)\cdot(f_t + r(x_t,t)\cdot g(t))\,dt\}$$

Now we may get rid of y and return to previous notations.

$$\mathrm{I}+\mathrm{II}=\exp(-\int_0^{\tau}(r(x_t,t)+x_t)dt)\cdot g(\tau)+\{\int_0^{\tau}\exp\left(-\int_0^{t}(r(x_s,s)+x_s)ds\right)\cdot(f_t+r(x_t,t)\cdot g(t))\,dt\}$$

The price of a bond with given call τ is a mathematical expectation of contributions of individual trajectories

$$v_{\tau}(x,0)=E(\mathrm{I}+\mathrm{II})$$

Effective market will price the bond by choosing the call time to a maximum disadvantage of a bondholder. Therefore we obtain price of the bond as a result of the

Optimization Problem 4

$$v(x,t)=\min_{\tau} v_{\tau}(x,t)$$

or

$$v(x,t)=\min_{\tau} E[\exp(-\int_0^{\tau}(r(x_t,t)+x_t)dt)\cdot g(\tau)+$$
$$\{\int_0^{\tau}\exp\left(-\int_0^{t}(r(x_s,s)+x_s)ds\right)\cdot(f_t+r(x_t,t)\cdot g(t))\,dt\}]$$

Applying Equation 3 to this problem we obtain a Differential Equation for $v(x,t)$

Equation 5

$$g-v(t,x)+[\frac{1}{2}\cdot\sigma^2\frac{\partial^2 v}{\partial x^2}+b\cdot\frac{\partial v}{\partial x}+\frac{\partial v}{\partial t}-(x+r)\cdot v+f_t+rg+v-g]_-=0$$

where again $[a]_-=\min(a,0)$.

Pricing with Irrational Cause and Quasi-Optimality Assumption

Our quasi-optimality assumption stipulates inability of the bondholder to call the bond exactly at optimal time. Assume that the issuer may call the bond before or after the optimal time T^{τ}. Probability distribution of the calling time is set by the intensity function ρ. It says that the conditional probability that the bond will be called by the issuer himself on the interval [t, t+Δt] is $\rho\cdot\Delta t+o(\Delta t)$. If issuer has the ability to pick any intensity functions, his choice will be the approximation of T^{τ} by the intensity function which has infinitely high values in the small neighborhood of T^{τ}. To illustrate the point consider an example. Assume that the issuer calls the bond in accordance with the intensity function

$$\rho^{\varepsilon}(x,t)=\rho^{\varepsilon}(t)=\begin{cases}0 & t\le T-\sqrt{\varepsilon}\\ \frac{1}{\varepsilon} & t\ge T-\sqrt{\varepsilon}\end{cases}$$

Probability that the bond will be called on the interval $[T-\sqrt{\varepsilon}, T]$ is $1-e^{-\frac{1}{\sqrt{\varepsilon}}}$. It approaches one as ε approaches zero. We see that with ε close to zero, the bond is almost certain to be called at the optimal time. On the other hand with positive ε, call will be spread randomly in the vicinity of T, making price higher than optimal.

Now we have two intensity functions. The difference between them is that quasi-optimality intensity function ρ is controlled by the issuer while irrational cause intensity function r(x,t) is imposed on him. Denote $v^{\varepsilon}(x,t)$ the value of the optimization problem 4 given that the issuer is not allowed to stop exactly at the optimal stopping time τ but rather he is allowed to use any intensity functions ρ with values not higher than $\frac{1}{\varepsilon}$. Apparently, $v^{\varepsilon}(x,t) \Rightarrow v(x,t)$ as $\varepsilon \Rightarrow 0$. It is shown in [1] that under some assumptions v^{ε} is a solution of the system of the following equations.

Equation 6

$$\varepsilon \cdot ((Lv + f + g \cdot r) - (g - v)) + g - v = 0 \quad \text{and} \quad g \le v \qquad (A)$$

or

$$Lv + f + g \cdot r = 0 \quad \text{and} \quad g - v \ge 0 \qquad (B)$$

where L is a differential operator

$$Lv = \frac{1}{2} \cdot \sigma^2 \frac{\partial^2 v}{\partial x^2} + b \cdot \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} - (x + r) \cdot v$$

Equation 6 is not easy to solve. One of the complications comes from the uncertainty and incorrectness of the boundary conditions. We may overcome this problem by imposing rather innocent (from market value point of view) assumptions. We assume that the bond will be called unconditionally if the interest rate falls below *low bound* level x_L. similarly we assume that the bond is sold at the market price of a non-callable bond if the interest rate is higher than *high bound* level x_H. Those assumptions should not have significant impact in most reasonable pricing areas. Thus it is not unreasonable to assume that the bond with 7% coupon and sufficiently low call price *g* will be called if the interest rate falls below 0.5% and would be considered non-callable by a hypothetical market in a 30% interest rate environment. These assumptions create the needed simple boundary conditions

$$v(x_L, t) = g(t) \quad \text{and} \quad v(x_H, t) = P_B(x_H, t)$$

where $P_B(x_H, t)$ is the price of a non-callable bond with the same characteristics as the callable one. For such a problem in its entire complexity it is quite unlikely that a closed form solution exists even for simplest term structure models. At the same time with the added boundary conditions the finite difference methodology works almost the same way as in a regular problem involving Partial Differential Equations.

Pricing with stationary processes

Assume that x_t is a stationary process. This means that coefficients σ and b in equation 1 do not depend on time. Assume also that intensity function $r(x,t)$ is a function only of state x, $r(x,t) = r(x)$. Then an equation for price v has the same appearance as equation 6

Equation 7

$$\varepsilon \cdot ((Lv + f + gr - (g - v)) + g - v = 0 \quad \text{and} \quad g \le v \qquad (A)$$

or

$$Lv + f + gr = 0 \quad \text{and} \quad g - v \ge 0 \qquad (B)$$

There is a crucial difference, however. v does not depend on t, and operator Lv becomes an ordinary differential operator

$$Lv = \frac{1}{2} \cdot \sigma^2 v'' + b \cdot v' - (x + r) \cdot v$$

To simplify the matter we further assume that the issuer is 100% optimal. In this case we may use equation 5 for Pricing with Irrational Cause (without Quasi-Optimality assumptions). This time equations for price become quite simple. There exists a constant x_c such that

Equation 8

$$\frac{1}{2}\sigma^2 v'' + bv' + (x + r)v + f + rg = 0$$

$$\text{where} \quad x > x_c$$

$$\text{and} \quad v(x_c) = g; \quad v'(x_c) = 0; \quad v(\infty) = 0;$$

It is a second order ordinary differential equation with somewhat bizarre boundary conditions. In case of CIR model with constant coefficients (equation 2) we have

Equation 9

$$\frac{1}{2}\sigma \cdot xv'' + k \cdot (\theta - x_t)v' + (x + r)v + f + rg = 0 \qquad \text{(A)}$$

$$\text{where} \quad x > x_c \quad \text{and} \quad v(x_c) = g; \quad v'(x_c) = 0; \quad v(\infty) = 0; \qquad \text{(B)}$$

Let us make one more simplification. Let $r(x) = \frac{1}{T}$ where T is the assumed time to maturity. By doing this we ensure that the average time to a call due to the irrational cause is equal to maturity.

Equation 10

$$\frac{1}{2}\sigma \cdot xv'' + k \cdot (\theta - x_t)v' + (x + \frac{1}{T})v + f + \frac{1}{T}g = 0 \qquad \text{(A)}$$

$$\text{where} \quad x > x_c \quad \text{and} \quad v(x_c) = g; \quad v'(x_c) = 0; \quad v(\infty) = 0; \qquad \text{(B)}$$

Equation 10 has a closed form solution which can be obtained from traditional sources [5],[6].

Summary

The idea of distributing maturity using irrational cause may well be implemented in building pricing methodology for Mortgage Backed Securities. As it is shown above additional "stochasticity" may improve pure mathematical characteristics of a problem. This of course cannot be done without sacrificing some features. The partial differential equation approach "implies" Markovian Property for underlying processes. This means that *burnout* or *seasonality* effects in modeling of prepayment functions should be reconsidered accordingly. Theoretically speaking it is not a severe restriction because any non-Markovian process may be converted to Markovian by adding a dimension. Unfortunately for practitioner, an added dimension may be not just a problem but an insurmountable obstacle.

The approach discussed in this paper takes into account some unavoidable aberrations in the issuer's decision while calling a bond. Depending on assumptions the user may prefer a full scale model described by the set of equations 6 , or the most simplified version shown by equations 9 and 10 where the general solution is available.

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