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Equilibrium in Competitive Insurance Markets Under Adverse Selection and Yaari's Dual Theory Of Risk

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Abstract

Under Yaari's dual theory of risk, we determine the equilibrium separating contracts for high and low risks in a competitive insurance market, in which risks are defined only by their expected losses, that is, a high risk is a risk that has a greater expected loss than a low risk. Also, we determine the pooling equilibrium contract when insurers are assumed non-myopic. Utility theory generally predicts that optimal insurance indemnity payments are nonlinear functions of the underlying loss due to the nonlinearity of agents' utility functions. Under Yaari's dual theory, we show that under mild technical conditions the indemnity payment is a piecewise linear function of the loss, a common property of insurance coverages.

1. Introduction

By assuming adverse selection in a competitive insurance market in which agents are expected utility maximizers, one can explain many common provisions found in insurance policies—deductibles, (nonlinear) coinsurance, and maximum limits, (Young and Browne, 1997) and (Fluet and Pannequin, 1997). Two issues remain unresolved under utility theory: (1) Most insurance policies provide an indemnity benefit that is a piecewise linear function of the underlying loss. Under utility theory, however, nonlinearity of the utility function in wealth forces optimal insurance to be nonlinear, in general (Young and Browne, 1997). (2) Utility theory predicts that a risk-averse agent will buy less than full coverage when the premium charged is greater than the actuarial expected value (e.g., Mossin, 1968; or Smith, 1968). Mossin, however, observes that many of his 'otherwise rational' friends purchase full insurance.

Yaari (1987) develops a theory of risk, parallel to utility theory, by modifying the independence axiom of von Neumann and Morgenstern (1947). In Yaari's theory, attitudes toward risks are characterized by a distortion applied to probability distribution functions, in contrast to utility theory in which attitudes toward risks are characterized by a utility function of wealth. Segal and Spivak (1990) show that, under Yaari's dual theory of risk, a risk-averse agent may purchase full coverage even if the premium charged is greater than the agent's expected loss. Thus, under Yaari's dual theory, the second puzzle mentioned above is solved. This paper primarily addresses the first problem.

In this paper, we apply Yaari's dual theory of risk to determine equilibrium insurance policies in the presence of adverse selection. We show that, in this model, low risks may prefer to pool with high risks and purchase full coverage, a result anticipated by Segal and Spivak (1990). More importantly, we show that, under mild technical conditions, optimal insurance provides an indemnity benefit that is a piecewise linear function of the loss. This occurs because under Yaari's theory, the value functional is linear in wealth, as opposed to utility theory, in which the value functional, or expected utility, is nonlinear in wealth. In addition, by reconsidering optimal insurance under Yaari's dual theory, we examine the robustness of optimal insurance results obtained under utility theory. We find several qualitative similarities between optimal insurance contracts under the two theories.[1]

In Section 2, we state our basic assumptions and, in Section 3, formulate our problem concerning equilibrium separating and pooling insurance contracts in a competitive, anticipatory market. We analyze the equilibrium separating and pooling policies in Section 4. First, we consider the simple case in which the loss amount is fixed but the probabilities of loss differ between the risk classes, the case that Rothschild and Stiglitz (1976) study. Then, we consider the case of a general random loss and obtain conditions satisfied by the optimal separating and pooling contracts. We show that under mild technical requirements, these conditions are sufficient to uniquely determine the optimal contracts. In this case, we find that the equilibrium contract is a piecewise linear function of the underlying loss. We present an illustrative example in Section 5 and show how piecewise linear coverage with a deductible and a maximum limit can be realized as an equilibrium policy in our framework. In the last section, we summarize our results and compare them with ones obtained under utility theory.

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[1] Also Doherty and Eeckhoudt (1995) consider optimal insurance under Yaari's dual theory of risk.

2. Market for Insurance

We assume that insurers compete for the business of low and high risks in the market, with the proportion of high risks equal to ρ. Low and high risks are identical except for their distributions of loss $X \geq 0$. A risk is defined to be *low* versus *high* if the expected loss of a low risk is less than or equal to the expected loss of a high risk.

We assume that individuals are rational as determined by Yaari's dual theory of risk (1987): Risk preferences under Yaari's theory can be represented by the expected value with respect to a distorted probability. Let $\tilde{g}$ denote the *distortion* function that distorts the probability that a random variable will exceed a given value: $\tilde{g}$ maps the unit interval [0, 1] onto itself and is nondecreasing. In particular, $\tilde{g}(0) = 0$ and $\tilde{g}(1) = 1$. The certainty equivalent of a random outcome Y is equal to (Yaari, 1987)

$$\int_{-\infty}^{0} \left(\tilde{g}\left[S_Y(t)\right] - 1\right)dt + \int_{0}^{\infty} \tilde{g}\left[S_Y(t)\right]dt = \int_0^1 S_Y^{-1}(q)\,d\tilde{g}(q),$$

in which S_Y is the decumulative distribution function of Y, $S_Y(t) = \Pr\{Y > t\}$, $t \in \mathbf{R}$, and S_Y^{-1} is its inverse, defined by $S_Y^{-1}(p) = \inf\{t\colon S_Y(t) \leq p\}$, $0 < p < 1$.[2] Thus, an individual is rational under Yaari's theory if he or she orders random outcomes Y according to the value $\int_0^1 S_Y^{-1}(q)\,d\tilde{g}(q)$ and chooses Y to maximize this value.

One can think of $\int_0^1 S_Y^{-1}(q)\,d\tilde{g}(q)$ as a generalized expected value, in which one distorts the probabilities before calculating the expectation. Note that for $\tilde{g}$ equal to the identity, $\int_0^1 S_Y^{-1}(q)\,dq = E[Y]$.

We assume that $\tilde{g}$ is strictly increasing; thus, individuals value more highly outcomes that are more likely to occur, that is, they preserve the ordering of first stochastic dominance. In addition, we assume individuals are risk-averse; that is, they order risks according to second stochastic dominance, or equivalently, $\tilde{g}$ is convex (Yaari, 1987).

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[2] When referring to a particular risk type, we further subscript S_Y and S_Y^{-1} by L or H, to represent low or high risk, respectively.

Let $I:\mathbf{R}^+ \to \mathbf{R}^+$ denote an insurance policy which pays $I(x)$ to an insured if the insured suffers a loss of size x. We assume that I is continuous and piecewise differentiable with $0 \le I' \le 1$ and $I(0) = 0$. Such an insurance policy I is called *feasible*. The inequality $I' \ge 0$ restricts the indemnity benefit to be a nondecreasing function of the underlying loss in order to prevent a policyholder misrepresenting a loss downward. Similarly, the inequality $I' \le 1$ restricts the indemnity benefit to increase at a slower rate than the underlying loss in order to prevent a policyholder from misrepresenting a loss upward. Insureds can buy at most one insurance policy to cover their potential losses. Denote the initial wealth of an insured by w.

The expected *value* V_i to an individual of type i, $i = L$ (for low risk) or H (for high risk), who buys an insurance policy I for a premium P is given under Yaari's theory by

$$\begin{aligned} V_i(I,P) &= \int_0^1 S^{-1}_{w-P+I(X)-X,i}(p)\,d\tilde{g}(p) \\ &= w - P - \int_0^1 S^{-1}_{X-I(X),i}(p)\,dg(p) \\ &= w - P - \int_0^1 \left[S_i^{-1}(p) - I\left(S_i^{-1}(p)\right)\right] dg(p) \\ &= w - P - \int_0^\infty g\left[S_i(t)\right]dt + \int_0^\infty g\left[S_i(t)\right]dI(t), \end{aligned} \tag{2.1}$$

in which g is the distortion defined by $g(p) = 1 - \tilde{g}(1-p)$, $0 \le p \le 1$.[3] Note that g is an increasing, concave distortion. When the random variable is the underlying loss random variable X, then we simply write S_i for $S_{X,i}$, $i = L$ or H.

3. Separating and Pooling Equilibria

We assume that insurers behave as if they are risk neutral and that administrative expenses and investment income are zero. We assume that the insurance market is a competitive market, as in Rothschild and Stiglitz (1976), and that each insurance policy

[3] In simplifying the value V, we rely on the following: Two random variables X and Y are *comonotonic* if there exist nondecreasing real-valued functions h_1, h_2 and a random variable Z, such that $X = h_1(Z)$ and $Y = h_2(Z)$. The restrictions on the rate of growth of a feasible insurance policy result in the random variables X, $I(X)$, and X-$I(X)$ being pairwise comonotonic. In the case of comonotonic random variables, we have that $S^{-1}_{X+Y} = S^{-1}_X + S^{-1}_Y$. Note that a constant is comonotonic with respect to any random variable. Also, the integral $\int_0^1 S^{-1}_X(p)\,d\tilde{g}(p)$ is asymmetric in that $\int_0^1 S^{-1}_{-X}(p)\,d\tilde{g}(p) = -\int_0^1 S^{-1}_X(p)\,dg(p)$. These integrals are special cases of Choquet integrals with respect to nonadditive measures. See Denneberg (1994) for more background in nonadditive measure theory.

earns nonnegative profits. A set of contracts is in *equilibrium* if there does not exist an additional contract which, if also offered, would make a positive profit. Information asymmetries exist in the market because of regulatory prohibitions on underwriting or the inability of insurers to acquire relevant information.[4]

Under asymmetric information, a separating equilibrium consists of a set of contracts (I_H, P_H) and (I_L, P_L) satisfying the conditions stated above, together with the self-selection constraints:

$$V_H(I_H, P_H) \geq V_H(I_L, P_L); \tag{3.1}$$

$$V_L(I_L, P_L) \geq V_L(I_H, P_H). \tag{3.2}$$

The nonnegative profit constraints, or premium constraints, are written

$$P_i \geq E_i[I_i(X)], \tag{3.3}$$

for $i = L$ or H. Competition will force the premium constraints to hold in equilibrium. Note that, in terms of the (inverse) decumulative distribution function, we can write an expectation, $E[I(X)]$, as $\int_0^1 I\left[S^{-1}(p)\right]dp = \int_0^\infty S(t)dI(t)$.

Full coverage at an actuarially fair price for both low and high risks constitutes a separating equilibrium if and only if $E_L[X] = E_H[X]$. Indeed, when insurers cannot observe a risk's type, full insurance is not feasible if $E_L[X] \neq E_H[X]$, because only the less expensive policy would be bought. If, however, $E_L[X] = E_H[X]$, then full coverage at an actuarially fair price of $E_L[X]$ would constitute an equilibrium.

In what follows, we show that, under Yaari's dual theory, if equilibrium is achieved by a pair of separating policies and if $E_L[X] < E_H[X]$, then the equilibrium coverage for high risks is full insurance at the actuarially fair price of $E_H[X]$, and the equilibrium coverage for low risks is less than full coverage. Let (I_S, P_S) denote the solution to the following problem:

$$\max_{I,P} V_L(I,P) = \max_{I,P}\left[w - P - \int_0^\infty g\left[S_L(t)\right]dt + \int_0^\infty g\left[S_L(t)\right]dI(t)\right], \tag{3.4}$$

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[4] It is straightforward to show that in the case of full information, equilibrium will consist of each risk receiving full coverage at an actuarially fair price. This duplicates a result from expected utility theory. Both results are consequences of the fact that risk-averse decision makers in each theory preserve the ordering given by second stochastic dominance (Yaari, 1987).

subject to

$$P \geq \int_0^\infty [S_L(t)] dI(t), \tag{3.5}$$

$$0 \leq I' \leq 1; \qquad I(0) = 0, \tag{3.6}$$

and

$$\begin{aligned} w - E_H[X] &= V_H(I^F, E_H[X]) \\ &\geq V_H(I, P) = w - P - \int_0^\infty g[S_H(t)] dt + \int_0^\infty g[S_H(t)] dI(t), \end{aligned} \tag{3.7}$$

in which I^F denotes full coverage. The last constraint (3.7) ensures high risks will not prefer the policy of low risks to full coverage.

Lemma 3.1 *If $E_L[X] < E_H[X]$, then (I_S, P_S) is a partial coverage contract satisfying*

$$V_H(I_S, E_L[I_S(X)]) = V_H(I^F, E_H[X]) = w - E_H[X],$$

and

$$V_L(I_S, E_L[I_S(X)]) \geq V_L(I^F, E_H[X]) = w - E_H[X].$$

Proof: See Appendix.

The contracts $(I^F, E_H[X])$ and $(I_S, E_L[I_S(X)])$ satisfy the self-selection constraints (3.1) and (3.2), as well as the nonnegative profit constraints (3.3). If a separating equilibrium exists, then they are the equilibrium contracts, as shown in the following proposition.

Proposition 3.2 *Assume that a separating equilibrium exists. The equilibrium pair of contracts is $(I_H, P_H) = (I^F, E_H[X])$ and $(I_L, P_L) = (I_S, E_L[I_S(X)])$.*

Proof: See Appendix.

Note that Proposition 3.2 tells us that the expected loss of a risk is sufficient to characterize a risk as low or high in order to determine the optimal separating contracts. The same is true in the case of a pooling equilibrium, as we demonstrate next.

Further assume that insurers are non-myopic, as in Wilson (1977)—that is, they will not offer policies that will become unprofitable if other policies are removed from the market in response to the introduction of a new policy. It may occur that both risks will prefer a pooling policy to their optimal separating policies. In that case, any pooling policy which is priced actuarially fair and that does not maximize the expected value V of the low risks can be improved upon for the low risks. That the old policy will be removed from the market is a result of insurers being non-myopic and of the following lemma.

Lemma 3.3 *Suppose that $E_L[X] < E_H[X]$. For any pooling policy I with an actuarially fair premium P, such that the high risks prefer I to full coverage with an actuarially fair premium of $E_H[X]$, we have that $P < E_H[I(X)]$.*

Proof: See Appendix.

Therefore, if the equilibrium is a pooling equilibrium, then the optimal pooling contract (I_P, P_P) solves the following problem:

$$\max_I V_L(I,P) = \max_{I,P}\left[w - P - \int_0^\infty g[S_L(t)]dt + \int_0^\infty g[S_L(t)]dI(t)\right], \tag{3.8}$$

subject to

$$P \geq (1-\rho)E_L[I(X)] + \rho E_H[I(X)], \tag{3.9}$$

and

$$0 \leq I' \leq 1; \qquad I(0) = 0. \tag{3.10}$$

For the optimal separating policies to constitute an equilibrium, we also have the self-selection constraints that at least one risk class will prefer its optimal separating policy to the optimal pooling policy. Similarly, for the optimal pooling policy to constitute an equilibrium, we have the self-selection constraints that both risk classes will prefer the optimal pooling policy to their optimal separating policies. To state this more simply, if at least one of the risks prefers to separate, then they will separate.

In the next section, we determine necessary conditions satisfied by the optimal separating and pooling insurance policies for general loss distributions. First, we look at a simple case: the amount of loss is nonrandom, and risks differ only in their probabilities of loss.

4. Design of Optimal Insurance Contracts

4.1 Nonrandom Loss Severity, Differing Probabilities of Loss

In this section, we examine the model studied by Rothschild and Stiglitz (1976). Specifically, risks are identical except for their probabilities of loss, with $p_i = S_i(0)$, $i = L$ or H, and $p_L < p_H$. We also assume that the loss amount is fixed at, say, x. We first present a graphical approach for understanding this problem which highlights the similarities between our results and those of Rothschild and Stiglitz. Then, we solve the problem algebraically.

Consider Figure 4.1 in which we graph premium P versus coverage I. The lines denoted $p_H I$, $p_P I$, and $p_L I$, are the fair-odds lines for the high, pooled, and low risks, respectively. The lines marked V_H and V_L are the indifference lines for the high and low risks, respectively.[5] These indifference lines are given by $P = g(p_i) I + [w - g(p_L) x - V_i]$, for $i = L$ or H, (2.1). The line V_H is steeper than the indifference line of the low risks because $g(p_L) < g(p_H)$. By using standard arguments, we see that the risks separate with high risks buying full coverage, denoted by I^F, and low risks buying the amount of coverage on their fair-odds line where V_L intersects V_H, denoted by I_S. In this case, the equilibrium is separating because V_L does not intersect the fair-odds pooling line.

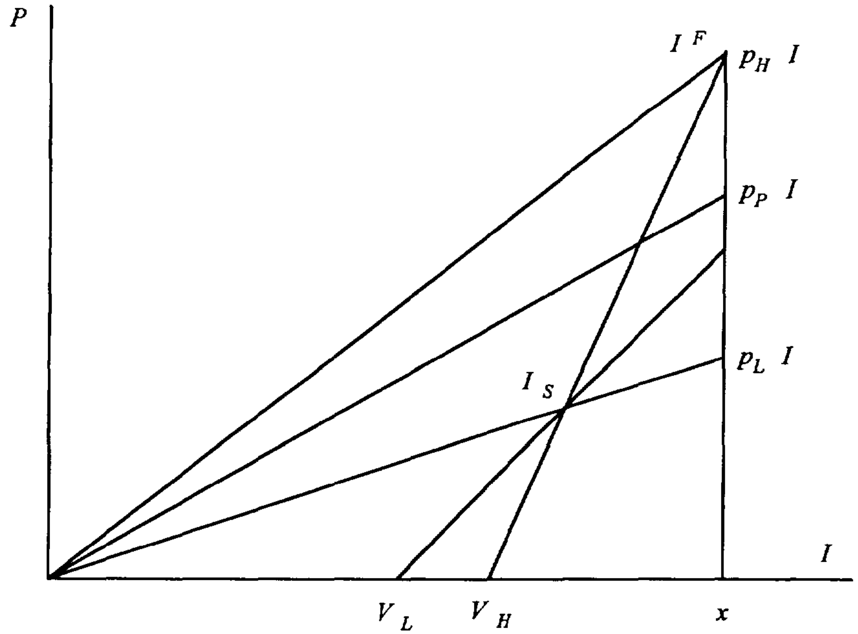

Figure 4.1 Separating Equilibrium

[5] We use the notation V_i, $i = L$ or H, to denote both an indifference line and the value (2.1) to a risk of being on that line.

In Figure 4.2, we see that the risks prefer to pool at full coverage, I_P, because the indifference line of the low risks, V_L, intersects the fair-odds pooling line. This result is anticipated by Segal and Spivak (1990) who show that under Yaari's dual theory of risk, optimal insurance may be full coverage even when the premium is loaded. The pooling policy will be in equilibrium if one assumes that insurers are non-myopic, as in a Wilson anticipatory market (1977).

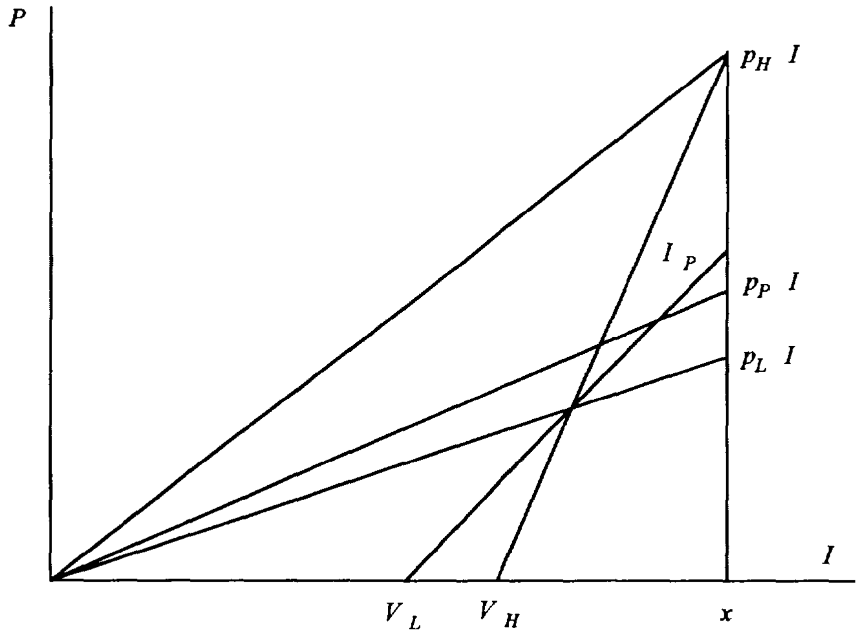

Figure 4.2 Pooling Equilibrium

I_S solves the self-selection equation $V_H(I_S,\ p_L I_S) = w - p_H x$. It follows that $I_S = \frac{g(p_H) - p_H}{g(p_H) - p_L} x$. The line V_L intersects the line $p_P I$, or equivalently the risks pool, if and only if the proportion of high risks ρ is less than $\frac{g(p_L) - p_L}{g(p_H) - p_L}$. Thus, we see that if the high risks make up a sufficiently small portion of the market, then the risks prefer to pool. Also, if p_L and p_H are sufficiently close, then the risks will pool because the ratio

$\frac{g(p_L) - p_L}{g(p_H) - p_L}$ increases to 1 as the probabilities of loss approach each other. Finally, if g_1 is a concave distortion, then so is $g_1 \circ g$, and $\frac{g(p_L) - p_L}{g(p_H) - p_L} \le \frac{g_1[g(p_L)] - p_L}{g_1[g(p_H)] - p_L}$. Thus, as g becomes more concave, the risks are more likely to pool.

To summarize, if the costs of pooling are not too great, as described above, then the risks will pool at full coverage. Otherwise, the risks separate with high risks purchasing full coverage and low risks purchasing $I_S = \frac{g(p_H) - p_H}{g(p_H) - p_L} x$, each at an actuarially fair price. It is straightforward to show that these results also follow from Theorems 4.1 and 4.5, given below.

4.2 *General Loss Random Variable*

In this section, we describe the optimal separating and pooling contracts. We assume that $E_L[X] < E_H[X]$, as in Section 3. In Section 4.2.1, we describe the optimal separating contracts. First we determine conditions satisfied by the optimal separating contract for the low risks, for general loss random variables. Then, we show that the conditions are sufficient to determine the optimal contract under mild technical requirements. By further restricting the loss distributions, the distortion function, or both, we obtain conditions under which the optimal contract for the low risks is either a deductible policy or a policy with a maximum limit. In Section 4.2.2, we parallel Section 4.2.1 for the optimal pooling contract.

4.2.1 Optimal separating contracts

In this subsection, we give conditions satisfied by the optimal separating insurance contracts for general loss random variables. The optimal separating insurance contract for high risks is full coverage, I^F, at an actuarially fair premium, $E_H[X]$, as shown in Proposition 3.2. Conditions for the optimal separating policy for low risks are given in the following theorem.

Theorem 4.1 *The optimal separating insurance contract* (I_S, P_S) *for the low risks, that solves* (3.4)-(3.7), *satisfies the following: There exists a nonnegative constant* λ *such that*

(*a*) *If* $g[S_L(x)] < (1-\lambda)S_L(x) + \lambda g[S_H(x)]$ *in a neighborhood of* x_0, then $I_S'(x_0) = 0$.

(*b*) *If* $g[S_L(x)] > (1-\lambda)S_L(x) + \lambda g[S_H(x)]$ *in a neighborhood of* x_0, then $I_S'(x_0) = 1$.

The premium P_S *equals the expected indemnity benefit,* $P_S = E_L[I_S(X)]$, *and the self-selection constraint* (3.7) *holds at the optimum.*

Furthermore, if the following technical condition (*C*1) *holds, then* (*a*) *and* (*b*) *above are sufficient to determine* I_S *and* λ *uniquely*:

$$(C1) \qquad \frac{g[S_L(x)] - S_L(x)}{g[S_H(x)] - S_L(x)} \text{ is not constant on any interval in } \mathbf{R}^+.$$

Proof: Maximizing $w - P - \int_0^\infty g[S_L(t)]dt + \int_0^\infty g[S_L(t)]dI(t)$, subject to the constraints $P \geq \int_0^\infty [S_L(t)]dI(t)$, and $w - E_H[X] = w - P - \int_0^\infty g[S_H(t)]dt + \int_0^\infty g[S_H(t)]dI(t)$, is equivalent to minimizing

$$M(I,\lambda) = \left[\int_0^\infty S_L(t)dI(t) - \int_0^\infty g[S_L(t)]dI(t)\right] - \lambda\left[\int_0^\infty S_L(t)dI(t) - \int_0^\infty g[S_H(t)]dI(t)\right]$$

over feasible I and $\lambda \geq 0$ because the premium constraint binds. Because a feasible I is piecewise differentiable, we can write the objective function M as

$$M(I,\lambda) = \int_0^\infty \{(1-\lambda)S_L(t) - g[S_L(t)] + \lambda g[S_H(t)]\} I'(t)dt.$$

This expression for M leads directly to the conditions for I_S.

If condition (C1) holds, then clearly conditions (a) and (b) are sufficient to determine I_S, for a given λ. To show that λ is unique, suppose that it is not. Let $\lambda_1 \leq \lambda_2$ be such that

$$\begin{aligned} E_H[X] &= \int_0^\infty \left[S_L(t)I_1'(t) + g[S_H(t)](1 - I_1'(t))\right]dt \\ &= \int_0^\infty \left[S_L(t)I_2'(t) + g[S_H(t)](1 - I_2'(t))\right]dt, \end{aligned} \tag{4.1}$$

in which I_j, $j = 1$ or 2, is the insurance policy determined by conditions (a) and (b) with $\lambda = \lambda_j$. Because $\lambda_1 \leq \lambda_2$, $I_2' \leq I_1'$ and let A be the subset of $\mathbf{R}^+$ given by

$$A = \{x: I_1'(x) = 1, I_2'(x) = 0\}$$
$$= \left\{x: \lambda_1 \le \frac{g[S_L(x)] - S_L(x)}{g[S_H(x)] - S_L(x)} \le \lambda_2\right\}.$$

By (4.1), we have that

$$\int_A S_L(t)\,dt = \int_A g[S_H(t)]\,dt,$$

which implies that A has measure zero because $S_L < S_H$ on A. Thus, $\lambda_1 = \lambda_2$, and λ is uniquely determined by the self-selection constraint (3.7). □

One can interpret the conditions in Theorem 4.1 economically. Indeed, the left-hand side of each expression in (a) and (b) is the marginal benefit to the low risks of receiving additional coverage beyond a given loss x. The corresponding right-hand sides are the marginal costs of paying additional premium for that coverage, adjusted for the net marginal gain to the high risks for the extra coverage. Therefore, if the marginal benefit of receiving extra coverage beyond x is outweighed by the adjusted marginal cost, then $I_S'(x) = 0$, an intuitively appealing result.

From condition $(C1)$ in Theorem 4.1, we learn that the optimal indemnity payment for the low risks is a piecewise linear function of the underlying loss, a result that is not generally true under utility theory (Young and Browne, 1997). The reason that the optimal indemnity is a piecewise linear function is that the value function under Yaari's dual theory is linear in wealth, while expected utility is not generally linear in wealth. Condition $(C1)$, in particular, implies that $I_S' = 0$ or 1, except at most a countable number of points. In this case, the optimal coverage for the low risks looks like a 'terrace' with alternating flat portions and portions that increase with slope 1. A special case of such an insurance policy is deductible coverage combined with a maximum limit, $I_S(x) = \min[\max(x-d, 0), u]$.

The following corollary follows directly from condition (b) in Theorem 4.1 and from the fact that $g(p) \ge p$, for all $p \in [0, 1]$.

Corollary 4.2 *If $S_L(x) > S_H(x)$, for all x in a neighborhood of x_0, then $I_S'(x) = 1$ in a neighborhood of x_0. In particular, let $x_0 = 0$; if the loss random variable of the low risks is dominated by the loss random variable of the high risks under second stochastic dominance, then I_S is full coverage up to some loss amount.*

Next, we obtain conditions that lead to optimal policies with deductibles, maximum limits, or both. We state the following without proof.

Corollary 4.3 *(a) If* $\dfrac{g[S_L(0)] - S_L(0)}{g[S_H(0)] - S_L(0)} < \lambda$, *then the optimal separating policy for the low risks has a nonzero deductible; that is, there exists a* $d \geq 0$, *such that* $I_S(x) = 0$, *for* $x \leq d$; *and* $I_S(x) > 0$, *for* $x > d$.

(b) *If* $\lim_{x\to\infty} \dfrac{g[S_L(x)] - S_L(x)}{g[S_H(x)] - S_L(x)} < \lambda$, *then the optimal separating policy for the low risks has a maximum limit; that is,* $I_S(x) = u$, *for some* u, $m > 0$, *and for all* $x > m$.

The condition in part (b) holds when, $g'(0) < \infty$ and $\lim_{x\to\infty} \dfrac{S_H(x)}{S_L(x)} = \infty$, for example. It also holds when $g(p) = p^c$, for some $0 < c < 1$, and when $\lim_{x\to\infty} \dfrac{S_H(x)}{S_L(x)} = \infty$. The condition $\lim_{x\to\infty} \dfrac{S_H(x)}{S_L(x)} = \infty$ indicates that the loss distribution of the low risks is heavily dominated by the one of the high risks for large losses. In that case, the low risks are willing to give up coverage at large loss amounts because they are much less likely to incur large claims than the high risks. For the special case in which the severity densities are equal, we have the following result.

Corollary 4.4 *Suppose the loss severity densities of the risks are identical with* $S_L(0) < S_H(0)$.[6] *Then, optimal separating insurance for the low risks* I_S *may have a deductible or may have a maximum limit, depending on the risk aversion embodied by the distortion function g. Specifically, define the relative risk aversion for probabilities by*

$$R_r(p) = -\frac{pg''(p)}{g'(p)}.$$

(a) *If* $R_r(p)$ *is nonincreasing with respect to p, then the optimal* I_S *is deductible coverage, given by* $I_S(x) = \max(x - d, 0)$, *for some* $d \geq 0$, *and for all* $x \geq 0$.

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[6] Note that, in this case, condition (C1) of Theorem 4.1 holds.

(*b*) *If* $R_r'(p) \geq \dfrac{R_r(p)}{g(p)-p} > 0$, *for all* $p \in (0, 1)$, *then the optimal* I_S *has a maximum limit with full coverage below the limit; that is,* $I_S(x) = \min(x, u)$, *for some* $u \geq 0$, *and for all* $x \geq 0$.

Proof: See Appendix.

If R_r is decreasing, then as insurable events get more rare (p decreases), the decision maker (d.m.) becomes more risk-averse relative to the probability of the event (R_r increases). In this case, Corollary 4.4 implies that a d.m. will insure rare events, namely, losses over a deductible, and self-insure losses under the deductible. A family of distortions for which R_r is nonincreasing is the collection of power distortions g_c given by $g_c(p) = p^c$, for a fixed, but arbitrary $c \in (0, 1)$. R_r for the distortion g_c is identically $1-c$.

On the other hand, if a d.m. is relatively less risk-averse as events get more rare, then the d.m. will self-insure rare events, namely, losses over a maximum limit. A family of distortions for which $R_r'(p) \geq \dfrac{R_r(p)}{g(p)-p} > 0$, $p \in (0, 1)$, is the collection of dual power distortions g_d given by $g_d(p) = 1 - (1 - p)^d$, for fixed, but arbitrary $d \geq 2$. Indeed, $R_r(p) = (d-1)\dfrac{p}{1-p}$, from which the inequality follows.

4.2.2 Optimal pooling contract

In the following theorem, we state conditions satisfied by the optimal pooling insurance contract. The proof parallels the one of Theorem 4.1, so we omit it.

Theorem 4.5 *The optimal pooling insurance contract* (I_P, P_P), *that solves* (3.8)-(3.10), *satisfies the following conditions:*

(*a*) *If* $g[S_L(x)] < (1-\rho)S_L(x) + \rho S_H(x)$, *in a neighborhood of* x_0, *then* $I_P'(x_0) = 0$.

(*b*) *If* $g[S_L(x)] > (1-\rho)S_L(x) + \rho S_H(x)$, *in a neighborhood of* x_0, *then* $I_P'(x_0) = 1$.

The premium P_P *equals the (pooled) expected indemnity benefit,*

$$P_P = (1-\rho)\, E_L[I_P(X)] + \rho\, E_H[I_P(X)].$$

Furthermore, if the following technical condition ($C2$) *holds, then* (a) *and* (b) *above are sufficient to determine* I_P *uniquely*:

($C2$) $\dfrac{g[S_L(x)] - S_L(x)}{S_H(x) - S_L(x)}$ *is not equal to* ρ *on any interval in* $\mathbf{R}^+$.

One can interpret the conditions in Theorem 4.5 economically, as for Theorem 4.1. For example, if the marginal benefit of receiving extra coverage beyond x outweighs the adjusted marginal cost for the low risks, then $I_P'(x) = 1$. If the condition ($C2$) holds, then optimal pooling insurance is a piecewise linear function of the loss. Again, this result is not generally true under utility theory (Young and Browne, 1997) for utility functions that are nonlinear in wealth.

We have the following straightforward corollary which is weaker than Proposition 3.2.

Corollary 4.7 *If* $S_L = S_H$, *then the optimal pooling policy* I_P *is full coverage.*

Note that Corollary 4.2 also holds for the optimal pooling contract. The next corollary parallels Corollary 4.3.

Corollary 4.8 (a) *If* $\dfrac{g[S_L(0)] - S_L(0)}{S_H(0) - S_L(0)} < \rho$, *then the optimal pooling policy has a nonzero deductible*; *that is, there exists a* $d \geq 0$, *such that* $I_P(x) = 0$, *for* $x \leq d$; *and* $I_P(x) > 0$, *for* $x > d$.

(b) *If* $\lim_{x \to \infty} \dfrac{g[S_L(x)] - S_L(x)}{S_H(x) - S_L(x)} < \rho$, *then the optimal pooling policy has a maximum limit*; *that is,* $I_P(x) = u$, *for some* u, $m > 0$, *and for all* $x > m$.

The discussion following Corollary 4.3 holds in this case, too. In particular, if the loss distribution of the high risks heavily dominates the one of the low risks for large losses, then, quite generally, the condition in (b) will hold. In the pooling case, we can assert something stronger than Corollary 4.4.

Corollary 4.9 *Suppose the loss severity densities of the risks are identical with* $S_L(0) < S_H(0)$.[7] *Then, optimal pooling insurance* I_P *is deductible coverage, given by*

$$I_P(x) = \max(x - d, 0),$$

[7] Note that, in this case, condition ($C2$) of Theorem 4.5 holds.

for some $d \geq 0$.

Proof: See Appendix.

In the next section, we present an illustrative example to show how piecewise linear coverage with a deductible and a maximum limit can be realized as an equilibrium policy in our framework.

5. Illustrative Example

Suppose $\rho = 0.10$; that is, high risks constitute 10% of the market. Assume that low risks have probability of loss $S_L(0) = 0.80$ and have loss severities distributed according to the exponential distribution with mean \$1000. Similarly, high risks have probability of loss $S_H(0) =$ 1.00 and have loss severities distributed according to the exponential distribution with mean \$2000. Assume that both the low and high risks have a power distortion, $g(p) = p^{1/1.1}$. It follows that low and high risks are willing to pay up to \$898 and \$2200 for full coverage, respectively.

The optimal separating policy to the low risks I_S is a piecewise linear function of the loss, given by

$$I_S(x) = \min\big(\max(x - 290, 0), 919 - 290\big), \quad x \geq 0.$$

That is, the I_S has a deductible of \$290 with a maximum limit of \$919, or maximum benefit of \$628. The value to the low risks of this policy, ignoring wealth w, is -877, while the value to the high risks is -2000.

The optimal pooling policy I_P is a piecewise linear function of the loss, given by

$$I_P(x) = \min\big(\max(x - 159, 0), 1662 - 159\big), \quad x \geq 0.$$

That is, the I_P has a deductible of \$159 with a maximum limit of \$1662, or maximum benefit of \$1503. The value to the low risks of this policy is -894, while the value to the high risks is -1762. We see that the low risks want to separate. Thus, the equilibrium is a separating one with the high risks receiving full coverage and the low risks receiving I_S, each at an actuarially fair price.

In the next section, we conclude our work by summarizing our results and by comparing them with ones obtained under utility theory.

6. Summary

We showed that if the two risk classes have equal expected losses, $E_L[X] = E_H[X]$, then equilibrium is full coverage at an actuarially fair price. If $E_L[X] < E_H[X]$ and if a equilibrium is achieved by a pair of separating contracts, then the equilibrium coverage for high risks is full insurance at the actuarially fair price of $E_H[X]$, and coverage for low risks is less than full. These results are also true under utility theory, (Fluet and Pannequin, 1997) and (Young and Browne, 1997). We showed that the optimal separating contract for the low risks solves the optimization problem given by (3.4-7). Under utility theory, the optimal contract for the low risks solves the corresponding problem with the value function replaced by expected utility.

We showed that if insurers are non-myopic, then a Wilson anticipatory pooling equilibrium might exist. The optimal pooling contract solves the problem given by (3.8-10). Under utility theory, the optimal contract solves the corresponding problem with the value function replaced by expected utility (Young and Browne, 1997). Utility theory predicts that the optimal pooling coverage is never full coverage, (Mossin, 1968) and (Smith, 1968). However, under Yaari's dual theory, we showed that the optimal pooling coverage might be full coverage if the costs of pooling are not too great, Section 4.1.

We found conditions satisfied by the optimal contracts. Qualitatively similar conditions hold for the optimal contracts under utility theory: In that case, one compares marginal utility benefits and costs, whereas under Yaari's theory, one compares marginal benefits and costs which arise from a value functional which is an expected value with respect to a distorted probability.

We showed that under mild technical conditions, the optimal separating and pooling coverages are piecewise linear functions of the underlying loss because the value function is linear in wealth. This result does not hold under utility theory (Young and Browne, 1997) for utility functions that are nonlinear in wealth. Thus, Yaari's theory predicts a common property of most indemnity contracts—namely, piecewise linearity—which utility theory does not predict.

References

Borch, K. (1968), *The Economics of Uncertainty*, Princeton University Press, Princeton, New Jersey.

Denneberg, D. (1994), *Non-Additive Measure and Integral*, Kluwer, Dordrecht.

Doherty, N. A. and L. Eeckhoudt (1995), Optimal insurance without expected utility: The dual theory and the linearity of insurance contracts, *Journal of Risk and Uncertainty*, 10: 157-179.

Fluet, C. and F. Pannequin (1997), Insurance contracts under adverse selection with random loss severity, working paper, University of Quebec at Montreal.

Mossin, J. (1968), Aspects of rational insurance purchasing, *Journal of Political Economy*, 76: 553-568.

Rothschild, M. and J. Stiglitz (1976), Equilibrium in competitive insurance markets: An essay on the economics of imperfect information, *Quarterly Journal of Economics*, 90: 629-649.

Segal, U. and A. Spivak (1990), First order versus second order risk aversion, *Journal of Economic Theory*, 51: 111-125.

Smith, V. L. (1968), Optimal insurance coverage, *Journal of Political Economy*, 76: 68-77.

von Neumann, J. and O. Morgenstern (1947), *Theory of Games and Economic Behavior*, second edition, Princeton University Press, Princeton, New Jersey.

Wilson, C. (1977), A model of insurance markets with incomplete information, *Journal of Economic Theory*, 97: 167-207.

Yaari, M. E. (1987), The dual theory of choice under risk, *Econometrica*, 55: 95-115.

Young, V. R. and M. J. Browne (1997), Explaining insurance policy provisions via adverse selection, working paper, University of Wisconsin-Madison.

Appendix

Proof of Lemma 3.1:

First, note that the premium constraint will always bind. Because $E_L[X] < E_H[X]$, $V_H(I_S, E_L[I_S(X)]) > w - E_H[X]$, if I_S were full coverage. Thus, the self-selection constraint

$V_H(I_S, E_L[I_S(X)]) \leq w - E_H[X]$ binds. To prove that $V_L(I_S, E_L[I_S(X)]) \geq w - E_H[X]$, it is enough to find a contract I compatible with the constraints that satisfies this inequality. Let (I_α, P_α) be a coinsurance contract; that is, $I_\alpha(x) = \alpha x$, for some $\alpha \in [0, 1]$, and $P_\alpha = \alpha E_H[X]$. Let

$$\alpha^* = \arg\max_{\alpha \in [0,1]} V_L(I_\alpha, P_\alpha) = \arg\max_{\alpha \in [0,1]}\left[w - \int_0^\infty g[S_L(t)]dt + \alpha\int_0^\infty \left(g[S_L(t)] - S_H(t)\right)dt\right].$$

Then, $\alpha^* = 0$ or 1, which implies that $V_L(I_S, E_L[I_S(X)]) \geq V_L(I_{\alpha^*}, P_{\alpha^*}]) \geq w - E_H[X]$.

□

Proof of Proposition 3.2:

From the properties of the solution I_S, we have that if full coverage is the optimal separating contract for the high risks, then I_S is the optimal separating contract for the low risks. Therefore, it is enough to show that full coverage is the optimal separating contract for the high risks.

Suppose not; that is, suppose that the optimal I^* is not the identity on $\mathbf{R}^+$. Then, $V_H(I^*, E_H[I^*(X)]) < w - E_H[X]$, by continuity of I^*, because full coverage I^F at an actuarially fair premium is optimal for any risk. Consider a contract which provides full coverage at a premium $P_\varepsilon = E_H[X] + \varepsilon$, for $\varepsilon > 0$. For ε small enough, $V_H(I^*, E_H[I^*(X)]) < V_H(I^F, P_\varepsilon) = w - E_H[X] - \varepsilon < w - E_H[X]$. The contract (I^F, P_ε) would attract all the high risks and make positive profits. If it attracts the low risks, it still makes positive profits because $E_L[X] < E_H[X]$. Thus, (I^F, P_ε) overturns the equilibrium if $I^* \neq I^F$. It follows that the separating equilibrium for high risks is full coverage I^F at a premium $E_H[X]$. □

Proof of Lemma 3.3:

We are given that high risks prefer the pooling policy I to full coverage, both at actuarially fair prices. That is,

$$\begin{aligned} w - E_H[X] < V_H(I, P) &= w - P - \int_0^\infty g\left[S_{X-I(X),H}(t)\right]dt \\ &\leq w - P - E_H[X - I(X)], \end{aligned}$$

in which the second inequality follows from Jensen's inequality. It follows that $P < E_H[I(x)]$.

□

Proof of Corollary 4.4:

First suppose $R_r(p)$ is nonincreasing, or more simply, decreasing with respect to p. To show that I_S is deductible coverage, by Theorem 4.1, it is enough to show that the ratio

$$\frac{g[S_L(x)] - S_L(x)}{g[S_H(x)] - S_L(x)} \tag{A.1}$$

is increasing with respect to x. Write $p = p_L\, S(x)$, then $p_H S(x) = \frac{p_H}{p_L} p_L S(x) = \alpha p$, in which $\alpha = p_H/p_L \in [1, 1/p]$. Then, the ratio in (A.1) is increasing in x if and only if

$$r(p) = \frac{g(p) - p}{g(\alpha p) - p}$$

is decreasing in p.

Now, $r'(p) \le 0$ if and only if $\frac{g'(p) - 1}{g(p) - p} \le \frac{\alpha g'(\alpha p) - 1}{g(\alpha p) - p}$, for all $\alpha \in [1, 1/p]$. This inequality holds with equality at $\alpha = 1$; thus, it holds for all $\alpha \in [1, 1/p]$, if the right-hand side is increasing in α. The derivative of the right-hand side with respect to α is proportional to $\alpha p g(\alpha p) g''(\alpha p) - \alpha p^2 g''(\alpha p) + g(\alpha p) g'(\alpha p) - \alpha p g'(\alpha p)^2$. Substitute q for αp and β for $1/\alpha$; then, the derivative is proportional to

$$d(q,\beta) = q g(q) g''(q) - \beta q^2 g''(q) + g(q) g'(q) - q g'(q)^2,$$

with $\beta \in (0, 1]$. To verify that $d(q)$ is greater than or equal to zero, it is enough to verify this for β as small as possible, namely 0. Then, we are left with checking whether

$$\begin{aligned} 0 &\le q g(q) g''(q) + g(q) g'(q) - q g'(q)^2 \\ &\propto \{g(q) - q g'(q)\} - R_r(q) g(q). \end{aligned}$$

Assuming that the limit exists as q approaches 0, we have $d(0, 0) = 0$. Thus, finally, it is enough to show that $d(q, 0)$ is increasing in q. Its derivative is equal to $-R_r'(q) g(q)$, which is positive by the assumption that R_r is decreasing. Thus, we have verified part (a) of the proposition; part (b) follows by similar reasoning. □

Proof of Corollary 4.9:

By Theorem 4.5, it is enough to show that

$$\frac{g[S_L(x)]-S_L(x)}{S_H(x)-S_L(x)}=\frac{g[p_LS(x)]\Big/p_LS(x)-1}{p_H\Big/p_L-1}$$

is increasing with respect to x, or equivalently, that $g(p)\big/p$ is decreasing with respect to p, which is true because g is concave. □