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The Ownership of the Pension Plan Surplus Using Cooperative Game Theory

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Abstract

Private defined-benefit pension plans must, by law, pre-fund the retirement benefits. This requires the periodic valuation of liabilities and calculation of contributions. Because valuation assumptions are seldom borne out by reality, a surplus (or deficit) ensues and varies over time. When the surplus becomes sizable, there often arises a desire to distribute part of it so as to bring it down to a more reasonable level.

We first provide a brief overview of cooperative game theory and, in particular, sharing rules. Then, using certain sharing rules, we determine how to split the surplus or deficit among the plan participants and sponsor. In general, the participant's share in the surplus depends on the total of his contributions. Other considerations, such as the actual investment income received on his contributions or the years of service, may influence the distribution of the surplus. It may even appear desirable to take account of the accrued liability.

1 Introduction

There are two main types of pension plans: defined-contribution and defined-benefit. As the names imply, the former sets the contribution while the latter sets the benefit. By their very nature, defined-contribution plans are always fully funded. Hence, they will not be of interest here.

Within the class of defined-benefit pension plans, we can distinguish between two ways of funding the plan: pay-as-you-go and pre-funded. In the first case, today's contributions pay for today's benefits, with no or little fund existing at any point in time. In the second case, today's contributions pay for tomorrow's benefits, with a consiberable fund likely to emerge over time. These two cases are two extremes and funding levels can actually fall anywhere in between. Of the two funding methods, the one that is of interest to us is pre-funding.

Pre-funding can be accomplished in two ways. The pension plan may be insured or trusteed. On the one hand, if the plan is insured, we basically have that the administration of the plan is assigned to the insurance company. On the other hand, if the plan is trusteed, the administration of the plan is in the hands of a pension plan committee. It is only in the latter case that surpluses may appear and that some distribution of it may have to considered.

In summary, we focus on trusteed, pre-funded, defined-benefit pension plans. By their very nature, they are the only ones that may generate a surplus which would then have to be split in some fashion among the sponsor and participants.

2 Emergence of Surplus and Claims to It

By law, pre-funded plans must, periodically, determine the contributions to be made and calculate their liabilities. In order to do that, several long-term assumptions have to be made. Rarely does reality reproduce these assumptions. As a result, the accumulated funds will not necessarily be equal to the liabilities. We will then have a surplus (or deficit). The surplus itself will vary over time.

As long as the surplus is modest in size relative to the size of the plan, there is little interest on the part of the participants and sponsor to get a share of it. However, when the surplus gets large, sometimes so large that contributions would not be needed over the next few years, the sponsor is likely to express a desire to appropriate part of the surplus for, say, some investment in the company. At the same time, participants are likely to ask for their share of the surplus since the assets are invested and accumulated on their behalf.

To be more precise, the *potential* claimants are the following:

- sponsor;
- active participants;
- past participants who left with vested benefits;
- disabled participants;
- annuitants;
- beneficiaries entitled to some current or future benefits.

In fact, other than the sponsor, anyone whose name is in the current records of the pension plan is a potential claimant. Out of those, we still have to determine who the *rightful* claimants are.

Most of the controversy lies with the legitimacy of the sponsor's claim. On one side, some claim that only the benefits are guaranteed and that the pension plan is effectively part of the company's balance sheet. Hence, the sponsor would be the *sole* owner of the surplus. On the other side, others claim that retirement benefits are deferred wages and that assets belong entirely to the current and past participants as well as their beneficiaries. As a result, the sponsor would *not* be entitled to any share of the surplus.

For all of the other potential claimants, their inclusion in, or exclusion from, the set of rightful claimants can be based on practical, rather than ideological, considerations. In particular, the pension plan may wish to distribute surplus only to those who could have paid for a deficit had the experience been unfavorable. In that case, the so-called rightful claimants would be the active participants along with, perhaps, the sponsor. Since all participants are or have been active over some period of time, this would ensure that they are entitled to some share in the surplus at some point in time.

It is actually not the aim of this paper to deal with the issue of whom to include and whom to exclude from the distribution[1]. It is not its aim

[1]For more discussion of the legitimacy of the sponsor's claim, see Adell(1988), Ascah(1991), and Bodie(1988).

either to determine, given liabilities and assets, the actual amount of surplus to distribute. Rather, this paper offers a way of calculating the shares of the rightful claimants, *given* who is to get a share and what amount is to be shared.

3 Cooperative Game Theory

Cooperative game theory provides one such way. It has been applied to other actuarial problems in the past. Jean Lemaire has been a pioneer in that area, applying the idea to premium calculation and allocation of costs in an insurance company ((1984), (1991)). Alegre and Claramunt (1995) have extended the idea to the allocation of the solvency cost in group annuities[2].

Many readers may be familiar with non-cooperative game theory, and it is important to distinguish between both kinds of games.

In *cooperative* games, every player knows what he can achieve by himself as well as with any other group of players. Hence, players seek to do as well as they can by cooperating whenever it is beneficial to them.

In contrast, in *non-cooperative* games, every player knows what he can achieve dependent on the other players' actions. Hence, players seek to do as well as they can individually, accounting for the potential negative impact of others' actions.

In practice, whether or not the game is cooperative is often determined by whether or not cooperation can be enforced or even take place. For example, chess is a non-cooperative game whereas car pooling is a cooperative game.

As far as pension plans are concerned, we consider them to be cooperative games in which participants collaborate by pooling their contributions together and administering the pension plan as a whole rather than as several little entities.

Cooperative games may be of two kinds. Utility may be transferable or non-transferable. The most general case, but also the most difficult to handle, is the one with non-transferable utility. So as to simplify the computations, we assume that the pension plan actually is a transferable utility (TU) game.

The game is a TU game only if the good (or goods) to be allotted to the players is divisible, and its unit is worth the same to all players, regardless of their current basket of goods. This means that, in a TU game, every player

[2]This is not the objective here. While Alegre and Claramunt were interested in premium-type calculations, we are considering "after the fact" calculations.

has a quasi-linear utility function, which is the sum of a linear function of the good to be divided plus some function of all other goods. As is the case here, the good to be allotted often is money.

Non-TU games would be harder to deal with since, instead of coming up with a single value for each player, we would have to consider the whole spectrum of attainable utility vectors when a given number of agents decide to cooperate.

4 Determinants of a TU Game

To fully define a TU cooperative game, of which type we assume the pension plan to be, we need to know:

- the set of agents N;
- the characteristic function v.

In a general TU cooperative game, there are n agents, indexed by $i \in N = \{1, 2, \ldots, n\}$. Any non-empty subset of agents $S \subseteq N$ can decide to form a coalition. The set of all coalitions is $\mathcal{P}(N)$, the power set of N, and $\emptyset$ is the empty coalition, with no agents in it. Also, N is called the grand coalition.

The function v is called the characteristic function, with domain $\mathcal{P}(N)$ and range $\Re$, which specifies what any given subset of agents can achieve, independent of what the other players do. For instance, if the coalition S forms, they receive collectively the surplus generated by it, denoted by $v(S)$. This surplus $v(S)$ is called the worth of the coalition S. Besides, all agents know the surplus $v(S)$ associated with each $S \in \mathcal{P}(N) \setminus \{\emptyset\}$. What makes such a game cooperative is the possibility for agents to form a coalition and thus enjoy greater returns through cooperation.

In terms of notation, a cooperative game is denoted by (N, v). The set of all TU games is denoted by Γ. Also, if the set of agents N is fixed, Γ^N denotes the set of all TU games having N as grand coalition.

It may be that we also need to know the player's contributions in order to determine the players' shares of the surplus. If that is the case, we will denote player i's contribution by c_i. Then, to make the relationship between v and c stand out, we denote the worth of coalition S by $v(c; S)$, which is equivalent to $v(\{c_i : i \in S\})$. Hence, such a cooperative game shall be identified by (N, v, c). Besides, the subset of all such games will be denoted by Γ_R. Similarly, the smaller subset of such games which have N as grand coalition will be denoted by Γ_R^N.

5 Types of TU Games

Depending on the characteristics of v, we can define two types of games: superadditive and convex[3].

Type of game 1 *A game is* ***superadditive*** *if and only if the worth of the union of any two disjoint coalitions is greater than or equal to the sum of their worths. In symbols, for any game* $v \in \Gamma^N$*, this translates to requiring that* $v(S \cup T) \geq v(S) + v(T)$, $\forall S, T \in \mathcal{P}(N)$, $S \cap T = \emptyset$.

Superadditive games have the particularity that any two disjoint coalitions have a common interest to form only one coalition. The administration costs and investment policy associated with a pension plan usually depend on the size of the plan in such a way that we consider the pension plan to define a cooperative game.

Type of game 2 *A game is* ***convex*** *if and only if the sum of the worths of the union and intersection of any two coalitions is greater than or equal to the sum of their individual worths. In symbols, for any game* $v \in \Gamma^N$*, this translates to requiring that* $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, $\forall S, T \in \mathcal{P}(N)$.

So to speak, convex games are games with increasing returns to cooperation. Such games are very particular in that shares can always be found such that each coalition recovers at least its worth. That is not the case for superadditive games. In simple terms, while superadditive games make joining forces natural, convex games make withdrawing unnatural provided shares are determined appropriately.

Whereas it appears reasonable to claim that the game defined by the pension plan is superadditive, it does not seem as likely that it be convex, at least not over the entire range of possible contributions. In fact, it could be that returns to cooperation are increasing as the plan goes from small to medium, but then decreasing for any further increase in size.

[3]Both types of games can be redefined for the subset Γ_R by replacing $v(S)$ by $v(c; S)$ for all coalitions.

6 What Allocations and Sharing Rules Are

We have already mentioned that we want to distribute the surplus. In fact, we want to determine how to allocate the surplus amont the players.

Definition 1 *An **allocation** is a vector of shares* $(x_1, x_2, \ldots, x_n) \in \Re^n_+$ *satisfying the equality* $\sum_{i=1}^n x_i = v(N)$. *In other words,* x_i *is the amount received by agent* i*, and all of the surplus is being distributed.*

Implicit in the definition is an assumption of efficiency, which ensures that none of the surplus to be distributed is wasted. Implicit is also the requirement of feasibility, so that we do not allot any more than what we actually have.

We will focus on the grand coalition N since we ultimately want to determine how to split the surplus among all the potential claimants. As indicated earlier, potential claimants may not receive anything from a certain allocation. Indeed, zero shares do satisfy the definition of an allocation.

Not only do we need to find how to allocate the surplus in some particular situation, we would also like to come up with a rule that would determine the allocation in any situation. Hence, we also need to define the broader concept of a sharing rule.

Definition 2 *A **sharing rule** is a mapping*

$$\begin{aligned} \phi: \ \Gamma^N &\to \ \Re^n_+ \\ v &\to \phi(v), \end{aligned}$$

that is, one that associates to any game v *an allocation* $\phi(v)$ *of the total surplus* $v(N)$. *Hence,* $\phi(v)$ *satisfies the equality* $\sum_{i \in N} \phi_i(v) = v(N)$. *(Sharing rules are also called value operators.)*

7 Examples of Sharing Rules

We are now ready to introduce the four sharing rules that we will apply in the next section. They have been chosen for their properties as well as for their popularity in the economics literature.

Sharing rule 1 *[Shapley (1953)] Given the set of agents N and a TU game $v \in \Gamma^N$, where n is the number of agents, the* ***Shapley value*** *ϕ^S allocates $v(N)$ as follows:*

$$\phi_i^S = \sum_{S:i \notin S} \frac{(n-s-1)!\ s!}{n!}[v(S \cup \{i\}) - v(S)], \forall i \in N.$$

In the formula, s denotes the size of the coalition S and, by convention, $0! = 1$ and $v(\emptyset) = 0$.

Upon looking at the formula, we recognize the Shapley value as a weighted average of the individual's marginal contributions. The weight assigned to the marginal contribution of i to coalition S is the probability that, in a random ordering of the elements of N, the elements of S come first (in any order), then i, and then the elements of $N \setminus (S \cup \{i\})$ (again, in any order).

Before defining the nucleolus, we need to know how the *leximin ordering* works. Also, before establishing preference or indifference between vectors y and z (of the same size n), we first have to rearrange their coordinates in increasing order. Denote by y^* and z^* the corresponding ordered vectors. We then say that y and z are *leximin indifferent* if $y^* = z^*$. Also, we say that y is *leximin preferred* to z if there exists an integer $k < n$ such that $y_i^* = z_i^*, \forall i \leq k$, but $y_{k+1}^* > z_{k+1}^*$.

The leximin ordering firstly concerns itself with the elements that are worst off, then the next worst off, and so on. Underlying it is a search for equality. This ordering is at the heart of the following definition.

Sharing rule 2 *[Schmeidler (1969)] Given are the set of agents N and a TU game $v \in \Gamma^N$, where n is the number of agents. We denote by $\mathcal{B}$ the set of allocations $x \in \Re_+^n$. We then define the excess vector $e(x) \in \Re^{2^n-2}$ associated to the allocation $x \in \mathcal{B}$ as*

$$e(x; S) = \sum_{i \in S} x_i - v(S), \forall S \subset N, S \neq \emptyset.$$

The **nucleolus** *is the unique allocation $\phi^N \in \mathcal{B}$ such that, for every other $x \in \mathcal{B}$, $e(\phi^N)$ is leximin preferred to $e(x)$.*

The nucleolus is a core selection. The core is made up of all the allocations for which $\sum_{i \in S} \geq v(S), \forall S$, and a core selection is a sharing rule which picks an allocation in the core whenever the core is nonempty.

As a matter of fact, the allocation chosen by the nucleolus lies at the center of the core whenever it is nonempty. (Here, we mean "center" in the physical sense. Hence, we consider the core as a hyperplane made of matter having uniform density and find its center of gravity.) This centrality comes about because of the egalitarian distribution of the excesses among all coalitions. The excesses measure the benefit enjoyed by each coalition S beyond its own opportunity surplus $v(S)$.

Equality of excesses should in no way be confused with equality of shares. It all depends on the location of the core. In fact, the Shapley value usually leads to shares which lie closer to the center of the simplex than those of the nucleolus.

Sharing rule 3 *[adapted from Moulin and Shenker (1991)] Given the set of agents N and a TU game $(v, c) \in \Gamma_R^N$, where n is the number of agents, denote by c_i agent i's contribution and by c_N the sum of the contributions. The* ***average sharing*** *rule ϕ^A allocates $v(c; N)$ as follows:*

$$\phi_i^A(v, c) = \frac{c_i}{c_N} v(c; N)$$

Indeed, the shares determined by the average sharing rule are very easy to calculate. They are proportional to what each agent puts in. Because of that feature, this sharing rule is also sometimes called the proportional sharing rule.

Sharing rule 4 *[adapted from Moulin and Shenker (1991)] Given the set of agents N and a TU game $v \in \Gamma^N$, where n is the number of agents, denote by c_i agent i's contribution and by c_N the sum of the contributions. We assume $c_1 \leq c_2 \leq \cdots \leq c_n$. The* ***serial sharing*** *rule ϕ^C allocates $v(c; N)$ as follows:*

$$\begin{aligned}
\phi_1^C(v, c) &= \frac{v(c; \{1, 1, \ldots, 1\})}{n} \\
\phi_i^C(v, c) &= \phi_{i-1}^C(v, c) \\
&\quad + \frac{v(c; \{1, 2, \ldots, i, \ldots, i\}) - v(c; \{1, 2, \ldots, i-1, \ldots, i-1\})}{n + 1 - i}, \forall i > 1
\end{aligned}$$

A very interesting property characterizes the serial sharing rule. Under that rule, each agent i receives a share based on all agents' contributions up to what i contributes. In other words, in determining agent i's share, we calculate the surplus based on the actual contributions for agents with index less than or equal to i, and on contributions equal to c_i for agents with index greater than i. That is what is meant by the notation $v(c; \{1, 2, \ldots, i, \ldots, i\})$.

8 Application of the Sharing Rules to a Pension Plan

8.1 Characteristics of the Pension Plan

As a preliminary example of how to apply the four sharing rules introduced in the previous section, we will work with a three-person pension plan. The benefit formula used here is 2% times the number of years of service times the annual salary in the final year. No indexation is provided. The three participants started in 1978, then aged 25, 30 and 35 with starting salaries 30,000, 32,500 and 35,000, respectively.

We look at the experience of the plan from 1978 to 1981, using average Canadian experience for rates of return and wage inflation. The salary scale is fixed and none of the three participants dies or leaves in the interval. For valuation purposes, we assume an interest rate of 10% and a wage increase of 8%. When determining the surplus of smaller coalitions, we decrease the rate of return by 0.5% and the valuation interest rate by 0.1%, per fewer person. Contributions and liabilities are determined using the PUC (projected unit credit) method.

We will calculate the shares using the sharing rules in the order that they were presented. Moreover, we assume that the whole surplus is being distributed and that none of it goes back to the sponsor. We will index the participants from youngest to oldest, so that agent 1 was 25 in 1978.

8.2 Shares Determined by the Four Sharing Rules

First of all, for the Shapley value, we need to determine what the surplus would have been for all one- and two-people pension plans, subsets of the "big" one. Using the assumptions made, we obtain the following worths:

$$\begin{array}{lclclclclcl} v(\{1\}) & = & 77.18 & v(\{2\}) & = & 355.87 & v(\{3\}) & = & 618.77 \\ v(\{1,2\}) & = & 646.54 & v(\{1,3\} & = & 787.34 & v(\{2,3\}) & = & 1174.86 \\ v(\{1,2,3\}) & = & 1286.63 & & & & & & \end{array}$$

Using these values and applying the formula found in Sharing rule 1, we obtain $\phi_1^S = 139.52$, $\phi_2^S = 472.63$, and $\phi_3^S = 674.48$.

For the nucleolus, we use the worths already derived and we obtain $\phi_1^N = 123.60$, $\phi_2^N = 511.12$, and $\phi_3^N = 651.91$. Since the nucleolus is defined in terms of the excesses, it is interesting to see what they turn out to be with these shares:

$$\begin{array}{llllll} e(\phi^N,\{1\}) & = 46.42 & e(\phi^N,\{2\}) & = 155.25 & e(\phi^N,\{3\}) & = 33.14 \\ e(\phi^N,\{1,2\}) & = -11.82 & e(\phi^N,\{1,3\}) & = -11.83 & e(\phi^N,\{2,3\}) & = -11.83 \end{array}$$

Were it not for rounding errors, the excesses for the two-agent coalitions should all be the same. Furthermore, the fact that they are negative tells us that the core is empty.

When it comes to average sharing, we have to decide what we mean, here, by contributions. We can simply sum the actual contributions made. This however ignores their timing. We can also look at the accumulated contributions, using actual rates of return or some type of valuation interest rate. We could also use the accrued liability as a proxy for contributions but, whereas liabilities are calculated prospectively, contributions are calculated retrospectively. Another possible proxy might be the accrued benefit: it would be simple to determine based on the records, but it too would fail to account for the timing of the contributions. (This problem may not be as important in the case of a career-average plan.)

Here are the accumulated contributions, which we prefer to the other alternatives presented: $c_1 = 9622.94$, $c_2 = 11,644.99$, and $c_3 = 13,938.73$. These lead to the following shares: $\phi_1^A = 351.67$, $\phi_2^A = 425.57$, $\phi_3^A = 509.39$.

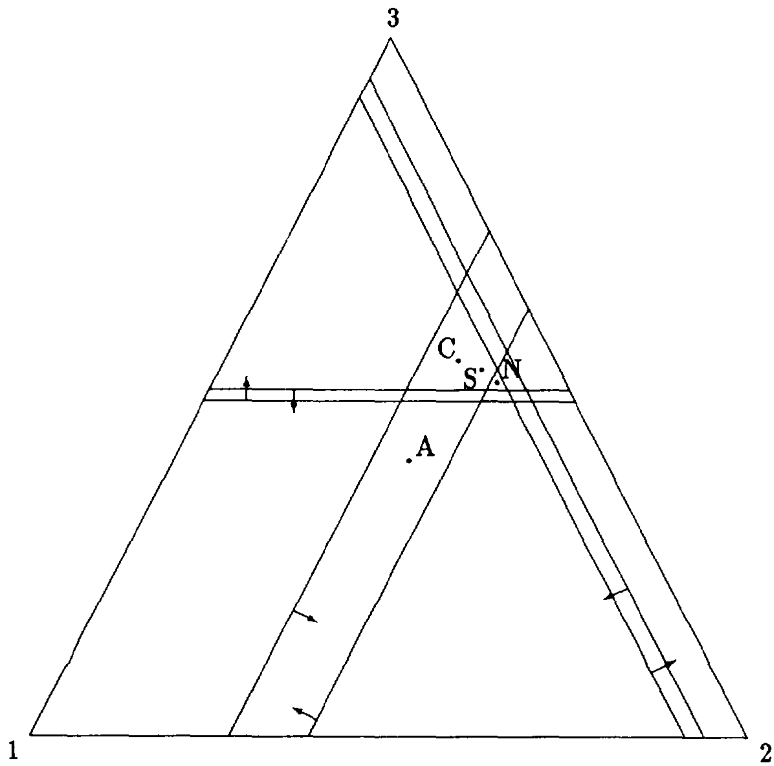
Finally, for the serial sharing rule, we need to calculate what the surplus would have been under different combinations of the three participants. We calculate $v(\{1,1,1\}) = 516.15$, $v(\{1,2,2\}) = 1017.15$, and $v(\{1,2,3\}) = 1286.63$. We then find $\phi_1^C = 172.05$, $\phi_1^C = 422.55$, and $\phi_2^C = 692.03$.

8.3 Graphical Representation of the Results

It is possible to represent graphically the results obtained above. When we only have three players, the set of possible allocations always is the triangle with vertices (0,0,$v(N)$), (0,$v(N)$,0), and ($v(N)$,0,0) in the three-dimensional system of co-ordinates. In two dimensions, we call the triangle a simplex. (In general, a simplex is a figure in m dimensions determined by $m + 1$ points.)

Figure 1 is the pictorial representation of our simple example. First of all, the simplex is the set of all possible allocations. The number at each vertex identifies the agent which would receive all the surplus if that was the chosen allocation. The interior lines indicate the constraints that would have

Figure 1: Game Depicted by the Three-Member Pension Plan



to be satisfied in order for the allocation to be in the core; the arrows indicate on which side of the line the allocation would have to be to achieve that. A careful look confirms that it is impossible to meet all the constraints. Rather, we obtain a "shadow" core, whose boundaries are given by the constraints imposed by the two-person coalitions.

The four allocations are given by the four points, and the letter used to identify each of them matches the superscript. It is interesting to confirm that the nucleolus is exactly at the center of the "shadow" core. Moreover, whereas the allocation given by the Shapley value and the serial sharing rule are not too far from the core – in fact, they satisfy the same three constraints, plus one more, as the nucleolus, the average sharing rule picks an allocation outside that cluster.

9 Conclusion

In practice, certain steps precede the actual distribution of the surplus. Indeed, we first need to decide how we wish to value the assets and the liabilities. We then need to choose the percentage of the surplus that is to be distributed. These two steps can be accomplished using a variety of methods which deserve further consideration.

In this paper, we have focused our attention on the following step: the splitting of the surplus of a pension plan among the claimants. Hence, we have presented a new approach to an existing situation. Our approach rests on the theory of cooperative game theory which has as its general objective the sharing of surplus (or cost).

This alternative to *ad hoc* rules has the advantage of explicitly recognizing each player's contributions. Moreover, because it is an axiomatic approach, we can determine which properties each sharing rule has. These properties may guide our choice of a particular sharing rule.

Unfortunately, this approach is, for most sharing rules, very computationally intensive. As a matter of fact, we need to perform as many valuations as there are coalitions for which we need to know the worth. Furthermore, there still remains subjectivity, be it in the choice of a sharing rule or in determining the impact of the size of the plan on the valuation assumptions.

Despite these shortcomings, we believe that this approach is of interest in its own right. Further research may unveil simplifications (or rules of thumb) which make sharing rules more readily applicable to the problem at hand.

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