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UNIQUENESS OF YIELD RATES

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ABSTRACT: When does a financial transaction have a unique yield rate, or a unique rate in some interval , such as the positive reals? We survey various results concerning this problem, and provide a unifying approach together with some new insights.

1. INTRODUCTION

Given a financial transaction we are often interested in the number of internal rates of return, (yield rates) and their location. From an actuarial point of view, the fundamental question is to identify those transactions with a single yield rate, or a single rate in some interval such as the positive reals. In this paper we are concerned with this counting problem . We will not deal here with possible interpretations when there are multiple yield rates. This issue has been extensively discussed elsewhere

This question of identifying single rate transactions was raised recently by the Canadian Institute of Actuaries in an attempt to clarify a section of the Canadian criminal code which makes it a offense to advance money at a effective interest rate exceeding 60% per annum. While this involves an old mathematical problem, the basic facts are not that well known among the actuarial profession. The actuarial literature contains some but not all of the major results. Various papers appeared in the finance literature in the l970's, often reproving known facts. The mathematical literature of course has a great deal of material, as at one time, the problem of finding roots of equations played a dominant role in mathematics. However it is not that simple to seek out and identify the important results from a financial perspective. The main purpose of this paper a to survey the pertinent facts and to provide some unifying approach. A key result is Theorem 3.4. Although the formula given there was known to Laguerre [3] we are able to provide a financial setting for the statement and relate it to the main facts on identifying unique yield transactions. In section 4 we describe various methods which have been used for counting roots, and indicate how they are all related to a certain fundamental array.

2. THE BASIC PROBLEM

Given a sequence of periodic cash flows

$$c_0, c_1, \ldots, c_N \tag{2.1}$$

we associate the polynomials

$$f(x) = \sum_{i=0}^{N} c_i x^i$$

and

$$g(x) = \sum_{i=0}^{N} c_i x^{N-i} = x^N f(x^{-1}) \tag{2.2}$$

For $x > -1$, $f(x)$ equals the present value of the payments at a periodic effective interest rate of $x^{-1}-1$, and $g(x)$ equals the accumulated value at a periodic effective rate of $x-1$. A *yield rate* (also called *internal rate of return*) of the transaction is a number $i > -1$ such that $(1+i)^{-1}$ is a root of $f(x) = 0$, or equivalently such that $(1+i)$ is a root of $g(x) = 0$. We are therefore interested in positive zeros of $f(x)$ or $g(x)$. For certain applications one may wish to confine attention to *positive* yields These correspond to zeros of $f(x)$ which lie in the interval $(0,1)$ or zeros of $g(x)$ which lie in the interval $(1,\infty)$.

We will sometimes want to write g expanded about a general point. Recall that for any real number r

$$g(x) = \sum_{i=0}^{N} \frac{g^{(i)}(r)}{i!} (x-r)^i \tag{2.3}$$

and replacing g by $g^{(k)}$ gives

$$g^{(k)}(x) = \sum_{i=0}^{N-k} \frac{g^{(i+k)}(r)}{i!} (x-r)^i \quad , \ k = 0, 1, \ldots, N \tag{2.4}$$

An important role in our approach is played by the balance functions. For any $x > -1$, and nonnegative integer k we let

$$b_k(r) = \ = \sum_{i=0}^{k} c_i \ r^{N-i}$$

the outstanding balance at interest rate $i = r-1$. We consider this defined for *all* nonnegative integers k by taking $c_k = 0$ for k larger than N. It is often convenient to invoke the recurrence relation

$$b_{k+1}(r) = r \ b_k(r) + c_{k+1} \tag{2.5}$$

We close this section with some general remarks about locating zeros of polynomials .

Note that a procedure which locates zeros in $(0,1)$ is sufficient to locate all zeros , Applying the procedure to the polynomial with the coefficients reversed, locates zeros in $(1,\infty)$. Then, reversing the sign of the odd coefficients gives us the polynomial $f(-x)$ and the positive zeros of this are the negative zeros of the original.

The best known method for estimating the number of zeros is Descartes rule of signs which says that the number of sign changes in (2.1) exceeds the number of *positive* zeros by an even integer. Lower bound tests are available as well. Suppose, for example, that you want to count zeros in $(0,1)$. Choose any sequence of points y_1, $y_2, \ldots, y_m$ in $(0,1)$ and look at the sequence

$$f(0), \ f(y_1), \ldots, \ f(y_m), \ f(1) \tag{2.6}$$

The number of sign changes in this sequence gives an obvious lower bound.

The reader should note that when counting sign changes in a sequence we ignore zero entries. For example, the sequence (-10,0,2,0,-3) has two sign changes.

It is important to realize that any method can run into difficulties in the case of multiple zeros. For example, if a polynomial has a zero of multiplicity two at a point x_0, a small perturbation of f can give a polynomial with no zero at that point, but any given numerical procedure could fail to distinguish these two cases if the change is sufficiently small.

3. YIELD RATES AND BALANCES

The number of yield rates is closely related to the pattern of balances. The following theorem summarizes the basic facts.

THEOREM 3.1

(a) (i) If for some $r > 0$, $b_k(r) \le 0$, $k = 0,1, \dots ,N-1$, and $b_N(r) = 0$, then $r-1$ is the unique yield rate

(ii) If for some $r > 0$, $b_k(r) \le 0$, $k = 0, 1, \dots N-1$, and $b_N(r) > 0$, then there is a unique yield rate i, and moreover, i is greater than $r-1$

(b) If there is exactly one sign change in the sequence of coefficients

$$c_0 , c_1 , c_2, \dots, c_N$$

there is a unique yield rate.

(c) If there is exactly one sign change in the sequence of partial sums

$$c_0 , c_0 + c_1 , c_0 + c_1 + c_2 , \dots , \sum_0^N c_i$$

there is a unique *positive* yield rate.

REMARKS Part (a)(i) is well known and intuitively obvious from an actuarial point of view. Multiple rates arise when there are mixed elements of both borrowing and lending present. In this case, the balance never becomes positive and the transaction is strictly a lending transaction. Part (a)(ii) follows by continuity. As we increase r, the nonpositivity of the balances together with (2.4) implies that the final balance will become zero. Part (b) of course is even better known and typifies the usual loan transaction. It follows immediately from Descartes rule of signs, but it can also be viewed as a special case of part (a). Suppose that the sign changes from negative to positive. There must be at least one yield r, and balances can never become positive, for this would necessarily occur after the sign change and then the final balance could never become zero. Part (c) is not as well known and is the source of some confusion. Note that the condition implies a unique *positive* yield, but there can be several negative yields. The statement appears a few times in the actuarial literature, but as far as we can see, without proof.

As a generalization of Theorem 3.1, we have the following. It appears to be new as far as the actuarial or financial literature goes, but a similar statement appears as a strictly mathematical theorem in [3].

THEOREM 3.2

Consider the sequence

$$b_0(r), b_1(r), \ldots, b_N(r)$$

(a) If there is exactly one change in the sequence there is exactly one yield rate greater than r-1.

(b) If all terms of the sequence are of the same sign, there are no yields greater than r-1.

REMARK . Theorem 3.2(a) implies all of Theorem 3.1. Taking $r = 1$, gives part (c) and taking $r = 0$ gives part(b). Part (a) follows as well, after noting that (2.4) and the nonpositivity of balances imply that for any $s < r$, we must have $b_s < 0$.

A unified proof of all these results will now be obtained. We first invoke a theorem of Laguerre who generalized the rule of signs from polynomials to power series.

THEOREM 3.3 (Laguerre)

Suppose that $f(x) = \sum_{i=1}^{\infty} c_i x^i$ has radius of convergence $\rho > 0$.

Suppose that C , the number of sign changes in the sequence (c_i) is finite. Then if Z is the number of positive zeros of $f(x)$ (including multiplicity)

(i) $Z < C$

(ii) Provided that $\rho = \infty$, or that the series diverges for $x = \rho$, $C-Z$ is even.

See [4] for details. One uses Rolle's theorem to show that the number of positive zeros is finite, and then invokes the fact that a function analytic in a disk has a powers series expansion which converges in the disk, to reduce the problem to the polynomial case.

In the classical case of a polynomial, $\rho = \infty$, and the parity check given in (ii) always holds.

The unifying result we mentioned above is

THEOREM 3.4

Given any polynomial f(x) and $r > 0$

$$\frac{f(x)}{(1-xr)} = \sum_{k=0}^{\infty} b_k(r)\, x^k \quad \text{for} \quad x < \frac{1}{r}$$

Proof. Straightforward calculations.

We now note that Theorem 3.4 together with Descartes rules proves all assertions, since the positive zeros of f(x) which as less than $1/r$, are exactly the same as the zeros of the function $f(x)/(1-xr)$. Note also that the power series has a singularity at the point $1/r$, so the parity check in Theorem 3.3 holds.

The case when $1/r$ is a root of f leads to part (a)(i) of Theorem 3.1 . The reason we are able to make a global statement here is explained by the fact that in this case the power series appearing in Theorem 3.4 is actually a polynomial.

4 A FUNDAMENTAL ARRAY

In this section we survey some methods for counting roots which are related to the following array of numbers. The first row is found by writing down the partial sums of the coefficients as in Theorem 3.1(c). We include all coefficients, taking $c_k = 0$ for k greater than N, so this row is really infinite , although it stabilizes at a constant. The second row is found by writing down the partial sums of the first row, and we continue to iterate the process.

These arrays appear in [3] where there are several examples. Laguerre also observed the connection with the well known Fourier-Budan method.(described below). They are used in by Uspensky in his textbook [7] to describe what he claims is a little known procedure attributable to Vincent. More recently, these arrays were rediscovered by J.R. Pratt, [5] [6]. Pratt's work, motivated by financial applications, provides the most extensive contributions to this tool. He develops new algorithms together with theoretical justifications.

Let c_{ij} denote the entry in the ith row and jth column where we number the row starting with 1 and the columns starting with 0. Then

$$c_{ij} = \text{the coefficient of } x^j \text{ in the expansion of } \frac{f(x)}{(1-x)^i}$$

and therefore, the number of sign changes in any row gives an upper bound to the number of zeros of $f(x) = 0$, in the interval $(0,1)$ This generalizes Theorem 3.1(c), which considers just the first row of the array

It is not necessary to compute complete rows to get an upper bound. We can verify that

$$c_{k+1,N-k} = \frac{g^{(k)}(1)}{k!} \tag{4.1}$$

so by (2.3) the (N+1)st-diagonal, that is running from $c_{1,N}$ to $c_{N+1,0}$ gives the coefficients of g when expanded about the point 1. Hence, the number of sign changes on this diagonal gives an upper bound to the number of zeros of g in $(1,\infty)$ which are the zeros of f in $(0,1)$.

Pratt uses more general paths, and shows in fact that we can get an upper bound by counting sign changes on any path which starts in column 0, and ends in row 1 at a column $k \geq N$. The key to this is his observation that in moving from one row to the next, we perform successively the following. Insert the sum between any two numbers, and then delete one of the numbers. Neither operation can add new sign changes so the number of changes decreases or stays the same as we continue adding rows. It follows that any row which lies beneath a path of the type described , has fewer sign changes than on that path. Pratt's procedure usually gives a very quick and efficient method of estimating the number of zeros .

As an example consider

$$f(x) = -1 + 3x - 5x^2 + 6x^3$$

We obtain the array

$-1 \quad 2 \quad -3 \quad 3 \quad 3 \quad 3 \; .....$

$-1 \quad 1 \quad -2 \quad 1 \quad 4 \quad 7 \; ......$

$-1 \quad 0 \quad -2 \quad -1 \quad 3 \quad 10 \; ......$

The first row has three sign changes indicating either one or three zeros in the interval (0,1). The second row adds no new information, but from the third, we see there is a unique root in this interval. We do not need all these entries however and can see this just from the path -1, 0, -2, 3 running from $c_{3,0}$ to $c_{1,3}$

Pratt shows that the exact upper bound will eventually be reached if one goes far enough in the array. Moreover, in the multiplicity free case he gives a method for choosing the sequence in (2.6) so that the lower bound will eventually equal the upper bound. This provides a stopping rule to the procedure which therefore becomes an exact algorithm for counting the zeros of multiplicity free polynomials.

As mentioned, this array can be applied to the Fourier-Budan method, a well known refinement of the sign rule. For any real number r, (2.3) shows that the sequence of sign changes in the sequence $\{f(r), f'(r), f''(r), \ldots, f^{(N)}(r)\}$ give an upper bound to the number of zeros in (r, ∞). In fact a stronger statement is possible, namely, that the quantity (sign changes - zeros) decreases as r increases. Hence given $r < s$, if we compute the number of sign changes in the derivatives at r, and subtract from this the number of sign changes in the derivatives at s, we get an upper bound for the numbers of zeros in the interval (r,s). (See [1] for a derivation and [2] for some financial applications). To implement the Fourier-Budan method for the interval (0,1) we compute the array of the polynomial with reversed coefficients, that is of g. From (4.1) we see that the (N+1)st diagonal gives the number of sign changes in the sequence $\{f^{(k)}(1)\}$, which we then subtract from the number of sign changes in the sequence of original coefficients , to arrive at an upper bound.

Vincent's method is also based on this observation regarding the (N+1)st diagonal Let us define for any polynomial p, of degree N the polynomials p_1 and p_2 given by

$$p_1(x) = p(x+1), \quad p_2(x) = (1+x)^N p([1+x]^{-1})$$

As noted above the coefficients of p_2 are on the $(N+1)$st diagonal of the array constructed for p, and those of p_1 are on this diagonal for the polynomial obtained from p by reversing the coefficients. The Vincent method simply iterates the calculation of diagonals to produce a branching procedure. Starting with f we calculate the polynomials

$$f_{i_1 i_2 \cdots i_k}$$

where each i_j is 1 or 2. (For example, f_{121} is the polynomial p_1 where p is f_{12}.) We terminate branches where there are no sign changes or one sign change, indicating either no positive zeros or exactly one positive zero respectively. If all branches terminate, we obtain an exact count on the positive zeros. Moreover, by working backwards we can narrow down their location. For example if $f_{i_1 i_2 \cdots i_k 2}$ has a zero in (a,b), then $f_{i_1 i_2 \cdots i_k}$ has a zero in $([1+b]^{-1}, [1+a]^{-1})$ while if $f_{i_1 i_2 \cdots i_k 1}$ has *a zero in* (a,b), *then* $f_{i_1 i_2 \cdots i_k}$ has *a zero in* $(a+1, b+1)$

Consider the following example, which appears in [7]

$$f(x) = 7 - 7x + x^3.$$

We produce the following sequences of coefficients.

		Sign changes
f_1 :	1, -4, 3, 1	2
f_2 :	1, 7, 14, 7	0
f_{11}:	1, 5, 6, 1	0
f_{12}:	1, -2, -1, 1	2
f_{121} :	-1, -1, 2, 1	1
f_{122}:	-1, -2, 1, 1	1

All branches terminate, and we know we have exactly two positive zeros. To locate them more precisely we note that

f_{121} has exactly one zero in $(0, \infty)$ implying that
f_{12} has exactly one zero in $(1, \infty)$ implying that
f_1 has exactly one zero in $(0, 1/2)$ implying that
f has exactly one zero in $(1, 3/2)$,

and

f_{122} has exactly one zero in $(0, \infty)$ implying that
f_{12} has exactly one zero in $(0, 1)$ implying that
f_1 has exactly one zero n $(1/2, 1)$ implying that
f has exactly one zero in $(3/2, 2)$.

Vincent proved that for the multiplicity free case, all branches terminate and an exact count of positive zeros is obtained.

As a final result based on the array, we have the following theorem, which appears to be new.

THEOREM 5.1 Suppose that the first row of the array for f has C sign changes. Then for any nonnegative integer m, and any $k \leq m+N$, the number of zeros of the k-th derivative of the function $x^m g$ in the interval $(1,\infty)$ is less than or equal to C.

Proof As we have noted, the sequence $\{ g^{(k)}(1): k = 0,1,... \}$ has at most C sign changes. and (2.4) establishes the result for $m = 0$. Observe now that the original definition of N and hence of g is ambiguous, since we did not specify that a_N was nonzero. By adding zero coefficients, we get the same result for $x^m g$ replacing g.

We see from this that the simple procedure of Theorem 3.1(c) cannot select all polynomials with unique zeros in (0,1) but only a very special class.

For completeness we mention another classical method of counting zeros, namely the procedure of Sturm, which we are not going to discuss here. The interested reader is referred to [1]. This procedure gives an exact count of the number of zeros, ignoring multiplicity. For example, the method would return a count of 1 for $f(x) =$

$(x-2)^m$. Aside from the fact that this method usually involves a great deal of calculation, the ignoring of multiplicity means that it is not that practical for the purpose of locating unique yield transactions.

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