ACTUARIAL RESEARCH CLEARING HOUSE
1998 VOL. 1

Multiple Period Contracts Contingent on Previous Contract Choices

Abstract: Rothschild and Stiglitz's (1976) single period model of the insurance industry can be extended to multiple periods. Insurers offer a sequence of single period contracts in which future contracts are conditioned on past contract choices. In a multiple period framework, once low risks have revealed their type, competition forces future contracts to be contingent on this event. However, this is not observed in the marketplace. Possible reasons given in the literature for this are that insurers cannot restrict the amount of insurance bought (Kunreuther and Pauly (1985)), or that insureds are unaware of their risk type.

Under the assumption of perfect competition, multiple period Rothschild-Stiglitz contracts are developed in this paper. These contracts are compared to a series of one period Wilson (1977) pooling contracts. It is shown that low risk consumers maximise their utility by pooling with high risk consumers. Thus a third reason as to why Rothschild-Stiglitz contracts are not observed; multiple period separating contracts do not exist because the cost of separation is too high.

Mary V. Kelly

Faculty of Management

2500 University Dr. N.W.

University of Calgary

Calgary, AB T2N 1N4

Phone: (403) 220-6168

Email: mkelly@mgmt.ucalgary.ca

08 September 1997

Multiple Period Contracts Contingent on Previous Contract Choice

Typically, policyholders do not purchase property/casualty insurance only once in their lifetimes, but make annual decisions concerning insurance purchases. In most personal and commercial lines of insurance, contracts are renewed annually and relationships between insurers and policyholders often include significant past history. To reflect this reality, it would be beneficial to have a theoretical multiple period model of the property/casualty insurance industry.

Rothschild and Stiglitz's (1976) one period model of the insurance industry predicts that equilibrium in the marketplace exists in which consumers with differing risk propensities purchase varying amounts of insurance. A multiple period extension of Rothschild and Stiglitz's model can be constructed in which a consumer's future contract choice is contingent on her past contract choice. As in the one period Rothschild-Stiglitz model, a consumer reveals her risk type through the amount of coverage purchased in the period of separation. Since the contract purchased reveals the consumer type, future policies must be conditioned on this information and as such future contracts are contingent on past contracts.

Unfortunately, in personal lines of property-casualty insurance, policies are not typically contingent on past policy choices. Future contract provisions, either the amount of insurance offered, or the price of insurance, are usually contingent on the past accident histories of the insureds and not on the amounts of insurance purchased. Two possible reasons given in the literature for this are that contracting insurers cannot restrict the amount of insurance bought from other insurance companies and so the amount cannot be used as a screening mechanism (Kunreuther and Pauly (1985)) and that insureds are unaware of their risk type. This paper provides another conjecture as to why these contracts, denoted Rothschild-Stiglitz contracts, are not observed in the market place. Quite simply, the cost of revealing one's type is so high that lower risk consumers prefer to pool with higher risk consumers.

In this paper, multiple period Rothschild-Stiglitz contracts are constructed. As in the single period framework, the separating menu of contracts is designed such that insurers earn zero profits and no consumer has the incentive to misrepresent her type. In the multiple period framework, until a consumer reveals her type, she may purchase either one of the contracts in the separating menu of contracts or a pooling contract satisfying Wilson's (1977) anticipatory equilibrium. After a consumer reveals her risk type, she receives full insurance priced for her risk type for the remaining periods.

The separation decision of low risk consumers in a multiple period world in which both Rothschild-Stiglitz contracts and pooling contracts are offered is examined. It is shown that the decision to separate is a function of the number of periods remaining in the model, the loss probabilities of the different consumer types, the mix of consumer types in the economy and the size of potential loss. The decision to separate is <u>not</u> a function of the total number of periods in the model.

Numerical examples are provided to assist understanding of the theoretical results. Through an examination of multiple period pooling and separating contracts, it is shown that the costs of separation are so high that low risk consumers are better off pooling with high risk consumers. Thus one conjecture as to why observed insurance contracts do not reveal a consumer's risk type is that the cost of separation for low risks is too high. That is, in general, utility-maximising low risk consumers would to pool with higher risk types than to reveal their own risk level.

It is shown that over a reasonable range of parameter values, there is no equilibrium amount of indemnity that can be offered by insurers that will prevent high risk consumers from misrepresenting their loss type several periods before the last period. That is, in many situations perfectly competitive insurers offer only pooling contracts. Even when both pooling and separating contracts are feasible, it is also shown that utility-maximising low risk consumers would never wish to reveal their type.

Finally, some empirical evidence against the classic theoretical result that low risk consumers tend to purchase less insurance than high risk consumers is provided. The amount of insurance purchased for third party liability private passenger automobile coverage is surveyed for consumers with differing accident histories. It is evident that consumers with better driving records tend to buy more insurance than those consumers that have incurred many claims, contrary to theoretical predictions.

In Section 1, both Wilson pooling contracts and multiple period Rothschild-Stiglitz contracts are constructed and analysed. In Section 2, numerical examples of the results derived in Section 1 are presented. Section 3 provides empirical evidence that low risk consumers purchase more insurance than high risk consumers do.

1. Multiple Period Rothschild-Stiglitz Contracts

Rothschild and Stiglitz's model provides great insight into informational problems surrounding a one period insurance model. Their analysis can be extended to a multiple period world. In a multiple period framework, once low risk clients have revealed their risk propensity, insurers are restricted to offering full insurance contracts to both consumer types. In this section, the optimal separation decision of a low risk consumer is defined.

The basic structure of the economy is as follows. These consumers live in a world where there are multiple time periods and 2 states of the world in each period. Events in each period are independent of other periods. It is assumed that there is no moral hazard in this framework.

In this world, there exist risk-averse consumers who differ only by their risk propensity. Low risk consumers, who comprise $1-\lambda$ of the population, will, in any one period, incur a loss of size d with probability ρ^{ℓ}. High risk consumers, who comprise λ of the population, will, in any one period, incur a loss of size d with probability $\rho^{h} > \rho^{\ell}$. It is assumed, for simplicity, that this probability of loss is uncorrelated across consumers and across time periods. Also, for simplicity, it is assumed that neither consumers nor firms discount future returns. Consumers are endowed with initial wealth W. This level of wealth is significantly large such that consumers face no wealth constraints over the entire time frame.

All consumers possess constant absolute risk aversion. This assumption is extremely valuable. First, it allows for the insurance purchasing decision of a consumer to be examined separately from her investment and consumption decisions. As long as changes in investment and consumption are uncorrelated with potential losses, then a consumer's insurance decisions can be examined in isolation. Secondly, the use of the negative exponential utility function ensures the period by period time consistency of the model. Insurance coverage is bought at the beginning of the period, before any losses have occurred. With the constant absolute risk aversion utility function, the optimal amount of insurance that would have been purchased *ex post*, after it is known whether or not a loss has incurred, is the same as the amount that was actually purchased.

Because of this, the optimal separation decision of the low risk consumer can be defined before insurance is purchased for the first time.

In each time period, an individual can insure against loss by purchasing a one-period insurance contract from perfectly competitive insurers. It is assumed that consumers receive greater utility from purchasing insurance than from foregoing insurance. Insurers offer repeated contracts to consumer, which is consistent with reality. The contracts are contingent only upon past contract choice and not past accident history. In each period, the moves are as shown in Figure 1.

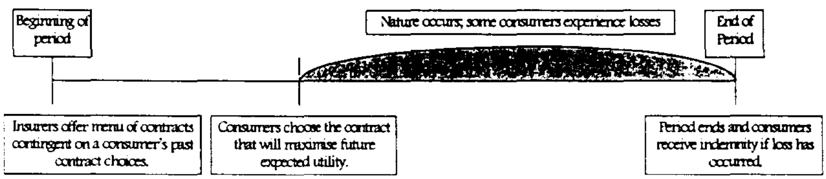

Figure 1 - Ordering of Movement within a Period

The two probabilities of loss in each period, p^h and p^l are know to the potential insurers, as is the size of the potential loss, d. Each consumer knows if she is high or low risk, but this information cannot be observed by any of the insurance companies. In selling insurance each period, the perfectly competitive insurers each incur an additive expense e to write a single policy.[1]

If all insurers could observe each consumer's true level of risk, then one set of contracts that could be offered in equilibrium to consumers consists of a series of single period full insurance contracts

[1] Because of the fixed cost involved with the purchase of insurance, the utility maximising individual would never purchase more than one policy. In the presence of a multiplicative expense loading all insureds would prefer to purchase less than full insurance (Arrow (1965), Mossin (1968), Szpiro (1985) and Borch (1990)). Eisenhauer (1993) shows that full insurance may be purchased in the presence of an expense loading if the insurer and the consumer have differing estimates of the probability of loss. In the model in this paper, the expense loading is additive. In this case, the utility maximising individual would always prefer to purchase a full insurance contract or purchase no insurance. Competition, whether realised or potential, constrains insurers to offering full insurance contracts in equilibrium. The additive expense behaves as a quasi-fixed cost, since it is only incurred if insurance is purchased. Thus it is possible that this expense could be so high that consumers would not purchase insurance. Despite this drawback of the additive expense loading, the additive expense loading is a realistic representation of the costs incurred in writing a policy. As noted by Wade (1973), the use of a constant expense to cover those costs which are incurred at a constant level per policy is a dominant pricing strategy.

with a single period price of $\rho^h d + e$ for the high risk consumers and $\rho^l d + e$ for the low risk insureds. The total utility earned is $-e^{-\alpha(W - n\rho^h d - ne)}$ by the high risk consumer and $-e^{-\alpha(W - n\rho^l d - ne)}$ by the low risk consumer.

Since consumers maximise the present value of future utility, it is not necessarily true that the only contracts offered in equilibrium are those contracts which earn insurers zero expected profit. The required condition is that insurers earn zero expected profit over the entire association between the consumer and the firm. If contracting insurers offer non actuarially fair contracts in some periods, consumers might have the incentive to switch to rival insurers at some point in the model. The possibility of switching by consumers increases the complexity without increasing the clarity of the model. If insurers are restricted to offering contracts that earn zero expected profits each period, then this complexity will not exist. Therefore, the analysis herein will focus on contracts that earn zero expected profits each period.

Contracting insurers observe the past policy choices of its consumers while rival insurers only know if a consumer has previously purchased insurance or if a consumer is new to the insurance market. Therefore rival insurers are constrained to offering full insurance priced for the high risk type to all consumers who switch insurers. Any other contract offering has the potential to earn negative expected profits for the rival insurer. Because of the types of contracts offered by rival insurers, only high risk consumers would ever have the incentive to switch insurance companies.

In any period, once a consumer has already revealed her risk type, *ex-ante* competition, whether realised or potential, restricts insurers to offering full insurance priced for each risk type. That is a low risk consumer would pay $\rho^l d + e$ and a high risk consumer would pay $\rho^h d + e$ to purchase full insurance. Because consumers maximise their utility over the entire time frame, during the initial selection of insurance, they would not choose a stream of contracts which promised less than full insurance once separation has occurred.[2] This precommitment of the insurer to a series of contracts is standard in multiple period insurance models (see, for example, Cooper and Hayes (1983), Dionne (1983) and Dionne and Doherty (1994)).

---

[2] This, of course, depends on the assumption that consumers choose policies that maximise utility over all future periods. Kunreuther and Pauly note that there has not been much empirical verification of the degree of foresight that policyholders possess, although they conjecture that policyholders behave myopically. Non-myopic behaviour of consumers is a necessary condition in a multiple period Rothschild-Stiglitz framework.

In equilibrium, low risk consumers will maximise expected utility by separating $k=k^*$ periods before the last period, where k is defined as follows:

- $k=0$ implies that consumers never separate. All consumers purchase single period pooling contracts for n periods
- $k=1$ implies that consumers separate in the last period. They never receive full insurance contracts.
- For any value of $k>1$, insureds purchase single period pooling contracts for the first $n-k$ periods and single period full insurance separating contracts for the last $k-1$ periods.

Figure 2 shows the stream of policies received if consumers separate k^* periods before the last period. Because high risk consumers are subsidised in the pooling contracts, they would never choose to separate. Therefore the low risk consumer makes the separation decision. As such, only the expected utility earned by the low risk consumer is examined.

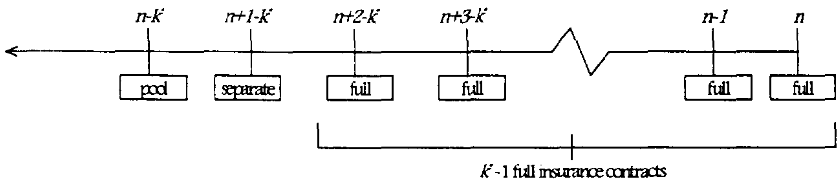

Figure 2 - Stream of Contracts if Separation Occurs in Period k*

Before separation occurs, in each period all consumers purchase a one period pooling contract. After separation has occurred, consumers receive full insurance contracts priced for their risk type. Figure 3 shows the possible paths for consumers in a three-period world. The number of paths grows exponentially as the number of periods increases. In an 'n' period model, there are $3*2^n-2$ possible outcomes.

Because consumers possess negative exponential utility functions, low risk consumers can *a priori* select the period in which they wish to separate. Since insurance companies know the preferences of insureds, they too know *a priori* in which period separation will occur.

Therefore, even if there are $3 * 2^n - 2$ possible outcomes, in equilibrium most of these paths will not be observed. If insureds separate k^* periods before the last period, 2^{n-k^*+1} outcomes are possible.

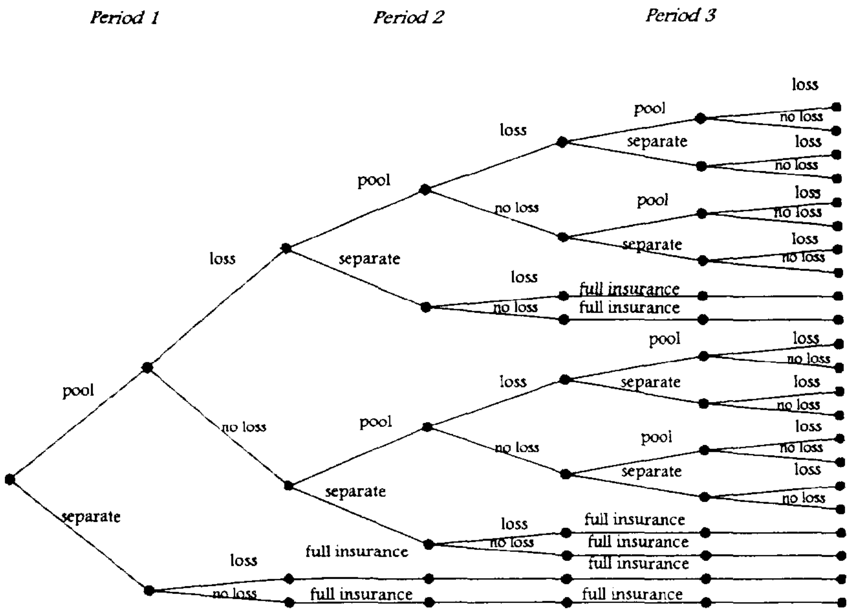

Figure 3 - Possible Outcomes in a 3 Period Model

If consumers have not yet revealed their risk type in a previous period, insurers may offer one of two contracts. The first contract is the pooling contract. The second type of contract is a menu of contracts; one policy is a full insurance contract priced at the expected cost of insuring a high risk consumer and the second is a partial insurance contract priced at the expected cost of insuring a low risk consumer. As in the one period Rothschild-Stiglitz model, the level of indemnity chosen is such that the high risk consumers receive no future benefit from misrepresentation. The two possible contracts are defined in Proposition 1.

Proposition 1: *The two contracts that insurers could offer consumers before type is revealed are a pooling contract and a separating menu of contracts. A pooling contract that satisfies Wilson's anticipatory equilibrium criteria is given by a level of coverage* I^0 *and a price*

$p^0 = \rho^0 I^0 + e$, where I^0 is given by $I^0 \equiv d - \frac{1}{\alpha}\log\left[\frac{\rho^0(1-\rho^\ell)}{\rho^\ell(1-\rho^0)}\right]$, and where ρ^0, the pooled probability of loss, is defined by $\rho^0 \equiv \lambda\rho^h + (1-\lambda)\rho^\ell$. A separating menu of contracts for period k, following from Rothschild and Stiglitz, is given by two contracts. The first is a full insurance contract priced at $\rho^h d + e$. The second contract is for a level of coverage I_k at a price $\rho^\ell I_k + e$, where I_k is the solution to $\rho^h e^{-\alpha((1-\rho^\ell)I_k - d)} + (1-\rho^h)e^{\alpha\rho^\ell I_k} + e^{\alpha(k-1)\rho^\ell d} = e^{\alpha k\rho^h d}$.

Proof: From Wilson, the pooling contract offered each period to both types of consumers is one that maximises the utility of the low risk consumer subject to a single period zero profit constraint on insurers. Solving the maximisation problem for the level of the indemnity yields I^0 as defined in the proposition.

As in the one period model, the separating menu of contracts consists of two policies: the first contract, which is chosen by the high risk consumer, is a full insurance contract priced at the expected cost of insuring a high risk consumer. The second policy is a partial insurance contract priced at the expected cost of insuring a low risk consumer. The level of indemnity is chosen so that there is no incentive for the high risk consumers to misrepresent their type. If high risk consumers truthfully reveal their type, they receive full insurance coverage for k periods. The incremental utility[3] earned over the last k periods is

$$-e^{\alpha k(\rho^h d + e)}. \tag{1}$$

Alternatively, high risk consumers could dissemble their risk type. They would purchase the partial insurance contract designed for the low risk consumers in period $n+1-k$, and for the remaining $k-1$ periods receive full insurance priced at the cost of insuring a low risk consumer. In this case, the incremental expected utility is

$$-\rho^h e^{-\alpha((1-\rho^\ell)I_k - d - e)} - (1-\rho^h)e^{\alpha(\rho^\ell I_k + e)} - e^{\alpha(k-1)(\rho^\ell d + e)}. \tag{2}$$

The high risk consumer will be indifferent between the two contracts when the two expected utilities are equal. Equating functions (1)and (2) gives

$$\rho^h e^{-\alpha((1-\rho^\ell)I_k - d)} + (1-\rho^h)e^{\alpha\rho^\ell I_k} + e^{\alpha(k-1)\rho^\ell d} = e^{\alpha k\rho^h d}, \tag{3}$$

which can be solved for I_k to yield the equilibrium level of insurance. ■

The amount of partial insurance offered in the separating menu of contracts is a function of the period in which separation occurs, the size of the potential loss and the risk propensities of the two types of insureds.

The amount of partial insurance offered, I_k, if separation occurs k periods from the end, increases as the probability of loss faced by the low risk consumers, ρ^ℓ, increases and decreases as the probability of loss faced by the high risk consumers, ρ^h, increases. This occurs because as the two types become more dissimilar (as the distance between ρ^h and ρ^ℓ increases), the benefits to the high risk consumer from misrepresentation increases, and as such, the partial indemnity offered in the period of separation must decrease.

From equation (3), which characterises I_k, define the implicit function

$$G(\alpha,\rho^\ell,\rho^h,d,I_k,k) = \rho^h e^{-\alpha((1-\rho^\ell)I_k - d)} + (1-\rho^h)e^{\alpha\rho^\ell I_k} + e^{\alpha(k-1)\rho^\ell d} - e^{\alpha k\rho^h d}. \tag{4}$$

The relationship between I_k and ρ^ℓ can be derived by straightforward differentiation of $G(\alpha,\rho^h,\rho^\ell,d,I_k,k)$. Specifically

$$\frac{\partial I_k}{\partial \rho^\ell} = \frac{I_k e^{\alpha k\rho^\ell I_k}\left(\rho^h e^{\alpha(d-I_k)} + 1 - \rho^h\right) + (k-1)d\; e^{\alpha(k-1)\rho^\ell d}}{e^{\alpha k\rho^\ell I_k}\left[\rho^h(1-\rho^\ell)e^{\alpha(d-I_k)} - \rho^\ell(1-\rho^h)\right]} > 0.$$

The relationship between I_k and ρ^h is easiest to show through numerical computation.

The amount of partial insurance offered is a decreasing function of k, the number of periods remaining in the world. This implies that the earlier low risk consumers decide to separate, the lower the amount of insurance that is offered in the partial insurance contract in that period. Since the benefit from misrepresentation to the high risks increases as the number of periods until the end increases, the partial indemnity offered in the contract designed for low risk consumers in

---

[3] This function ignores the expected utility gained from the initial wealth and the first $n-k$ periods of pooling

the period of separation must decrease to discourage misrepresentation. Straightforward differentiation of (4) with respect to I_k and k yields

$$\frac{\partial I_k}{\partial k}=\frac{-d\left[\rho^h e^{\alpha k\rho^h d}-\rho^\ell e^{\alpha(k-1)\rho^\ell d}\right]}{e^{\alpha k\rho^\ell I_k}\left[\rho^h\left(1-\rho^\ell\right)e^{\alpha(d-I_k)}-\rho^\ell\left(1-\rho^h\right)\right]}<0. \tag{5}$$

The amount of partial insurance is an increasing function of the size of potential loss. This can be shown numerically. As is observable from (3), the amount of partial insurance is independent of the number of periods for which the pooling contracts were purchased; all that matters is the number of periods until the end.

The relationship between I_k and α is much more complex. For small values of d, I_k first decreases and then increases in α. For larger and more realistic values of d, the relationship is monotonic, as is shown in Figure 4. The curves plotted in Figure 4 display the optimal amount of indemnity over a range of risk aversion coefficients for differing values of ρ^h, ρ^ℓ and d. The period of separation is arbitrarily selected to be 4; the graphs depict the relationship between I_4 and α. The range of α corresponds to the range suggested by Haubrich (1994) and the loss probabilities represent typical loss frequencies for personal insurance. The first value of d utilised, \$2036, is the average size of private passenger automobile property damage claim paid in the United States for 1996. The second value of d, \$11 161, is the average size of private passenger automobile bodily injury claim paid in the United States for 1996.[4]

Now that the possible contracts that could be offered by perfectly competitive insurers have been defined, the expected utility accruing to the low risk consumer and her subsequent maximisation problem can be developed. The expected utility earned by a low risk consumer who pools for the entire n periods is given by

[4] Both figures are taken from *The Fact Book 1997: Property/Casualty Insurance Facts* published by the Insurance Information Institute. The figure for property damage excludes claims histories from Massachusetts, Michigan, New Jersey and South Carolina. The figure for bodily injury excludes Massachusetts and all states with no-fault insurance.

$$V_0 = -e^{-a\left(W - n\rho^0 I^0 - ne\right)} \sum_{j=0}^{n} \binom{n}{j} \left(\rho^t\right)^j \left(1-\rho^t\right)^{n-j} e^{aj\left(d-I^0\right)}$$
$$= -e^{-a\left(W - n\rho^0 I^0 - ne\right)}\left(1-\rho^t+\rho^t e^{a\left(d-I^0\right)}\right)^n. \tag{6}$$

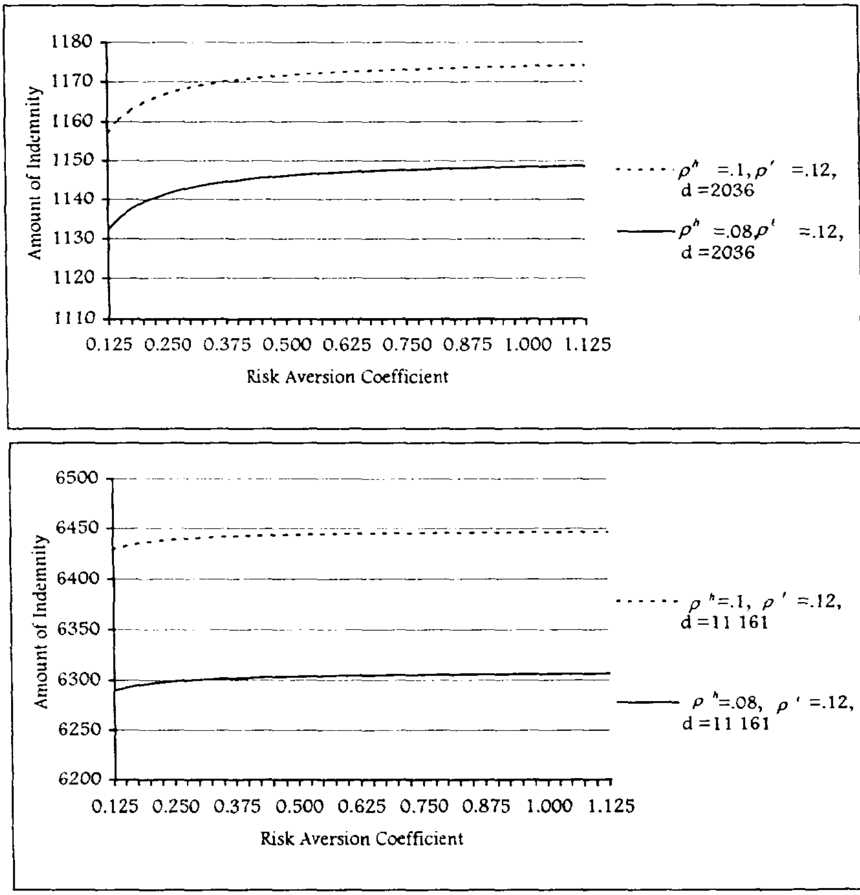

Figure 4 Relationships between the Indemnity Amount and the Risk Aversion Coefficient, a

The consumer is endowed with an initial wealth of W, and in each of the n periods pays $\rho^0 I^0 + e$ to purchase the pooling contracts. The summation represents the expected utility earned from the random losses over the n periods.

Alternatively, the low risk consumer could decide to separate k periods, where $k \geq 1$, before the last period. In this case, the utility earned by the low risk consumer is

$$V(k) = -e^{-\alpha(W-ne)} e^{\alpha(k-1)\rho^{l} d} \left[\rho^{l} e^{-\alpha((1-\rho^{l})l_k - d)} + \left(1-\rho^{l}\right) e^{\alpha\rho^{l} l_k}\right] \left\{ e^{\alpha(n-k)\rho^{o} l^{o}} \sum_{j=0}^{n-k} \binom{n-k}{j} \left(\rho^{l}\right)^{j} \left(1-\rho^{l}\right)^{n-k-j} e^{\alpha j(d-l^{o})} \right\}$$

$$= -e^{-\alpha(W-ne)} e^{\alpha(k-1)\rho^{l} d} e^{\alpha\rho^{l} l_k} \left[1-\rho^{l} + \rho^{l} e^{\alpha(d-l_k)}\right] \left\{ e^{\alpha(n-k)\rho^{o} l^{o}} \left[1-\rho^{l} + \rho^{l} e^{\alpha(d-l_o)}\right]^{n-k} \right\} . \tag{7}$$

In the last line of the function, the first term represents the utility from the original endowment less the expense loading that must be paid every period. The second term is the contribution from the $k-1$ periods of full insurance. Because full insurance has been purchased for these periods, it is irrelevant from a utility standpoint whether or not losses have occurred. The next two terms, $e^{\alpha\rho^{l} l_k}\left[1-\rho^{l} + \rho^{l} e^{\alpha(d-l_k)}\right]$, express the addition to expected utility arising from the period in which separation occurs and the term in the curly brackets arises from the $n-k$ periods in which the consumer purchased pooling contracts.

The maximisation problem of the low risk consumer is straightforward and is given in Theorem 1.

Theorem 1*: To maximise expected utility, low risk consumers choose the optimal period of separation, k^*, where k^* solves $k^* = \arg\max_k (V_0, V(k))$. An approximation to k^* is given by $\bar{k}^*$, where $\bar{k}^*$ is the solution to*

$$\left[Q(\bar{k}^*)\right]^{-1}\left(1-\rho^{h} + \rho^{h} Q(\bar{k}^*)\right) = \rho^{h} e^{\alpha(\bar{k}^*\rho^{h} - \rho^{l})d} - \rho^{l} e^{\alpha(\bar{k}^* - 2)\rho^{l} d},$$

and $Q(\bar{k}^*)$ *is given by*

$$\frac{\left(a\left((\rho^{l})^{2}(1-\rho^{h}) - \rho^{h}(1-\rho^{l})^{2}\right) + \alpha(1-\rho^{l})(1-2\rho^{h})\rho^{l} b(\bar{k}^*)\right)}{2(1-\rho^{l})\rho^{l}\rho^{h}\left(a - \alpha b(\bar{k}^*)\right)}$$

$$- \frac{\sqrt{\left\{a\left((\rho^{l})^{2}(1-\rho^{h}) - \rho^{h}(1-\rho^{l})^{2}\right) + \alpha(1-\rho^{l})(1-2\rho^{h})\rho^{l} b(\bar{k}^*)\right\}^{2} + 4(1-\rho^{l})^{2}(\rho^{l})^{2}(1-\rho^{h})\rho^{h}\left(a - \alpha b(\bar{k}^*)\right)^{2}}}{2(1-\rho^{l})\rho^{l}\rho^{h}\left(a - \alpha b(\bar{k}^*)\right)}$$

and where a and $b(\bar{k}^*)$ *are defined as follows:*

$a \equiv \alpha\left(-\rho^l d + \rho^0 I^0\right) + \log\left[1 - \rho^l + \rho^l e^{\alpha(d - I^0)}\right]$ *and* $b\left(\bar{k}^*\right) = \dfrac{d\left[\rho^h e^{\alpha \bar{k}^* \rho^h d} - \rho^l e^{\alpha\left(\bar{k}^* - 1\right)\rho^l d}\right]}{e^{\alpha \bar{k}^* \rho^h d} - e^{\alpha\left(\bar{k}^* - 1\right)\rho^l d}}$. *To ascertain the true value of k^*, the expected utility calculated at the two integral values of k on either side of $\bar{k}^*$ must be compared.*

Proof: Since k is integral, function (7) is not differentiable with respect to k, but it is possible to define the function

$$\bar{V}(k) = -e^{-\alpha(W - rc)} e^{-\alpha(k-1)\rho^l d} e^{\alpha\rho^l I_k}\left[\rho^l e^{\alpha(d - I_k)} + 1 - \rho^l\right]\left\{e^{\alpha(n-k)\rho^0 I^0}\left[\rho^l e^{\alpha(d - I_0)} + 1 - \rho^l\right]^{n-k}\right\},$$

for $k > 0$, which is continuously differentiable in k, identically equal to $V(k)$ for integral values of k and which possesses a unique maximum (see Appendix for details). Thus the value of k which maximises $\bar{V}(k)$ will also maximise $V(k)$. Since $V(k)$ evaluated at zero does not equal V_0, if $\bar{k}^*$ is less than one, $V(k = 1)$ must be compared with V_0 to ascertain the true value of k^*.

Maximising $\bar{V}(k)$ with respect to k, collecting terms, setting the derivative equal to zero, and dividing through by non-zero terms yields the equation

$$\begin{aligned}
\frac{\partial \bar{V}(k)}{\partial k} &= \left(1 - \rho^l\right)\left[\alpha\left(-\rho^l d + \rho^0 I^0\right) + \log\left(1 - \rho^l + \rho^l e^{\alpha(d - I^0)}\right)\right] \\
&\quad - \alpha\rho^l\left(1 - \rho^l\right)\left[1 - e^{\alpha(d - I_k^*)}\right]\left.\frac{\partial I_k}{\partial k}\right|_{I_k = I_k^*, k = \bar{k}^*} \\
&\quad + \rho^l\left[\alpha\left(-\rho^l d + \rho^0 I^0\right) + \log\left(1 - \rho^l + \rho^l e^{\alpha(d - I^0)}\right)\right] e^{\alpha(d - I_k^*)} \\
&\equiv 0 \quad .
\end{aligned} \tag{8}$$

For notational convenience, define $a = \left[\alpha\left(-\rho^l d + \rho^0 I^0\right) + \log\left(1 - \rho^l + \rho^l e^{\alpha(d - I^0)}\right)\right]$, which is constant with respect to k. From (5)

$$\left.\frac{\partial I_k}{\partial k}\right|_{I_k = I_k^*, k = \bar{k}} = \frac{d\left[\rho^h e^{\alpha \bar{k}^* \rho^h d} - \rho^l e^{\alpha\left(\bar{k}^* - 1\right)\rho^l d}\right]}{e^{\alpha k \rho^l I_k^*}\left[\rho^l\left(1 - \rho^h\right) - \rho^h\left(1 - \rho^l\right)e^{\alpha(d - I_k^*)}\right]}.$$

Substituting for $e^{\alpha\rho^l I_k}$ from (3) into $\frac{\partial I_k}{\partial k}$ yields

$$\left.\frac{\partial I_k}{\partial k}\right|_{I_k=I_k^*,k=\bar{k}^*} = b\left(\bar{k}^*\right)\frac{\left(1-\rho^h\right)-\rho^h e^{\alpha\left(d-I_k^*\right)}}{\rho^\ell\left(1-\rho^h\right)-\rho^h\left(1-\rho^\ell\right)e^{\alpha\left(d-I_k^*\right)}},$$

where

$$b\left(\bar{k}^*\right)=\frac{d\left[\rho^h e^{\alpha\bar{k}^*\rho^h d}-\rho^\ell e^{\alpha\left(\bar{k}^*-1\right)\rho^\ell d}\right]}{e^{\alpha\bar{k}^*\rho^h d}-e^{\alpha\left(\bar{k}^*-1\right)\rho^\ell d}}.$$

Substituting back into (8) yields

$$\left(1-\rho^\ell\right)\rho^\ell\left(1-\rho^h\right)\left(a-\alpha b\left(\bar{k}^*\right)\right)+e^{\alpha\left(d-I_k^*\right)}\left(a\left(\left(\rho^\ell\right)^2\left(1-\rho^h\right)-\rho^h\left(1-\rho^\ell\right)^2\right)+\alpha\left(1-\rho^\ell\right)\left(1-2\rho^h\right)\rho^\ell b\left(\bar{k}^*\right)\right)$$
$$-e^{2\alpha\left(d-I_k^*\right)}\left(1-\rho^\ell\right)\rho^\ell\rho^h\left(a-\alpha b\left(\bar{k}^*\right)\right)=0$$

which is a quadratic equation in $e^{\alpha\left(d-I_k^*\right)}$. Using the quadratic formula and ignoring the extraneous root yields

$$e^{\alpha\left(d-I_k^*\right)}=\frac{\left(a\left(\left(\rho^\ell\right)^2\left(1-\rho^h\right)-\rho^h\left(1-\rho^\ell\right)^2\right)+\alpha\left(1-\rho^\ell\right)\left(1-2\rho^h\right)\rho^\ell b\left(\bar{k}^*\right)\right)}{2\left(1-\rho^\ell\right)\rho^\ell\rho^h\left(a-\alpha b\left(\bar{k}^*\right)\right)}$$
$$+\frac{\sqrt{\left\{a\left(\left(\rho^\ell\right)^2\left(1-\rho^h\right)-\rho^h\left(1-\rho^\ell\right)^2\right)+\alpha\left(1-\rho^\ell\right)\left(1-2\rho^h\right)\rho^\ell b\left(\bar{k}^*\right)\right\}^2+4\left(1-\rho^\ell\right)^2\left(\rho^\ell\right)^2\left(1-\rho^h\right)\rho^h\left(a-\alpha b\left(\bar{k}^*\right)\right)^2}}{2\left(1-\rho^\ell\right)\rho^\ell\rho^h\left(a-\alpha b\left(\bar{k}^*\right)\right)}$$
$$\equiv Q\left(\bar{k}^*\right)$$

which is an equation in both k^* and I_k^*. Substituting for I_k^* in (8) gives

$$\left[Q\left(\bar{k}^*\right)\right]^{-\ell}\left(1-\rho^h+\rho^h Q\left(\bar{k}^*\right)\right)=\rho^h e^{\alpha\left(\bar{k}^*\rho^h-\rho^\ell\right)d}-\rho^\ell e^{\alpha\left(\bar{k}^*-2\right)\rho^\ell d}, \tag{9}$$

which then can be solved numerically for $\bar{k}^*$. Once $\bar{k}^*$ has been determined, the monotonic properties of $\bar{V}(k)$ imply that the optimal value k^* is one of the two integer values closest to $\bar{k}^*$. ■

It is interesting to note that equation (9) is independent of n, the number of periods in the model. The separation decision depends on the number of periods remaining in the model and not on the number of periods for which insurance has already been purchased.

In this section, it is assumed that the cost to the insurer of writing a policy does not change after the first period. In reality, a major portion of the expense associated with a policy accrues from the initial underwriting. Therefore contracting insurers may wish to lower the expense loadings charged in subsequent periods. In a perfectly competitive market, barring any collusive behaviour, if future expense loadings can be reduced, the premiums charged by firms will fully reflect this savings.

Any potential expense differentials between the first and subsequent periods will not affect the behaviour of low risk consumers who have no incentive to switch insurers in any period. In this model, high risk consumers are currently indifferent to switching after type has been revealed. Before consumer type is revealed, utility maximising high risk consumers would never switch because they are subsidised by the low risk consumers. If contracting insurers lower their expense loading in subsequent periods, then the high consumers also have no incentive to switch insurers, even after type is revealed.

2. Numerical Results

Based on the characteristics of the separating and pooling contracts, one hypothesis as to why the Rothschild-Stiglitz separating menu of contracts is not observed in reality is that the cost of separation is too high. For various values of α, ρ^h, ρ^l, and λ, Table 1 gives both k^*, the optimal period of separation and $\bar{k}^*$, the approximation to the optimal period of separation, where $\bar{k}^*$ has been defined in Theorem 1. The optimal amount of insurance offered in the partial insurance contract, I_k, if separation occurs in period k and the corresponding utility earned by low risk consumers if they choose to separate in period k are also given.

The range of α chosen corresponds with values suggested by Haubrich. The probabilities, ρ^h and ρ^l, examined are similar to loss probabilities observed in personal insurance. Two values of the proportion of high risk consumers, $\lambda = 0.25$ and $\lambda = 0.75$ were considered. The proportion of high and low risk consumers in the population does not affect the amount of insurance offered in the separating menu of contracts, but does affect the total expected utility

α	0.25		0.60		0.90		1.25	
ρ^h	0.12		0.12		0.12		0.12	
ρ^l	0.10		0.10		0.10		0.10	
λ	0.25		0.25		0.25		0.25	
$\bar{k}^*$	0.91		0.86		0.85		0.85	
k^*	0		0		0		0	
k	I_k	$V(k)$	I_k	$V(k)$	I_k	$V(k)$	I_k	$V(k)$
0		-1.8E-06		-1.3E-07		-1.2E-09		-5.3E-12
1	968.37	-6.4E-05	973.85	-8.9E-04	975.16	-7.2E-04	975.89	-6.2E-04
2	835.02	-1.9E+08	840.52	-8.2E+26	841.83	-6.4E+41	842.56	-1.7E+59
3	701.69	-6.0E+20	707.18	-7.6E+56	708.49	-5.7E+86	709.23	-4.5E+121
4	568.35	-1.8E+33	573.85	-7.1E+86	575.16	-5.1E+131	575.89	-1.2E+184
5	435.02	-5.6E+45	440.52	-6.8E+08	441.83	-1.1E+179	442.56	$-\infty$
6	301.69	-1.7E+58	307.18	-6.0E+146	308.49	-4.0E+221	309.23	$-\infty$
7	168.35	-5.2E+70	173.85	-5.6E+176	175.16	$-\infty$	175.89	$-\infty$
8	35.02	-1.6E+83	40.52	-5.2E+206	41.83	$-\infty$	42.56	$-\infty$

α	0.25		0.60		0.90		1.25	
ρ^h	0.12		0.12		0.12		0.12	
ρ^l	0.08		0.08		0.08		0.08	
λ	0.25		0.25		0.25		0.25	
$\bar{k}^*$	0.74		0.70		0.69		0.68	
k^*	0		0		0		0	
k	I_k	$V(k)$	I_k	$V(k)$	I_k	$V(k)$	I_k	$V(k)$
0		-5.1E-07		-8.3E-16		-9.1E-13		-7.4E-20
1	947.30	-6.2E-04	952.68	-3.6E-08	953.96	-3.2E-01	954.68	-9.6E-04
2	816.87	-5.4E+08	822.25	-1.7E+21	823.53	-3.2E+42	824.24	-5.0E+56
3	686.43	-4.8E+20	691.81	-7.7E+49	693.09	-3.2E+85	693.81	-2.6E+116
4	556.00	-4.2E+32	561.38	-3.6E+78	562.66	-3.1E+128	563.37	-1.3E+176
5	425.56	-4.2E+43	430.94	-1.6E+107	432.22	-3.1E+171	432.94	$-\infty$
6	295.13	-3.2E+56	300.51	-7.5E+135	301.79	-3.1E+214	302.50	$-\infty$
7	164.69	-2.8E+68	170.07	-3.5E+164	171.35	$-\infty$	172.07	$-\infty$
8	34.26	-2.5E+80	39.64	-1.6E+193	40.92	$-\infty$	41.63	$-\infty$

In all scenarios $d = 1000$.

Table 1 - Separating Contract Indemnity Offered and Utility Earned for Various Parameter Values

α	0.25		0.60		0.90		1.25	
ρ^h	0.12		0.12		0.12		0.12	
ρ^l	0.10		0.10		0.10		0.10	
λ	0.75		0.75		0.75		0.75	
$\bar{k}^*$	0.91		0.86		0.85		0.85	
k^*	0		0		0		0	
k	I_k	$V(k)$	I_k	$V(k)$	I_k	$V(k)$	I_k	$V(k)$
0		-3.1E-04		-3.6E-05		-2.4E-14		-4.3E-19
1	968.37	-9.2E-04	973.85	-6.1E-04	975.16	-1.8E-12	975.89	-1.9E-16
2	835.02	-2.3E+08	840.52	-1.4E+24	841.83	-2.0E+29	842.56	-1.9E+41
3	701.69	-5.8E+19	707.18	-3.2E+51	708.49	-2.2E+70	709.23	-1.9E+98
4	568.35	-1.5E+31	573.85	-7.4E+78	575.16	-2.4E+111	575.89	-1.9E+155
5	435.02	-3.7E+42	440.52	-1.8E-02	441.83	-6.6E+154	442.56	$-\infty$
6	301.69	-9.2E+53	307.18	-3.9E+133	308.49	-2.9E+193	309.23	$-\infty$
7	168.35	-2.3E+65	173.85	-8.9E+160	175.16	$-\infty$	175.89	$-\infty$
8	35.02	-5.9E+76	40.52	-2.0E+188	41.83	$-\infty$	42.56	$-\infty$

α	0.25		0.60		0.90		1.25	
ρ^h	0.12		0.12		0.12		0.12	
ρ^l	0.08		0.08		0.08		0.08	
λ	0.75		0.75		0.75		0.75	
$\bar{k}^*$	0.74		0.70		0.69		0.68	
k^*	0		0		0		0	
k	I_k	$V(k)$	I_k	$V(k)$	I_k	$V(k)$	I_k	$V(k)$
0		-1.4E-06		-1.9E-12		-1.8E-07		-1.0E-15
1	947.30	-1.1E-05	952.68	-5.0E-10	953.96	-9.6E-04	954.68	-1.8E-10
2	816.87	-6.7E+04	822.25	-1.4E+14	823.53	-1.5E+32	824.24	-1.3E+39
3	686.43	-4.0E+14	691.81	-4.1E+37	693.09	-2.2E+67	693.81	-9.7E+87
4	556.00	-2.3E+24	561.38	-1.2E+61	562.66	-3.3E+102	563.37	-7.0E+136
5	425.56	-1.6E+33	430.94	-3.3E+84	432.22	-5.1E+137	432.94	$-\infty$
6	295.13	-8.3E+43	300.51	-9.4E+107	301.79	-7.7E+172	302.50	$-\infty$
7	164.69	-4.9E+53	170.07	-2.7E+131	171.35	$-\infty$	172.07	$-\infty$
8	34.26	-2.9E+63	39.64	-7.6E+154	40.92	$-\infty$	41.63	$-\infty$

In all scenarios $d = 1000$.

Table 1 (con't) - Separating Contract Indemnity Offered and Utility Earned for Various Parameter Values earned by the low risk consumers. A relatively small value of total loss $d = 1000$, was used in place of the previously examined values of $d = 2036$ and $d = 11\,161$. This was done to mitigate computational problems without sacrificing the flavour of the indemnity amounts.

The results are indeed surprising. First of all, in all sixteen scenarios examined, if the low risk consumers wish to separate more than eight periods before the end, there is no positive amount of indemnity that can be offered by insurers that will prevent the high risk consumers from misrepresenting their risk types. The benefit of misrepresentation is so great that there exists no level of indemnity that will prevent this from occurring. Thus, for a world in which these parameter values are realistic, perfectly competitive insurers would never offer a separating menu of contracts before the last eight periods, regardless of the number of periods in the model.

Calculated for each scenario is the optimal value of $\bar{k}^*$, as defined in Theorem 1. For all sixteen scenarios, this value is less than one, implying that low risk consumers would either pool for all periods or separate in the very last period. An examination of the expected utility earned by low risk consumers by period of separation confirms the analytic result of Theorem 1. For all sixteen scenarios, low risk consumers maximise their expected utility by pooling for the entire time frame. The expected utility earned by low risk consumers decreases rapidly as the number of periods of full insurance increases. Note that comparisons cannot be made between the scenarios, but only within each scenario. For each scenario, the total wealth of the insured less the total expenses paid, $W - ne$, is chosen so that the expected utility earned by low risk consumers when $k = 1$ is standardised to zero.

3. Anecdotal Evidence

This section provides anecdotal evidence against the traditional theoretical result that low risk consumers purchase less insurance than high risk consumers do. Results from Table 2 suggest that low risk consumers purchase more insurance than high risk consumers do. Data in this table are the average amount of coverage purchased from 1987 to 1990 by at-fault accident history for private passenger third party liability coverage from the Insurance Corporation of British Columbia (ICBC). ICBC is the only insurer in British Columbia that provides this mandatory coverage. For example, for drivers with claims rated scale (CRS) of 27, an average of 69.17% of these drivers over the three years surveyed purchased one million dollars in liability insurance. The figures in brackets below each average are the standard deviations across the three years of data.

CRS	Amount of Liability Purchased					
	$200 000	$300 000	$500 000	$1 000 000	$2 000 000	>$2 000 000
<22	6.38% (0.59%)	0.69% (0.18%)	10.81% (2.28%)	69.68% (0.97%)	11.10% (3.66%)	1.35% (0.27%)
22	8.54% (0.99%)	0.56% (0.14%)	9.69% (2.16%)	70.75% (0.60%)	9.44% (3.38%)	1.01% (0.16%)
23	9.01% (1.00%)	0.56% (0.14%)	9.87% (2.32%)	70.02% (0.67%)	9.47% (3.53%)	1.07% (0.23%)
24	9.48% (1.00%)	0.60% (0.17%)	9.68% (2.33%)	69.08% (0.76%)	9.95% (3.77%)	1.20% (0.26%)
25	17.36% (2.16%)	0.47% (0.12%)	7.94% (1.62%)	64.28% (0.65%)	8.87% (3.61%)	1.10% (0.16%)
26	10.46% (1.02%)	0.59% (0.11%)	9.98% (2.41%)	69.75% (0.79%)	8.05% (2.76%)	1.18% (0.16%)
27	10.86% (1.09%)	0.60% (0.20%)	9.74% (2.01%)	69.17% (0.66%)	8.42% (3.12%)	1.22% (0.15%)
28	13.17% (1.51%)	0.68% (0.28%)	8.72% (2.12%)	68.04% (0.57%)	8.33% (3.25%)	1.05% (0.18%)
29	13.17% (1.73%)	0.85% (0.50%)	9.53% (2.06%)	68.42% (1.46%)	7.11% (2.72%)	0.93% (0.29%)
30	14.18% (1.65%)	0.69% (0.28%)	9.53% (2.76%)	66.13% (1.93%)	8.34% (2.94%)	1.14% (0.21%)
31	15.49% (1.34%)	1.00% (0.30%)	10.04% (2.24%)	64.21% (2.12%)	7.61% (2.52%)	1.64% (0.61%)
32	17.65% (1.99%)	0.63% (0.43%)	12.03% (5.41%)	62.01% (2.08%)	5.85% (2.60%)	1.83% (0.72%)
33	20.15% (2.04%)	0.95% (0.68%)	9.12% (3.00%)	59.95% (5.46%)	7.10% (2.72%)	2.72% (1.69%)
34	17.72% (4.04%)	0.76% (1.11%)	10.29% (3.28%)	61.41% (1.80%)	7.36% (2.99%)	2.46% (1.44%)
35	23.41% (6.78%)	0.00% (0.00%)	13.93% (7.60%)	56.57% (5.53%)	5.29% (3.36%)	0.80% (1.09%)
>35	21.29% (5.35%)	-0.06% (0.00%)	7.69% (0.98%)	61.63% (2.57%)	7.24% (3.09%)	2.21% (1.31%)

Bracketed numbers represent the standard deviation of each cell

Table 2 - Average Earned Exposure Frequencies by Claim Rated Scale (CRS) and Amount of Liability Insurance Purchased for 1987 - 1990

Movement within the claims rated scale (CRS) is as follows. In the scale, 25 represents the base level (zero years of no at-fault claims reported). For each year of no reported at-fault claims, the insured moves down one level. Until 1988, 21 was the lowest level recorded. ICBC now records up to ten years of accident free history. If an insured submits an at-fault claim, she moves up three levels. Due to scarcity of data in the table, all classes above 35 have been combined. Lower CRS values represent drivers with very few accidents, and higher numbers, drivers with higher observed accident frequencies. $200 000 is the minimum amount of third party coverage required by law. Insurance brokers typically recommend that consumers buy at least $1 000 000 in coverage. Even without any formal statistical analysis, it is quite evident that the amount of insurance purchased varies across driving record. As already noted, these results are opposite to what is predicted in the literature. Results in the literature predict that low risk consumers purchase less insurance than higher risk consumers, whereas data in this table suggest the opposite. Theoretical models have assumed that consumers are identical except for probability of loss. Violations of this assumption are the most plausible reasons as to why empirical findings contradict theoretical results. The two most likely explanations are that both wealth and risk aversion vary with the probability of loss.

Drivers with lower CRS are possibly more risk averse. Not only does this affect their driving - making them more cautious and thus reducing the probability of loss - but they also carry more insurance. Wealth almost certainly varies across the insureds. As anecdotal evidence, ICBC offers a monthly payment plan for private passenger automobile insurance. One of the best indicators that a consumer is high risk is that she has defaulted on at least one monthly payment. Therefore it is plausible that high risk insureds have less wealth than low risk consumers do. Perhaps the high risk drivers would like to purchase more insurance but cannot do so because of the expense.

And finally, accident history is just a signal of a consumer's true risk propensity. So it is possible that many drivers with poor driving records are actually low risk drivers and that many drivers with no reported at-fault claims are, in fact, high risk drivers. The latter seems more plausible. The average claim frequency is estimated to be between 10% and 15% and so there is a 52% to 66% probability that an insured will have four consecutive accident free years. Even if an insured had a 20% probability of a claim, there is still a greater than 40% probability that she will have no claims in four consecutive years. The results from the literature may indeed hold, and conflicting evidence exists because accident history is an imperfect signal of an insured's risk propensity.

4. Conclusions

Consumers tend to purchase property/casualty insurance contracts repeatedly in their lifetimes. In most personal and commercial lines of insurance, contracts are renewed annually and contracts written by insurers are often contingent on past accident history. This paper introduces a plausible multiple period extension of Rothschild and Stiglitz' one period model of the insurance industry in which future policies must be conditioned on past contract choice. Specifically future policies are contingent upon the amount of insurance purchased in previous contracts. However, in personal lines of property-casualty insurance, policies are not typically contingent on past policy choices. The purpose of this paper is to examine why this is not the case.

Multiple period Rothschild-Stiglitz contracts were constructed and compared with a pooling contract satisfying Wilson's anticipatory equilibrium. The separation decision of low risk consumers in a multiple period world in which both Rothschild-Stiglitz contracts and pooling contracts are offered was examined. It was shown that the decision to separate is a function of the number of periods remaining in the model, the loss probabilities of the different consumer types, the mix of consumer types in the economy and the size of potential loss. The decision to separate was shown not to be a function of the total number of periods in the model.

Numerical examples were provided to assist understanding of the theoretical results. It was shown that the costs of separation are so high that low risk consumers are better off pooling with high risk consumers. One reason why observed insurance contracts do not reveal a consumer's risk type is that the cost of separation for low risks is too high. It was shown that over a range of parameter values, there are no feasible Rothschild-Stiglitz separating menu of contracts that can be offered by insurers. Even when both pooling and separating contracts are feasible, it was also shown that utility-maximising low risk consumers would never wish to reveal their type.

References

Borch, K.H. (1990). *Economics of Insurance*, North-Holland, Amsterdam.

Cooper, R. and B. Hayes (1987). Multi-Period Insurance Contracts. *International Journal of Industrial Organization* 5, 211-231.

Dionne, G. (1983). Adverse Selection and Repeated Insurance Contracts. *The Geneva Papers on Risk and Insurance* 8, 316-322.

Dionne, G. and N.A. Doherty (1994). Adverse Selection, Commitment and Renegotiation: Extension to and Evidence from Insurance Markets. *Journal of Political Economy* 102, 209-235.

Eisenhauer, J. (1993). Asymmetries and Household Insurance: A Note. *Insurance: Mathematics and Economics* 12, 57-60.

The Fact Book 1997: Property/Casualty Insurance Facts. Insurance Information Institute, New York, NY, (published annually).

Haubrich J.G. (1994). Risk Aversion, Performance Pay, and the Principal-Agent Problem. *Journal of Political Economy* 102, 258-276.

Kunreuther, H. and M. Pauly (1985). Market Equilibrium with Private Knowledge: An Insurance Example. *Journal of Public Economics* 26, 269-288.

Mossin, J. (1968). Aspects of Rational Insurance Purchasing. *Journal of Political Economy* 76, 553-568.

Rothschild, M. and J. Stiglitz (1976). Equilibrium in Competitive Insurance Markets: An Essay in Economics of Imperfect Information. *Quarterly Journal of Economics* 90, 629-650.

Szpiro, G.G. (1985). Optimal Insurance Coverage. *Journal of Risk and Insurance* 52, 704-710.

Wade, R.C. (1973). Expense Analysis in Ratemaking and Pricing. *Proceedings of the Casualty Actuarial Society* 60, 1-10.

Wilson, C. (1977). A Model of Insurance Markets with Incomplete Information. *Journal of Economic Theory* 16, 167-207

Appendix

Proposition 2: *The function $\bar{V}(k)$, as defined in Theorem 1, possesses a unique maximum with respect to the variable k.*

Proof: From Theorem 1, $\bar{V}(k)$ is defined by

$$\bar{V}(k) = -e^{-\alpha(W-ne)} e^{-\alpha(k-1)\rho^t d} e^{\alpha\rho^t I_k}\left[\rho^t e^{\alpha(d-I_k)} + 1 - \rho^t\right]\left\{e^{\alpha(n-k)\rho^0 I^0}\left[\rho^t e^{\alpha(d-I_0)} + 1 - \rho^t\right]^{n-k}\right\}$$

To show that this function possesses a unique maximum, it is necessary to show that the second derivative of $\bar{V}(k)$ is strictly negative. Differentiating $\bar{V}(k)$ twice with respect to k and simplifying yields:

$$\frac{\partial^2 \bar{V}(k)}{\partial k^2} = \bar{V}(k)\left\{\left[-a + \alpha\rho^t \frac{\partial I_k}{\partial k} * \frac{e^{\alpha(d-I_k^*)} - 1}{e^{\alpha(d-I_k^*)} + \frac{\rho^t}{1-\rho^t}}\right]^2 + \alpha\rho^t \frac{\partial^2 I_k}{\partial k^2} * \frac{e^{\alpha(d-I_k^*)} - 1}{e^{\alpha(d-I_k^*)} + \frac{\rho^t}{1-\rho^t}} - \alpha^2 \rho^t \frac{\partial I_k}{\partial k} * \frac{e^{\alpha(d-I_k^*)}}{(1-\rho^t)\left(e^{\alpha(d-I_k^*)} + \frac{\rho^t}{1-\rho^t}\right)^2}\right\},$$

where a has been previously defined as $a = \left[\alpha\left(-\rho^t d + \rho^0 I^0\right) + \log\left(1 - \rho^t + \rho^t e^{\alpha(d-I^0)}\right)\right]$. Since $\bar{V}(k)$ is strictly negative, then $\frac{\bar{V}''(k)}{\bar{V}(k)}$ must be strictly positive for $\bar{V}''(k)$ to be less than zero. Unfortunately, the terms in the curly bracket cannot be signed, and as such numerical methods are needed to ascertain the sign of $\frac{\bar{V}''(k)}{\bar{V}(k)}$.

Profile graphs of $\frac{\bar{V}''(k)}{\bar{V}(k)}$ with respect to the underlying variables, $\alpha, \lambda, \rho^h, \rho^t, d$ and k, are given in Figure 5. As can be seen in Figure 5. all the graphs are strictly positive over a moderate range of the underlying variables. Values for the variables in each graph that were not examined were set at $\alpha = 0.60$, $\lambda = 0.25$, $\rho^h = 0.12$, $\rho^\ell = 0.10$, $k = 4$ and $d = 1000$. A relatively small value of d was chosen to mitigate computational problems. In the profile graph with respect to α, the range of α examined corresponds to the range suggested by Haubrich. The proportion of high risk consumers in the population, λ, was examined over the

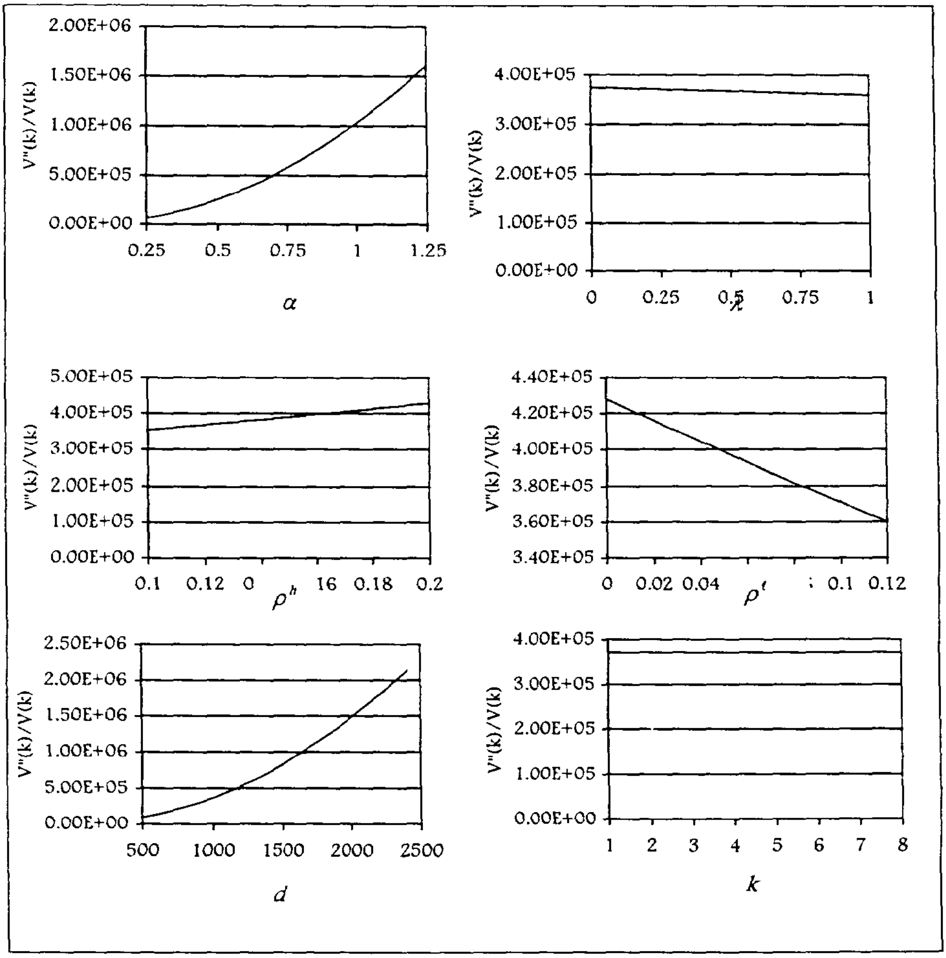

Figure 5 - Profile of $\frac{\bar{V}''(k)}{\bar{V}(k)}$ with respect to Underlying Parameters.

entire range from zero to one. In the profile graph with respect to the high risk's probability of loss, the range extends from the low risk's probability of loss upwards to 20%. The range examined for the low risk probability of loss extended from zero to ρ^h. Due to computation constraints, d was examined over a range of relatively small values. And finally, the range examined for k, the number of periods before the end in which separation occurs, corresponds to the results in Table 1.