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Value-at-Risk for Risk Portfolios

Working Paper

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Abstract

Value-at-Risk (VaR) methodologies are particularly useful to actuaries as methods to quantify investment and portfolio risk. Despite the recent developments in application and theory, there remains the problem of aggregating VaR measures among risk portfolios. In this paper, we use simple risk portfolios to discuss the abilities and shortcomings of the current methodologies for VaR, and suggest methodologies which can be used to facilitate the aggregation of VaR measures.

1 Introduction

Much debate has focused on how to measure risk. The goal for financial regulators is to find a measure that is simple to calculate, yet accurately identifies the level of risk that a company is exposed to. Value-at-Risk (VaR) is a recent approach that has been implemented into the risk management programs of many financial consulting firms. Most VaR techniques use price sensitivity methods to translate financial instruments into units of risk, or potential loss, based on a specified holding period, observation period, confidence interval and volatility of risk factors.

1.1 Additivity vs. Sub-Additivity

The Basle Capital Accord proposed a set of capital requirements for banks and other financial investment firms based on the inherent volatility of their individual assets. The requirements were determined separately for each asset, and then added to obtain the capital requirements of the portfolio. Unfortunately, this method did not permit a reduction in capital requirements for hedged or diversified portfolios. Thus, to account for the subadditivity of such risks, the Commission allowed the use of computer models to calculate the risk of a portfolio, as long as the models complied with Commission standards.

1.2 Other Uses of VaR

The importance of a risk measure is attributable to its ability to differentiate between different types of risk, as well as its ability to compare the severity of different risk portfolios. The application of VaR as a risk measurement technique has often been suggested for use as an aide to evaluate investment risk, to identify the optimal allocation of assets, to develop and evaluate portfolio strategies, to measure the quality of a portfolio, and to evaluate portfolio managers.

2 The Methodologies

The definition of a 95%, n-day, Value-at-Risk for portfolio P, with initial value P_0, is $VaR_{95\%}(P_0)$, such that

$$Pr(L_n < VaR_{95\%}(P_0)) = 0.95$$

or

$$Pr(L_n > VaR_{95\%}(P_0)) = 0.05$$

where L_n is the loss random variable for the portfolio, $L_n = P_0 - P_n$.
Market factors, such as, domestic and foreign interest rate structures, exchange rates, stock prices and inflation rates, are examples of risk factors which may have an impact on the financial risks of a portfolio. There are many methods which use these market factors to determine the distribution of L_n. These methods can be classified into three model types: the historical model, the analytic model and the simulation model.

2.1 Historical

In the historical approach, using a one day holding period, previous one day fluctuations in market factors are used to model possible fluctuations to current market factor values. Alternative profit/loss realizations are valued based on these fluctuations and a distribution for profit/loss can be obtained from these realizations to produce a confidence interval.

Historical Model:
$F_d = (f_{(1,d)}, f_{(2,d)}, \ldots, f_{(k,d)})$ the vector of observed risk factor values on day $d = 0, -1, \ldots, -n$
$P(F) =$ the value of a portfolio, using the factor values **F**.

$$F_0 = \quad \text{today's risk factor values}$$

$$\Delta F_d = F_d - F_{d-1} \quad \text{One day factor changes}$$

$$P_0 = P(F_0)$$

$$P_d = P(F_0 + \Delta F_d)$$

$$VaR_{1-\alpha}(P_0, n) = P_0 - (((1-\alpha) \times n)\text{th smallest value of} \quad P_d)$$

2.2 Simulation

An alternate method, the simulation method requires a distribution for changes in each market factor including correlations between factors. Normal and lognormal distributions are often used, with correlations derived from historical data. Given factor distributions, Monte Carlo simulation is used to obtain simulated changes in the market factors, which are used to obtain a profit/loss distribution and confidence intervals in the same way as in the historical method.

Simulation Model:
$Pr(F = (f_1, f_2, ..., f_k))$ = the joint density function of the risk factors.
P = the portfolio value.
Derive $Pr(P = p)$, the distribution of the portfolio value using simulation techniques and use this to determine $VaR_{(1-\alpha)}(P_0)$.

2.3 Analytic

A more restrictive approach, the analytic method decomposes the portfolio into elemental instruments each of which is exposed to only one market factor. A set of distributions for changes in the market factors is used to calculate the VaR and the portfolio variance. Since the portfolio is the sum of the elemental instruments, if the market factors have a joint normal distribution, then the portfolio is also normally distributed.

Analytic/Variance-Covariance Model:
Let $P_{f_1}, P_{f_2}, ..., P_{f_k}$ be the decomposition of Portfolio P into component securities.
Then,

$$P = \sum_{i=1}^{k} P_{f_i}.$$

Assuming that the component securities are related through a known covariance structure, using multivariate Normal techniques, the portfolio distribution can be approximated, and $VaR_{(1-\alpha)}(P_0)$ calculated.

3 The Problems

Even though these models seem intuitively reasonable and they are easy to explain, their tractability is based on the assumption that the percentile VaR measure is subadditive. Unfortunately, it is possible to show that VaR, as with all percentile measures, can be superadditive. This leads to the conclusion that VaR is not a consistent measure, and should be used with caution.

3.1 Super-Additivity

Consider two risks P_1 and P_2, with loss random variables L_1 and L_2 respectively. Suppose that the support of each loss covers more than three percent, but less than five percent of the risk factor distribution, and that the support of the two losses are disjoint. Equivalently, suppose that the following three conditions hold:

$$0.03 < Pr(L_1 > 0) < 0.05,$$

$$0.03 < Pr(L_2 > 0) < 0.05,$$

$$Pr(L_1 > 0 \ \ and \ \ L_2 > 0) = 0.$$

The first two conditions imply that $VaR_{95\%}(P_1) = 0$ and $VaR_{95\%}(P_2) = 0$. As well, the three conditions together imply that

$$0.06 < Pr(L_1 + L_2 > 0) < 0.10$$

and therefor, $VaR_{95\%}(P_1 + P_2) > 0$.
In the combined portfolio $(P_1 + P_2)$, the support covers more than five percent of the risk factor distribution, and the 95% VaR value is positive, and greater than the sum of the two individual VaR values. Thus, VaR as a percentile measure is superadditive.

3.2 Manipulating VaR

To show that this method allows the 95% VaR value to be reduced for any arbitrary portfolio, it must be possible to partition the portfolio so that each partition has a loss distribution with a support which covers less than 5% of the joint risk distribution. Using derivatives, simple combinations of calls and puts, it is possible to partition any portfolio, as long as the joint risk distribution is known in advance. In this way, it is possible to decompose any risk portfolio, with positive VaR value, into sub-portfolios, each having a VaR value of zero. As the percentile for the VaR value increases, more sub-portfolios may be needed; however, it is always possible to obtain a zero value for VaR. As a result, knowing the distribution of risk factors, the value for VaR may be arbitrarily chosen.

Even if VaR must be evaluated for an entire portfolio, there would be incentive to find other portfolios that would also benefit from trading risk partitions, so that each portfolio would show a large loss over a very small partition of the risk factors distribution. For example, when it is used as a regulation requirement, or an evaluation criterion, there is an incentive to obtain a specific VaR value by manipulating the portfolio. Unfortunately, this may work against the original purpose for having a risk measure, and the results may be to promote portfolios with extreme, but localized risk, as opposed to diversified portfolios with reduced tail risk.

4 Possible Solutions

As VaR has been accepted by the majority of risk managers, and has developed a large following, it would be difficult to radically change the risk measurement procedure. Although a change might me advantageous, there are tools available to investigate the validity of the value produced by VaR. These tools study the tail risk of a loss distribution and by using variations on these tools, it might be possible to ascertain the validity of a VaR measure.

4.1 The Extreme Value Measure [2]

VaR is an extreme value statistic. As such, it is important to understand the properties of this type of statistic in order to improve our understanding of our VaR results. The definition of a p-percentile, x_p is:

$$x_p = F^{-1}(p) = inf\{x \in \mathcal{R}; F(x) \geq p\}.$$

Based on a sample of n data points, $X_1, ..., X_n$, the empirical distribution of the random variable X is defined as:

$$F_n(x) = \frac{\#\{i \ : \ 1 \leq i \leq n \ and \ X_i \leq x\}}{n}, x \in \mathcal{R}.$$

Defining the order statistics for this distribution as

$$X_{(1),n} = max(X_1, ..., X_n) \geq X_{(2),n} \geq ... \geq X_{(n),n} = min(X_1, ..., X_n),$$

then x_p can be approximated by

$$\hat{x}_{p,n} = F_n^{-1}(p) = X_{(k),n}, \quad \frac{n-k}{n} \leq p \leq \frac{n-k+1}{n}.$$

By the Central Limit Theorem, it is possible to show that

$$\hat{x}_{p,n} \quad \sim \quad AN\left(x_p, \frac{p(1-p)}{nf^2(x_p)}\right)$$

from which we can obtain approximate confidence intervals for the estimated percentile.

As well, if $X_1, ..., X_n$ are iid, the binomial model for an order statistic can be used to produce percentile confidence intervals,

$$Pr(X_{j,n} \leq x_p < X_{i,n}) = \sum_{r=i}^{j-1} \binom{n}{r} p^{n-r}(1-p)^r \text{ for } i < j.$$

The resulting confidence intervals can help to identify the accuracy of a VaR value.

4.2 Mean Excess Function

Related to the mean future lifetime random variable of actuarial science, the Mean Excess Function (MEF) is a functional representation of the tail of the loss distribution. Using this function, it is possible to identify fat tails and anomalies that occur in the tail of the aggregate loss distribution.
The Mean Excess Function (MEF)[2] is defined as

$$e(u) = E(X - u|X > u),$$

and the Empirical Mean Excess Function is defined as

$$e_n(u) = \frac{\sum_{i=1}^{n}(X_i - u)^+}{\sum_{i=1}^{n} I_{\{X_i > u\}}}.$$

By plotting the MEF for some common distributions that have the same mean and 95th percentile, it is easy to see how the MEF can be used to identify long tailed distributions. The Pareto is has the fattest tail, and the Lognormal is also known for

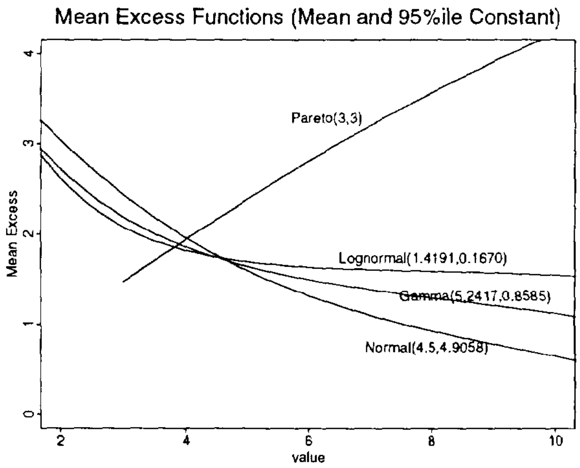

Figure 1: MEF for two parameter distributions with $Mean = 4.5$ and $x_{95\%} = 8.143$

its heavier tail, the Gamma has a lighter tail, and the Normal distribution is known for its short tail. Thus, the slope of the MEF in the tail of the distribution can be used to evaluate the severity of the tail risk.

5 Conclusions

It may be nice to have a risk measure that everyone is willing to implement; however, there is no point to having that risk measure if it can be manipulated. Thus we must find some way to verify the risk measure, or develop a new measure that does not allow itself to be manipulated. Extreme Value theory and the Mean Excess Function are suggested as ways to supplement the current VaR techniques. These suggestions may improve the current techniques; however, it is extremely important that we still search for consistent risk measures [1].

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