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An Unfinished Thesis from the 1950's

By
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Abstract

A reference to the thesis has been given already by Cecil J. Nesbitt in his discussion of Colin M. Ramsay's paper, Percentile Pension Cost Methods: A New Approach to Pension Valuation (see Transactions of the Society of Actuaries, XLV (1993), 420-422). Herewith is a more detailed reference to the thesis. It consists of:

I. Introduction

This provides, in the words of Robert W. Butcher himself, an overview of the thesis.

II. References

These were drawn upon by Robert W. Butcher for his thesis.

III. Table of Contents

For the thesis, the table provides a second overview of what Robert W. Butcher undertook.

IV. Commentary by Cecil J. Nesbitt

This attempts to evaluate the unfinished thesis and to explain why it would be difficult and untimely to reconstruct the thesis in its entirety. Some of the ideas from the thesis, particularly Hattendorf's Theorem, have been presented in more special contexts in *Actuarial Mathematics*, and in the paper by Cecil Nesbitt and Marjorie Rosenberg, Annuities for the Aged, 1992.1 Issue of the Actuarial Research Clearing House (ARCH), 251-289. Further developments await extension of our knowledge.

V. Statement by Marjorie V. Butcher

This offers some background about the thesis of her husband, Bob, and the role of his advisor and mentor, Cecil Nesbitt.

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I. Introduction

1.1 The General Problem

Any financial transaction with payments depending on the survival or death of a person or persons will involve risk, or possibility of loss, for the various parties concerned. The word "loss" is used here in a general sense to include negative loss or gain. A financial transaction of the type indicated can be classified as a life insurance where the payments are dependent on the death of a person, or as a life annuity, where the payments are dependent on the survival of a person. In practice, a transaction may have both life insurance and life annuity features.

The risks involved in such transactions may depend on mortality, interest, expense and possibly other factors such as terminations due to causes other than death. This thesis is concerned with the effect of the mortality factor only. It is assumed that: (1) all calculations are on a net basis, that is, expenses are not considered; (2) accumulated funds earn interest at a given fixed rate; and (3) death is the only cause of termination. The risk introduced by the mortality factor can be divided into two parts. First, there is the risk occasioned by the possibility that the assumed mortality may not be applicable to the group in question. This could result from the continuous improvement in mortality that has been observed in recent years. Secondly, even though the assumed mortality does operate in the population from which the given group may be regarded as a finite random sample, there is, on account of the finite size of the group, risk arising from the possibility that the actual experience may deviate from the expected. Most of the material in this thesis is concerned with this second risk.

Methods for assessing such mortality risk in regard to insurance and annuity transactions form the content of the dissertation. Although the basic methods and formulas are general, most of the applications to be made here deal with life annuity transactions, with particular emphasis on pension situations.

1.2 Two Approaches to the Theory of Risk

One approach to the subject is provided by the Collective Theory of Risk. The theory considers the operation of an insurance fund as a whole without direct reference to the individual transactions and persons involved. Only the risk portions of premiums and claims are considered. Attention is centered on the risk reserve accumulated from the excess of the risk premiums over the risk portions of the claims. Basic assumptions include: (1) there exists a fixed distribution for the risk portions of the claims, this distribution being independent of time; (2) the probability of a claim occurrence during any infinitesimal time interval (t, t+dt) is p dt where p is a constant, and the occurrence of more than one claim during such interval is of smaller order of magnitude than dt; and (3) the accumulation of the risk reserve is without accrual of interest. A stochastic process is obtained by assuming that the net risk premiums plus appropriate loadings are paid into the risk reserve continuously and risk claims are paid out of this reserve in discrete finite amounts. An analysis of the stochastic process will give the probability of ultimate ruin, that is, the probability that the risk reserve, assumed to contain some assigned amount initially, will at some time in the future become negative. More specific ruin situations have also been studied. The Collective Theory of Risk covers transactions with both positive and negative amounts at risk, that is, both life insurance and life annuity situations. It was originated by Lundberg and much work has been done on it in Scandinavia, [5] [10] [11].[1]

Another approach to the subject is provided by the Individual Theory of Risk. It is based on the theory of variances of independent random variables and builds up the variance of the losses involved in a group transaction from the variances of losses for the underlying individual transactions. Since the survival or death of one person has no effect on the survival or death of another person, except for the negligible effect of a common catastrophe, independent random variables can be defined to represent the losses incurred in a specific transaction. Once the variances of these random variables are evaluated, the normal distribution is used to determine probabilities of losses of given magnitudes. While the Individual Theory of Risk has been mainly developed by European actuaries, [2] [6] [12], some work has been done in America [8] [9] [14] [15].

The Collective Theory of Risk is almost totally distinct from the main stream of life insurance mathematics which, in general, approaches insurance and annuity problems by detailed consideration of the individuals involved. Although the idea of an aggregate approach is interesting, and not without promise, it is difficult to accept the theory at its present stage because of the basic assumptions which are, in general, unrealistic and inflexible. These assumptions seem especially unrealistic for most pension situations and, particularly, for the smaller pension groups for which risk theory might be of most value.[2] On the other hand, the Individual Theory of Risk is connected reasonably closely with standard actuarial methods and has some flexibility and adaptability. Accordingly, it was decided to explore in this thesis, as far as possible, what could be done with the Individual Theory of Risk. It is hoped that the formulas and methods developed in this thesis will help to overcome the usual criticism that the Individual Theory of Risk involves computations that are too long and laborious.

1.3 Pension Plan Risks

As is usual in pension work, we consider the participants in a pension fund as divided into an active group and a retired group. Of course, for a particular plan, one of these groups may be nonexistent. For members of the active group, that is, those with active employment status, contributions are paid into a fund from which retirement benefits are to be provided or purchased for those who retire. The contributions may be made by the individual himself, his employer, the government, etc. The identity of the contributor is immaterial for our purposes.

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[1] Numbers in brackets refer to the list of references found in part II

[2] Some attempts to make the application of the Collective Theory of Risk more practical have appeared recently, [11]. It should also be noted that the Collective Theory may be applied as well, if not better, to casualty insurance problems while the Individual Theory is better adapted to life insurance problems.

A particular pension plan will have a designated funding method which will indicate how the contributions for the active group are to be made into the fund. A complete spectrum of funding methods exists; the more important ones are discussed in a paper by Trowbridge, [16]. The risks in connection with several of these funding methods are considered in a later chapter of this thesis.

When an individual joins the active group of the pension plan at the initial age, the contributor can estimate the amount that he expects to pay into the pension fund for this individual. However, the exact amount that will be paid by the contributor will vary since contributions stop at retirement or prior death. Since the actual contributions will vary, the contributor incurs a certain risk when an individual joins the group.

On the other hand, there is risk involved in the pension fund itself. The contributions are determined to be sufficient to make the expected payments to the members of the retired group. Consider an individual's participation in the pension fund. If the individual dies during the active period, some or all of the contributions made on his behalf may remain in the fund and the fund will make a gain. Similarly, if the individual dies shortly after retirement, the fund may make a gain. On the other hand, if the individual lives for a long time after retirement, the fund may suffer a loss. Thus the pension fund involves a risk in respect to each individual since the actual gain or loss is unknown until the death of the individual.

1.4 A General Outline

The body of this thesis consists of a theory chapter, an applications chapter for retired groups and an applications chapter for active groups. In an appendix there appear a number of special tables, based on the α-1949 Mortality Table, [1], which may be used for applications where the α-1949 Table is the assumed mortality basis.

The first step in the theory chapter is the setting up of appropriate random variables and developing of formulas for the variances of loss in respect to an individual. Both discrete and continuous methods are used. Next, there is a development of Hattendorf's Theorem, [2] [6] [12], which connects the variance for an n-year term with the variances for the individual years in the n-year term. As a third step, the theory is extended to various groups with mortality expected to follow the given mortality table. Although considerable choice of groups is available, and the results are adaptable in various ways, it must be admitted that real pension situations are generally more complex and are only approximated by the theoretical groups. Nevertheless, the results and methods used for the theoretical groups may be sufficiently indicative for real groups. Also, variances of loss for real groups may be calculated by use of the formulas for individuals, applied in whatever detail appears adequate and desirable for the purpose in hand. The formulas will show the variances of loss caused by mortality deviations during a certain predetermined number of years, usually the first n years. The Hattendorf Theorem, which is the main tool used in this thesis, is particularly useful for formulas of this type because it produces results that are flexible, suitable for the definition of new commutation functions and not too laborious in application.

In the chapter for applications to retired groups a comparatively easy specialization of the general theory is carried through. The special commutation functions, in terms of which individual and group variances may be expressed by means of Hattendorf's Theorem, are readily computable in this case, and have been computed on two bases. Variances of losses are computed for various groups, and the additional funds or loadings required to offset losses up to three times the standard deviation are determined. Comparisons are made with the results obtained by Stone, [14], using special generating polynomials. In addition, comparison is made with the effect of projected decreases in mortality on annuity costs for various groups of annuities.

In the fourth chapter, the theory is applied to an active group, the members of which are covered by a simple illustrative pension plan for which various funding methods are examined. Special commutation functions may be developed in respect to these funding methods but, for some of the methods, the computations are quite complicated. Because of their more limited applicability, the commutation functions themselves were not computed for this situation, but instead there was developed a set of basic functions from which the commutation function values may be derived. Then, as in the case of retired groups, representative values of variances of losses for various groups are given and their significance discussed. Since there are a number of funding methods, and both the fund's loss and the contributor's loss are to be considered, the situation is more varied than in the retired group case, but the procedures follow as specializations of the general theory.

II. References

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III. Table of Contents

IV. Commentary by Cecil J. Nesbitt

As far as can be determined, I am in possession of the only full copy of the thesis by Robert W. Butcher, F.S.A. Unfortunately, that copy is handwritten and in preliminary form, awaiting further polishing. He had devoted much time, skill and energy to it, and obtained mathematical results which answered many questions I may have suggested to him. In 1956, he joined The Travelers Insurance Companies and did not finish the thesis. In September 1960, I wrote him:

> "My opinion is that it is an acceptable thesis in actuarial mathematics. It explores the Individual Risk Theory thoroughly, brings in new loss variables, and makes skillful application of new commutation functions to get the group variances. Its study of the active group is also new. Your proof of Hattendorf's Theorem is essentially the same as Harald Cramer's (which he got from Cantelli) but you have given it in more detail and in a more general context. Your organization of that proof, and development of the theory and calculations for the group variances (continuous as well as discrete) are good. The presentation needs some polishing and I enclose notes gleaned from my reading of the thesis."

I went on to indicate further steps he would need to complete doctoral requirements at Michigan, but that did not happen.

Recently, I tried to make a usable copy of the thesis, and had some limited success in Chapters 1 and 2, but have not checked all the formulas and computations, especially in Chapters 3 and 4 for the retired group, and for the active group under a pension plan. As you may judge from the introductory chapter by Robert Butcher, the Table of Contents, and the References, it would be a big job to reconstruct the thesis. I do not think this is the time for such reconstruction. My reasons are:

(a) The thesis is based on the aggregate payment technique and what I call the Life Insurance Approach.. This involves use of the time until death random variables, T (continuous) and K (discrete). Recently for annuities we have become interested in variables S (continuous) and J (discrete) which are related to the Life Annuity Approach and the current payment technique for annuities. The stress is then on survival times rather than time until death. There are problems of fitting S and J into the standard probability theory of random variables. They may fit better into growth (and decrement) theory. At any rate, there are open questions about the variables S and J. Studies of them are needed to test how significantly different are variances by the Life Annuity Approach from those by the Life Insurance Approach. To follow up on these ideas, see the paper by Cecil Nesbitt and Sarah Clark, m-Year Step-Wise Level Percent Financing; Life Insurance and Life Annuity Variances, Part II. It appears in the 1997.1 issue of Actuarial Research Clearing House (ARCH), 409-414. A separate background reference to the paper may be on the CD-ROM, What's New in Actuarial Education and Research, prepared by Arnold F. Shapiro (1997) as Highlights of the 31st Actuarial Research Conference. Also a printed copy of the paper may be obtained from Cecil Nesbitt, Department of Mathematics, 525 E.

University St., Ann Arbor, MI 48109-1109. Part II of the paper deals with some questions that came up when Cecil was scanning Section 18.6, More Reversionary Annuities, of the 2nd Edition of *Actuarial Mathematics*. This work is in a preliminary stage, and much more is needed to settle the questions.

(b) At this time of rapidly changing survival rates, long-term applications of Hattendorf's Theorem are questionable. They are, however, main tools in the thesis. Also, the thesis' general model for applying Hattendorf's Theorem considers insurance and annuity ideas together, and thereby is somewhat complex to discuss.

(c) The thesis develops special commutation functions and supplementary functions. The authors of the 2nd Edition of *Actuarial Mathematics* eschew the use of commutation functions, especially in this computer age.

(d) Collective risk theory and statistical analysis have developed strongly in the past 40 years since the thesis was prepared. Now modeling is a key concept, and has found a place for itself in actuarial applications. This has been one of the impacts of the computer revolution which is still going on.

(e) There exist more immediate calls on actuarial thinking. Among these are:

 1) Actuarial theory to a large extent has ignored the threat of nuclear disaster which still remains a heavy cloud over our long-term projections. So-called conventional weapons and terrorist actions are also daily threats.

 2) The obligations of Social Security, especially Medicare, demand critical actuarial attention.

 The public is aware of these items.

(f) Finally, this is a transition period for actuarial education and research. Until more stability emerges, a user of long-term applications should prepare to defend their foundations.

Robert W. Butcher did a remarkable job of actuarial theory and computations on what was known and used in the 1950's. Much development has occurred since then. As indicated in the foregoing, it may be inappropriate and difficult to reconstruct the thesis now. The development of more knowledge should help future actuaries to usefully build on the ideas presented by the thesis.

V. Statement by Marjorie V. Butcher

I deeply appreciate Professor Cecil Nesbitt's desire to bring the unfinished Ph.D. thesis of my late husband Bob to the attention of the actuarial community. Cecil has given much thought to the form and forum of its presentation. Bob and I are honored that he initiated and undertook this task. Since Bob and I attended the first, and many of the early, Actuarial Research Conferences (ARC's), I am happy that the 32nd ARC is the forum Cecil chose for this presentation of Bob's thesis research – a presentation tripartite in authorship, reminiscent of some joint projects in our University of Michigan years, 1950-56.

Bob's graduate school interests included actuarial mathematics, probability and mathematical statistics, and computer science. That Cecil suggested a thesis topic so perfectly suited to Bob's interests and abilities is one example of Cecil's mentoring. It's a measure of his perspective and insight that the general topic of the thesis, mathematical theory of risk, is today still alive and evolving. Bob's contribution advanced the subject in its day – a subject that has continued to evolve in the forty years since. But just think what has happened in those forty years to other aspects of actuarial theory and practice, or – within the memory of younger actuaries – the last thirty, twenty or even ten years!

Back in the 1950's exploring the mathematical, actuarial, statistical and computational aspects of individual theory of risk was an ideal assignment for Bob. And working under the stimulus of Dr. Nesbitt as thesis advisor was the ideal arrangement for Bob. Already he had developed some mathematical research maturity through the influence of (1) his father, C. Ward Butcher, principal and mathematics teacher at Paris (Ontario) High School, (2) the mathematics faculty, particularly chairman R. L. Jeffery, of Queen's University in Kingston, Ontario and (3) Professor L.A.H. Warren of the University of Manitoba who brought him there to teach actuarial mathematics, 1948-50. Coupled with a good analytical mind, Bob had the habit and capacity for hard, independent work. During the Ann Arbor years, studying, teaching and researching under Dr. Nesbitt, these fine qualities of Bob were enhanced and polished.

The two men were naturally, I believe, drawn to each other by a surprising set of similarities, which included Canadian roots, a certain shyness and humility, keen intelligence, skill at teaching mathematics, devotion to their students, quiet competence, ready sense of humor, drive and above all enthusiasm for their actuarial calling. They were a very good team of mentor and student, and very good friends for the rest of Bob's life.

Eventually, Bob felt that he should get some "real-world" actuarial experience; that thesis and degree were far enough along that they could be completed away from Ann Arbor – a miscalculation, as it turned out, since increasing responsibilities at The Travelers, Fellowship examinations, and establishing a home in Connecticut were all-consuming. But I believe, and I think Bob would agree, the thesis experience under Cecil significantly shaped his actuarial career and his adult life. Partly as a consequence, Bob became an actuary's actuary: a flexible, technical actuary/businessman who would tackle any problem – and usually solve it – whether in individual life insurance, group pensions, or property-casualty insurance; a friend and capable guide to younger colleagues; a researcher and director of researchers; and always, always a gentleman. He never lost his flair for, and love of, actuarial mathematics, mathematical statistics and computers. How, with a minimum of words, he could explain things! And enrich others by his encouragement!

So the unfinished thesis and Cecil Nesbitt profoundly influenced Bob Butcher's career and life; and in turn I believe Bob Butcher has similarly influenced others.

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