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A Karup-King Formula with Unequal Intervals

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Abstract

The Karup-King formula with equal intervals has an elegant symmetric structure providing for efficient calculation. It is a four point third degree polynomial passing through the two central points and having two slope conditions: (1) the slope at the second point is equal to the slope of the second degree polynomial passing the first three points; and (2) the slope at the third point is equal to the slope of the second degree polynomial passing through the last three points. The paper traces the algebra involved in a generalization of this formula to the case of unequal intervals. So far, this formula does not have an elegant structure. However, a degree of elegance can be obtained in terms of matrix notation which can readily used with a computer language such as APL.

Introduction

In working with his smoker - nonsmoker mortality data, J. M. Bragg found it necessary to interpolate pivotal mortality rates for duration groups. This involved interpolating rates located at unequal intervals. The intervals were not equal for two reasons. First, the duration groups were of different widths. Second, the "centers" of the groups were not at the midpoints of the range because the exposures reduced at higher durations for demographic reasons.

He wanted a good fit without running the risk of looping involved in passing polynomials through all points involved. The Karup-King formula would have been suitable if the pivotal points were equally spaced. Therefore, a formula with the characteristics stated in the **Abstract** was desired.

The Karup-King Formula for equal intervals in Everett form is

$$v_{x+s} = F(s)\, u_{x+1} + F(t)\, u_x \quad \text{where } F(s) = s + \tfrac{1}{2}s^2(s-1)\delta^2 \quad \text{and}$$
$t = 1 - s$

and in Lagrange form is

$$v_{x+s} = (-\tfrac{1}{2}s^3 + s^2 - \tfrac{1}{2}s)\, u_{-1} + (\tfrac{3}{2}s^3 - \tfrac{5}{2}s^2 + 1)u_o + (-\tfrac{3}{2}s^3 + 2s^2 + \tfrac{1}{2}s)u_1 + (\tfrac{1}{2}s^3 - \tfrac{1}{2}s^2)u_2$$

We want to find $F_a(x)$such that $F_a(b) = 0; \quad F_a(c) = 0; \quad F_a'(b) = \frac{b-c}{(a-b)(a-c)}$
$F_a'(c) = 0$

Let $F_a(x) = p_{ao} + p_{a1}x + p_{a2}x^2 + p_{a3}x^3$

Then $F_a'(x) = p_{a1} + 2p_{a2}x + 3p_{a3}x^2$

Where the p_{aj} $(j = 0, 1, 2, 3)$ satisfy the simultaneous equations:

$$p_{ao} + p_{a1}b + p_{a2}b^2 + p_{a3}b^3 = 0$$

$$p_{ao} + p_{a1}c + p_{a2}c^2 + p_{a3}b^3 = 0$$

$$p_{a1} + 2p_{a2}b + 3p_{a3}b^2 = \frac{b-c}{(a-b)(a-c)}$$

$$p_{a1} + 2p_{a2}c + 3p_{a3}c^2 = 0$$

These simultaneous equations can be expressed in matrix notation as: $\mathbf{A}\cdot\mathbf{P}_a = \mathbf{F}_a$
where

$$\mathbf{A} = \begin{bmatrix} 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 0 & 1 & 2b & 3b^2 \\ 0 & 1 & 2c & 3c^2 \end{bmatrix} \qquad \mathbf{P}_a = \begin{bmatrix} p_{ao} \\ p_{a1} \\ p_{a2} \\ p_{a3} \end{bmatrix} \qquad \mathbf{F}_a = \begin{bmatrix} 0 \\ 0 \\ \frac{b-c}{(a-b)(a-c)} \\ 0 \end{bmatrix}$$

This gives the solution: $\mathbf{P}_a = \mathbf{A}^{-1}\mathbf{F}_a$

Similarly, the coefficients, p_{bj}, p_{cj}, p_{dj} $(j = 0, 1, 2, 3)$can be found using the respective values of F_b, F_c, F_d where:

$$\mathbf{F}_b = \begin{bmatrix} 1 \\ 0 \\ \frac{2b-(a+c)}{(b-a)(b-c)} \\ \frac{c-d}{(b-c)(b-d)} \end{bmatrix} \qquad \mathbf{F}_c = \begin{bmatrix} 0 \\ 1 \\ \frac{b-a}{(c-a)(c-b)} \\ \frac{2c-(b+d)}{(c-b)(c-d)} \end{bmatrix} \qquad \mathbf{F}_d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{c-b}{(d-b)(d-c)} \end{bmatrix}$$

The problem of finding all the p_{ij} can be summarized by building an augmented matrix, $\mathbf{P}$, whose first column consists of p_{aj} $(j = 0, 1, 2, 3)$ and whose subsequent columns consist of p_{bj}, p_{cj}, p_{dj}. $\mathbf{P}$ would be equal to $\mathbf{A}^{-1}\mathbf{F}$ where $\mathbf{F}$ would be the augmented matrix whose columns consist of $\mathbf{F}_a$, $\mathbf{F}_b$, $\mathbf{F}_c$, $\mathbf{F}_d$. Thus,

$$\mathbf{P} = \mathbf{A}^{-1}\mathbf{F} \quad \text{gives the matrix} \begin{bmatrix} p_{ao} & p_{bo} & p_{co} & p_{do} \\ p_{a1} & p_{b1} & p_{c1} & p_{d1} \\ p_{a2} & p_{b2} & p_{c2} & p_{d2} \\ p_{a3} & p_{b3} & p_{c3} & p_{d3} \end{bmatrix}$$

The transpose of $\mathbf{P}$, $\mathbf{P}^T = \begin{bmatrix} p_{ao} & p_{a1} & p_{a2} & p_{a3} \\ \ddots & & \ddots & \end{bmatrix}$ can be multiplied by the vector $\mathbf{x}$, where $\mathbf{x}$ equals $\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$ giving $\begin{bmatrix} F_a(x) \\ F_b(x) \\ F_c(x) \\ F_d(x) \end{bmatrix}$. If the row vector, $\mathbf{u}^T$, where $\mathbf{u} = \begin{bmatrix} u_a \\ u_b \\ u_c \\ u_d \end{bmatrix}$, is multiplied by this last column vector, the result is the interpolated value, v_x.

In summary $v_x = \mathbf{u}^T(\mathbf{A}^{-1}\mathbf{F})^T\mathbf{x}$

A Significant Simplification

A simplification is possible by changing the origin and scale so that the two middle points have arguments 0 and 1 respectively and the points at the end have arguments m and $n\,(m < 0; n > 1)$. The interpolated value, v_s, is to be determined, where $s = \frac{x-b}{c-b}$; $m = \frac{a-b}{c-b}$ and $n = \frac{d-b}{c-b}$.

Using these values $\mathbf{A}$ becomes $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{A}^{-1}$is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$.

Further,

$$\mathbf{F} \text{ becomes } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{m(1-m)} & -\frac{(m+1)}{m} & \frac{m}{m-1} & 0 \\ 0 & \frac{1-n}{n} & \frac{2-n}{1-n} & \frac{1}{n(n-1)} \end{bmatrix}, \ \mathbf{s} = \begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix}, \text{ and } \mathbf{u} = \begin{bmatrix} u_m \\ u_0 \\ u_1 \\ u_n \end{bmatrix}$$

Thus the solution is $v_s = \mathbf{u}^T(\mathbf{A}^{-1}\mathbf{F})^T\mathbf{s}$. The inverse of the matrix $\mathbf{A}$, $\mathbf{A}^{-1}$, is given and is constant for ***all interpolations.***

Appendix A

The results given here can be generalized to include the Lagrange formula. As an example, consider a second degree polynomial passing through the points u_a, u_b, and u_c which can be expressed as

$$^{L}v_x = F_a(x)\,u_a + F_b(x)\,u_b + F_c(x)\,u_c \qquad \text{where}$$

$F_i(x)$ $(i = a, b, c)$ are determined so that

$$\begin{array}{lll} F_a(a) = 1 & F_a(b) = 0 & F_a(c) = 0 \\ F_b(a) = 0 & F_b(b) = 1 & F_b(c) = 0 \\ F_c(a) = 0 & F_c(b) = 0 & F_c(c) = 1 \end{array}$$

Let $F_a(x) = p_{ao} + p_{a1}x + p_{a2}x^2$ so that

$$\begin{array}{l} p_{ao} + p_{a1}a + p_{a2}a^2 = 1 \quad \text{or} \\ p_{ao} + p_{a1}b + p_{a2}b^2 = 0 \\ p_{ao} + p_{a1}c + p_{a2}c^2 = 0 \end{array}$$

$$\mathbf{A}\cdot\mathbf{P}_a = \mathbf{F_a} \text{ where } \mathbf{A} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}, \quad \mathbf{P}_a = \begin{bmatrix} p_{ao} \\ p_{a1} \\ p_{a2} \end{bmatrix}, \quad \mathbf{F}_a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, the coefficients p_{bj}, p_{cj} $(j = 0, 1, 2)$ can be found using the appropriate values for $\mathbf{F_b}$ and $\mathbf{F_c}$. Using an augmented matrix for $\mathbf{F}$, the system of equations for finding the coefficients of $F_i(x)$can expressed as $\mathbf{A}\cdot\mathbf{P} = \mathbf{F}$ where $\mathbf{F}$ turns out be the identity matrix and $\mathbf{P}$ is a 3-by-3 matrix whose columns are p_{aj}, p_{bj}, and p_{cj} $(j = 0, 1, 2)$.

Thus, the interpolated value, $^{L}v_x$, for the Lagrange polynomial passing through u_a, u_b, and u_c is

$$^{L}v_x = \mathbf{u}^T(\mathbf{A}^{-1})^T\mathbf{x} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}$$

This can be generalized to an n$^{\text{th}}$ degree polynomial. The "Significant Simplification" can also be used.

The following is an illustration of the use of the Lagrange polynomial in matrix form.

It is desired to find the slope of the second degree polynomial passing through u_m, u_o, and u_1 at u_o. The answer is

$$^{mo1}v'_{s|s=0} = [\,u_m \quad u_0 \quad u_1\,](\mathbf{A}^{-1})^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ where } \mathbf{A} = \begin{bmatrix} 1 & m & m^2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{s} = \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}.$$

Thus $\mathbf{s}' = \begin{bmatrix} 0 \\ 1 \\ 2s \end{bmatrix}$ and $\mathbf{s}'_{|s=0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Thus, the elements of the second row of $\mathbf{A}^{-1}$ are the coefficients of u_m, u_0,and u_1.

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{m(1-m)} & -\frac{m+1}{m} & \frac{m}{m-1} \\ \frac{1}{m(m-1)} & \frac{1}{m} & -\frac{1}{m-1} \end{bmatrix}$$

If $m = -1$, the formula ${}^{mol}v'_{s|s=0} = [\, u_m \quad u_0 \quad u_1 \,](\mathbf{A}^{-1})^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ becomes the familiar

$$v'_{s|s=0} = \frac{u_1 - u_{-1}}{2}$$

Appendix B

The formula $v_s = \mathbf{u}^T(\mathbf{A}^{-1}\mathbf{F})^T\mathbf{s}$ (or ${}^L v_x = \mathbf{u}^T(\mathbf{A}^{-1})^T\mathbf{x}$) can be executed in two different ways:

1. $\mathbf{u}^T(\mathbf{A}^{-1}\mathbf{F})^T$ results in a vector , the elements of which are the respective coefficients of s^o, s^1, s^2, etc. This can be useful when many values of s are required, such as, in plotting a graph for a given set of pivotal points.

2. $(\mathbf{A}^{-1}\mathbf{F})^T\mathbf{s}$ results in a vector of factors to applied to the respective pivotal points u_m, u_0, u_1 and u_n for a given value of s. If only a few values of s are required for many sets of pivotal points (each set spaced the same), $(\mathbf{A}^{-1}\mathbf{F})^T\mathbf{s}$ can be calculated for each value of s to be applied to the respective pivotal points.