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*C*omprehensive, *I*ntuitive, I*n*teractive *W*riting (*WINC*):

An Actuarial Pedagogical and Problem Solving Approach

by

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SECTION: *INTRODUCTION--The Problem*

Most instructors agree that students should be familiar with the *derivations* of formulae as well as the formulae themselves. Instructors would furthermore agree that it is preferable that students be able to justify formulae both *intuitively* as well as algebraically. Finally, most instructors would agree that a superior approach to problem solving is the use of a *core set of comprehensive techniques* by which to attack problem domains.

The obstacles to the implementation of these pedagogical goals are well known: For example, (a) it is difficult to find sufficient problem resources at current student levels for students to practice writing derivations; (b) student skills in writing derivations are typically so poor that *attempted* performance of routine exercise sets (even if they did exist) would not result in significant increases in competency levels.

The purpose of this presentation is to remedy these obstacles. More specifically this paper presents *semi-algorithmic* methods for the routine production of problem resources whose solutions require skillful writing of derivations. Furthermore, these same semi-algorithmic methods also enable students to routinely achieve success in *derivation-writing* exercises.

The semi-algorithmicity of the methods enables instructors without special training, by following some simple procedures, to routinely produce rich challenging *derivation-writing* problems. Students also, without special training, can, by following some simple procedures, routinely achieve significant success in these exercises and significantly increase their skill levels.

In summary, *the basic thesis of this paper is that student capability to iNteract with new problems can be significantly increased by frequently exposing them to Written derivations, verbal-Intuitive justification of formulae and core sets of Comprehensive techniques by which to approach problems. Both instructor composition of problem resources for presentation and student practice of these problem resources can be achieved by semi-mechanical procedures accessible to everyone. For brevity we shall refer to this type of instruction by the acronym WINC.*

An outline of the rest of this paper is as follows: In section 2 we rapidly review the educational process and current methods of pedagogy. In section 3 we very briefly explore the basic components of *WINC* and show how they contrast with the more traditional methods mentioned in section 2. Sections 4,5 and 6 respectively study *intuitiveness, comprehensiveness,* and *writing* in more detail using examples from the actuarial literature. Section 7 presents a list of topics from the actuarial literature that lend themselves naturally to *WINC* approaches. Section 7 also contains a collection of problems from recent actuarial exams which we hope will prove a valuable resource to both the instructor and student.

SECTION II: *EDUCATION & PROBLEM SOLVING:--What is known*

The basic educational process is familiar to everybody. Of the following four items---

EDUCATIONAL GOALS   *COURSE SEQUENCES/SYLLABI*
PEDAGOGY//TEXTBOOK   *ASSESSMENT//PROBLEM SETS//EXAM QUESTIONS*

this paper will concentrate on *pedagogy & assessment*. In other words, we suppose that the instructor is given actuarial educational goals as well as course sequences and syllabi. For any *given* syllabus topic we then describe the *presentation modes* (or *problem modes*) available to the instructor that relect his/her particular bias of pedagogy. In the sequel we shall equivalently use the terms *presentation (mode)* and *problem (mode)* to emphasize that our comments apply equally to classroom presentation and exams.

A rich literature exists--e.g.[Bloom], Piaget [Gredler], [Sutton & Ennis], [Van Hiele] ---on *levels of problem-presentation difficulty*. In this section we present 8 *levels* or *presentation modes* of difficulty using a process which we believe is more specific and less abstract than the Van Heile and Bloom hierarchies.

For purposes of illustration we suppose an instructor is teaching the syllabus topic--*compound interest*--which is connected with the well-known formula

$$A = P\,e^{rt}$$

where P,r,t and A represent a principle value, P, that accumulates at rate r over t units of time to accumulated value A. The eight problem levels are presented below in table 1 whose interpretation should be clear. We further clarify the table's meaning by interpreting the first few rows.

****

MODE	PROBLEM	VERBAL PROBLEM MODE
PLUG	GIVEN P,r,t FIND A	How much will P accumulate to in t years at rate r.
SOLVE	GIVEN P,r,A FIND t	How long will it take for P to accumulate to A at rate r.
DECIDE	Which is a better investment: Initial deposit P at rate r1 for t1 or at rate r2 for t2?	
MULTI-PART	You have P dollars. Invest p% at rate r1 and q% at rate r2. How much will you have after t years.	
MULTI-METHOD (& SOLVE)	Given P dollars, what compound interest rate r2 is equivalent to the simple interest rate r1 for t1 years	
ITERATION	If P is invested at rate r1 for t1 followed by a reinvestment at r2 for t2 find A.	
PROCEDURE/ SPREADSHEET	If you invest \$100, \$200, \$300 etc at the end of years 1, 2, 3 etc at interest rate r how much will have accumulated at the end of 25 years	

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TABLE 2.1: *Eight basic presentation/problem modes of pedagogy. Table interpretation is clear and further explained in the text. Note that for notational purposes several "cell entries" are absent.*

In the *plug mode* the instructor illustrates the syllabus topic by presenting problems where the numerical values of P,r and t are given and then computes A. By contrast in the *equation mode* the instructor illustrates the syllabus topic by presenting problems where the numerical values of say A,P and r are given and then computes t. So for example, to illustrate the *equation-verbal mode* the instructor might present the following problem: *"How long will it take for $1000 to double at an annual yield of 5%."*

REMARKS: (a) Dual classifications (e.g. MULTI METHOD-solve) are possible. In particular every mode can be algebraic (default) or verbal(e.g. equation-verbal).

(b) Further subclassifications could have been made based on e g. whether numbers or literals are used (e.g. *"Find the time for $1000 to double at 5%"* vs *"Find the time to double at 5%"*). This however will not be further pursued here.

(c) Table 2.1 is more specific than the Bloom and Van Heile classifications. For example it would appear that the *verbal-decide* mode is *Bloom, level 6---evaluation*---since the problem solver is being asked to make *decisions* and evaluations. Closer examination however shows that all the problem solver is doing is *numerically comparing* A1 and A2. The application of the previously learned skill of numerical comparison to the newly learned compound interest formula should therefore be classified as *Bloom, level 3--application.*

(d) The fundamental thesis of this paper can now be restated in terms of the presentation mode hierarchy given in table 2.1:

Many textbook illustrations occur in the modes of table 2.1 --Bloom levels1,2 and 3. Routine problem resources for higher problem modes seem to be absent. Furthermore even when superior problem resources exist student skill levels are frequently too low to benefit from them. The purpose of this paper is to remedy these deficiencies by introducing three new problem modes: The Writing, Intuitive and Comprehensive modes. We further provide semi-algorithmic methods for both routinely generating rich challenging writing problems (Bloom levels 4,5) as well as enabling students to routinely solve these problems successfully. Our thesis asserts that the resulting increased skill levels in writing, intuitiveness and comprehensiveness should significantly increases student iNteractiveness with new problem situations.

SECTION 3: *THE THREE BASIC IDEAS–WINC*

In section 1 we briefly mentioned the three components of *WINC: Writing, Intuitiveness* and *Comprehensiveness* and indicated how skill competency in these components should increase *iNteractiveness* with new problem situations. Prior to giving detailed examples of these methods in the next three sections, we very briefly review here the meanings of these three components by contrasting them with other modes of pedagogical presentation.

W: WRITING: By writing we refer to the use of derivations in both instruction and assessment. In contrast to the *formula method* which presents formulae and only assesses the student ability to use these formulae correctly, the derivation method seeks problems where the student is *forced to* use derivations. By derivations we refer to ordered sequences of equations which formally justify some formula. Clarifying examples will be presented in section 6, along with methods to produce enriched problems. Examples 6.1 and 6.2 demonstrate that use of *writing* -- derivations vs. formulae-- significantly increases a person's breadth of knowledge..

I: INTUITIVE: In contrast to a methodology that derives formulae *algebraically*, an *intuitive* methodology emphasizes use of verbal-intuitive equations in derivations. Clarifying examples will be presented in section 4. Example 4.3 demonstrates that *intuitiveness* can significantly accelerate the presentation and learning of complex formulae. Example 4.2 demonstrates that *intuitiveness* can be used to unify superficially diverse results.

C: COMPREHENSIVENESS: To explain the meaning of *comprehensiveness* we illustrate using trigonometric identities. One type of instruction methodology would present basic methods and do enough examples so that students feel comfortable proving identities. However, an alternative method would be to present an algorithmic procedure: We would start by clearly defining the problem domain---the trigonometric identities---and then proceed to give a step by step procedure for proving or disproving any identity [Dabs and Hanks]. Of course, the algorithmic method does not always give the most elegant proof and room for human creativity is still present.

Not every domain of thought can be summarized by a comprehensive set of principles. However very often it is useful to give comprehensive principles for important subdomains of knowledge, thus allowing more time to be spent on the nonalgorithmical parts of the subject. As a simple example when dealing with the problem domain of *logical arguments* it is useful to know the algorithmic method of truth table methods which apply only to 0th order logic thus allowing more time to be spent on the non 0th order logical elements of a problem domain.

Clarifying examples of *comprehensiveness* will be presented in section 5. Example 5.3 demonstrates that comprehensiveness can be an extremely useful tool in presenting and solving verbal problems. Another use of comprehensiveness is for student projects. That is, a challenge to good students is to precisely define a domain of problems and then give the step by step procedures and the class of identities that can be universally applied. Such studies enrich appreciation of a subject. A recent example of such a search for *comprehensiveness* occurs with the Fibonacci numbers [Dresel, Hendel -1, Rabinowitz].

N: InTERACTIVENESS: We stated that our overall goal is to increase student *interactiveness*. The fundamental assessment question is NOT, *"How often can students solve the types of problems we have instructed them in,"* but rather, *"How often can students solve new problems only resembling the ones we have instructed in."* A crucial component of *interactiveness* is a precise account of *resembling*. This will be given in section 6.

We now proceed to give mature examples of the above three presentation modes..

SECTION 4: *INTUITIVE*: *Illustrative examples*

EXAMPLE 4.1: The following multiple life identity

$$ {}_nq^1_{xy} = {}_nq^2_{xy} + {}_np_y\ {}_nq_x $$

is derived both verbally (Exercise 8.30 of [Bowers et al]) and algebraically (Equation 8.8.4). The verbal derivation interprets the *meaning* of the actuarial symbols and uses the law of the excluded middle:

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*The life (x) can die first before (y) and n years (*left side*)*

by either

(i) *(y) dying second before n years* (first summand on the right side)

or (ii) *(y) surviving n and (x) dying before n* (second summand on right side).

****

TABLE 4.1: *Verbal derivation/interpretation of the above multiple life identity.*

Other multiple life formulae can be derived similarly. This formula illustrates the principle that *intuitiveness* enhances recall ([Hendel-2]). Note the skillful pedagogic technique of presenting the algebraic derivation in the text--equation 8.8.4--and requiring a verbal derivation in the exercises--exercise 8.30. Such algebraic-verbal pairings are common in [Bowers et al]; other texts are encouraged to follow suit.

EXAMPLE 4.2: The well known *reserve formulae* can be derived by purely algebraic means. The following verbal derivation is also possible:

****

RESOURCES EQUALS REQUIREMENTS

RESOURCES======>*New premiums, previous reserves*

REQUIREMENTS====>*death benefits, future reserves*

EQUALS==>*with respect to forces of interest,mortality*

New premiums	=	π_{h-1}
Previous Reserves	=	${}_{h-1}V$
New Reserves	=	${}_hV$
Benefits	=	b
Present value (forces of interest)	=	v
Mortality/Survival	=	q_{x+h-1} , p_{x+h-1}

****

TABLE 4.2a: *A verbal derivation of the basic reserve formula. The derivation uses the form of a context free grammar.*

The above derivation is written in the form of a *context free grammar*[Lewis and Papadimitriou]. It could also be written as a *(formal) derivation* [Mendelson].

****

RESOURCES		EQUALS	REQUIREMENTS		
New-premiums	+ *Previous-reserves*	=	*Benefits*	+	*future reserves*
π_{h-1}	$+ {}_{h-1}V$	=	$b\ v\ q_{x+h-1}$	+	${}_hV\ p_{x+h-1}$

****

TABLE 4.2b: *Table 4.2b rewrites table 4.2a in the form of a formal derivation.*

We briefly introduce some technical terminology that will prove useful in the sequel: Table 4.2b is a *3 line derivation*. *Line 1* is the bold equation of table 4.2a. The left side of *Line 2* is *derived* from the left side of *line 1* using the following *derivation* (or *reason*) from table 4.2a:

RESOURCES ===> *new premiums, previous reserves*

In the sequel both the *grammatical derivation* (with *yield rules* represented by "="and "======>") as well as the *formal logic derivation* with *line numbers* and *reasons* will be freely used. Although more formal accounts are possible the above should suffice.

The reserve example allows a restatement of the utility of *intuitiveness*: An *algebraic* derivation of reserves can only derive one particular formula (in this case the above reserve formula). However the more plastic *intuitive* derivation can be used to derive other reservè formulae such as those with return of premium or expense reserves: In all cases the same initial verbal equation--- **RESOURCES=REQUIREMENTS** --- is used.

EXAMPLE 4.3: For the more skeptical reader we offer equation 7.3.4 [Bowers et al]

$$ {}_s\bar{V}(\bar{A}_{x:n}) + \bar{P}(\bar{A}_{x:n})\,\bar{a}_{x+s:t} = {}_tE_{x+s}\ {}_{s+t}\bar{V}(\bar{A}_{x:n}) + \bar{A}^{\ 1}_{x+s:t} $$

as a "convincing example" of a formula that can "only" be understood *intuitively* (but not algebraically).

SECTION 5: *COMPREHENSIVE*: *Illustrative Examples*

The basic attributes of comprehensiveness were mentioned in section III in connection with the trigonometric identities. Illustrations of comprehensiveness require three items:

(a) A precise definition of the *problem domain*,

(b) A (complete) *set of techniques* by which to approach them,

(c) A *procedure* for using the complete techniques *to solve* the problem.

EXAMPLE 5.1: (MULTIPLE DECREMENT LIFE TABLE PROBLEMS): 5.1(a) Problems in this domain are characterized by the fact that they give numerical values for certain values of x and n for the five functions-- $l_{x+n}^{(j)}$, ${}_nd_x^{(j)}$, ${}_nq_x^{(j)}$, ${}_np_x^{(j)}$, ${}_{n|}q_x^{(j)}$--and request computation of the numerical value of one of these five functions for some other specific value of x and n and j. (j may be an integer or τ (all decrements); we however do not allow superscript primes (single decrement probabilities in multiple decrement environments)).

5.1(b): The three fundamental identities governing *l* are the following: (i) $l_{x+n} = l_x - {}_n d_x$ (ii) $l_{x+n} = l_x p_x p_{x+1} \cdots p_{x+n-1}$ (iii) $l_{x+n} = 0$ if x+n is at least equal to the maximal attainable age.

For notational convenience we have omitted superscripts. Similar rules can be written down for the other life functions. In total nine rules suffice (the other five rules are the following: *the relations between: (iv) p and q (v) l, d and ${}_n q_x$ (vi) l,d and ${}_{n|} q_x$ (vii) $q^{(\tau)}$ and the $q^{(j)}$ (viii) $d^{(\tau)}$ and the $d^{(j)}$ and (ix) ${}_n d_x$ and d_{x+i}, $0 \le i \le n-1$.* These nine rules form a *comprehensive* rule set for this domain.

5.1(c): The procedure for "solving" a life table problem is to "write down" in the table the problem's given values and then repeatedly use the above 9 rules to generate "new table values" until the problem's requested value is found.

REMARKS: We deliberately defined the problem class in such a way so as to avoid "superscript primes" (single decrement survival probabilities in multiple decrement environments). If we allowed "primed survival probabilities" then the above set of rules would not suffice; they would have to be augmented with the various single-multiple decrement "ratio rules" (e.g.[Bowers et al, 9.5.9]

EXAMPLE 5.2: (INSURANCE ANNUITY PROBLEMS) 5.2(a): Problems give values of insurances and annuities (but not reserves and premiums) for various ages and periods and request calculating the value of an insurance or annuity for some other specific age and period. 5.2(b): *A comprehensive set of six identities are the relationships between the following: (i) insurance and annuities, (ii) pure and ordinary endowments, (iii) continuous and discrete insurances (under UDD), (iv) recursive formulae (e.g. relating $A_{x:n}$ and $A_{x+1:n-1}$), (v) "initial values" of annuities and insurances (e.g. $a_{x:1}$, $A_{x:1}$---both continuous and discrete), (vi) whole and term insurances.* 5.2(c): The procedure to solve problems is to repeatedly use applicable identities on the *given* values to derive new values until the value requested in the problem is found.

The next example combines *comprehensiveness* with *intuitiveness*.

EXAMPLE 5.3: The following table gives a *comprehensive* set of correspondences (i.e. a "dictionary") useful for functional interpretation in verbal maximum likelihood problems. The meaning of the table should be clear. The analysis into problem domain, rules and procedures (a,b and c) should be clear and will be omitted.

VERBAL:	**FUNCTIONAL:**		**LIFE FUNCTIONS:**
DIED AT t	f(t)		${}_t p_x \, u_{x+t}$
DIED BY t	F(t)		${}_t q_x$
SURVIVED TO x+t	S(x+t)		S(x+t)
ENTERED STUDY AT x	DIVIDE BY S(x)	e.g.	S(x+t)/S(x)

TABLE 5.3: *A comprehensive list of verbal-functional correspondences for maximum likelihood problems. The usage of this table as an aid in solving problems should be clear. Note that the last two lines of the table could be combined into one; they were separated to emphasize the difference between problems on newborns and other 'entries at an age.'*

The use of comprehensive correspondence tables for verbal problems may be controversial (e.g. the argument "*let the students think for themselves*"). However I have found such devices extremely useful with students at *all* levels. Furthermore comprehensiveness does *not* preclude further problem complexity. For example, a standard procedure to enrich maximum likelihood problems is to assume a uniform force model with different forces of mortality operating in different years.

REMARKS: (a) (Historical note) Comprehensive identity lists for the proof of trigonometric identities may be found in [Dabs and Hanks]

(b) In any given problem domain there is no one "right" set of comprehensive rules. Thus the finding of such comprehensive sets of rules makes excellent student projects. The hard part is the accurate definition of the problem domain.

(c) In example 5.1 we had to expand the traditional multiple life table with its four functions (l_x, d_x, q_x, p_x) to include $_{n|}q_x$. Such "minor changes" are common in these problems. (Provisions for $_nd_x$ and $_nq_x$ for n>1 should also be made).

SECTION 6: *WRITING: W: Illustrative Examples*

This is the most fruitful of the presentation modes that are studied in this paper. The terms *writing*, *derivation* or *essay* shall be used interchangeably.

EXAMPLE 6.1:

****

$q_x^{(1)}$ $= \int {}_tp_x^{(\tau)}\ u_{x+t}^{(1)}\ dt$ .... *Density-cumulative*

$= \int {}_tp'^{(1)}_x\ u_{x+t}^{(1)}\ p'^{(2)}_x\ dt$ .... *MD-SD*

$= q'^{(1)}_x \int {}_tp'^{(2)}_x\ dt$ .... *UDD-density*

$= q'^{(1)}_x \int (1-tq'^{(2)}_x)\ dt$ .... *UDD-fractional*

$= q'^{(1)}_x (1-q'^{(2)}_x/2)$ .... *Integration*

****

TABLE 6.1(a): *Example of a derivation (equation 9.6.3 in [Bowers et al]) The notation and terminology for derivations was illustrated in example 4.2 above. Thus we could say that line 4 of the above derivation is derived from line 3 by applying the formula* $_tp'^{(2)}_x = 1-tq'^{(2)}_x$ *whose justification (or reason) is the UDD assumption on fractional ages.*

A crucial component of *writing problem* composition is the skillful construction of problems whose solution requires a *similar equation derivation* that differs from a given one in say one or two *lines*. Heuristically we may say that the constructed problem *breaks the derivation* by *modifying a given line*. We immediately clarify with problem 150-8-21-- (Problem 21, exam 8, course 150 in [ACTEX --*Course 150*])---which requires *modifying line 4* of the above derivation by replacing the *UDD assumption* with a *point mass assumption*.

PROBLEM 150-8-21(slightly reworded): *A multiple decrement table has 2 decrements, (1) and (2). DECREMENT (2) OCCURS ONCE A YEAR 3/4-ths OF THE WAY THRU THE YEAR OF AGE. Decrement (1) in the associated single decrement table is uniformly distributed over each year of age. You are given* ${}_tp_x^{(\tau)}$ *and* $q'^{(2)}_x$. *Find* $q_x^{(1)}$.

(The actual problem asks for $d_x^{(1)}$ and gives ${}_tl_x^{(\tau)}$ and ${}_tl_{x+1}^{(\tau)}$. It further uses deaths and withdrawals vs. decrements (1) & (2))

SOLUTION:

****

$q_x^{(1)}$ $=\int {}_tp_x^{(\tau)} u_{x+t}^{(1)} dt$ .... *Density-cumulative*

$=\int {}_tp'^{(1)}_x u_{x+t}^{(1)} p'^{(2)}_x dt$ .... *MD-SD*

$= q'^{(1)}_x \int {}_tp'^{(2)}_x d$ .... *UDD-density*

$= q'^{(1)}_x \int_{0\le t\le .75} 1\, dt \quad +$

$\int_{.75<t\le 1} (1-q'^{(2)}_x)\, dt$ .... *Point-Mass*

$= q'^{(1)}_x (0.75+0.25(1-q'^{(2)}_x))$ .... *Integration*

****

Table 6.1(b): *Solution to problem 150-8-21. The similarity between tables 6.1(a) and 6.1(b) is further pursued in the text.*

It is immediately seen that the problem solution presented in table 6.1b *modifies line 4* of the derivation presented in table 6.1a by replacing the UDD assumption with a Point-Mass assumption. Equivalently we can say that table 6.1b *breaks* table 6.1a at *line 4*.

REMARKS: (a)Pedagogy theory would describe the above problem-solution as an example of the challenging *analysis stage--Bloom, level IV*--perceiving a learned item in terms of component parts.

(b) Our methodology is *semi-algorithmic*---Problem 150-8-21(reworded version) simply takes an equation *derivation* and creates a new problem by an appropriate assumption that *breaks* the *derivation* by *modifying a line* at an appropriate point. And nevertheless, it is a challenging problem.

(c) The actual version of problem 150-8-21 (vs our reworded version) has the added attribute of *naturality*--the multiple decrement derivation nature of the solution is partially disguised by consideration of withdrawals and deaths coupled with a plausible real world withdrawal assumption. While we posit that the construction of challenging, pedagogically useful writing problems is semi-algorithmic, nevertheless, the construction of *natural* problems requires the artistic skill and creativity traditionally associated with such problems.

EXAMPLE 6.2: Table 6.2 below gives the standard intuitive verbal derivation for the continuous whole life insurance formula under constant force assumptions. The solution to problem 150-9-5 below modifies this derivation.

****

Fair value for

Whole life insurance = Expected cost of insurance. .... *Fair=Expected*

= Actuarial value.. *Expected=Actuarial*

= Expected (present value)..................................... *Actuarial=Expected present*

=E(PV(t) f(t)).. *Functional notation*

$=\int e^{\delta t} ue^{-ut}\, dt$.. *PV and density formulae*

= $u / (\delta + u)$.. *Integration*

****

TABLE 6.2: *Verbal intuitive derivation of the whole life insurance formula under constant force assumptions*

PROBLEM 150-9-5 (reworded): *Find the value of a continuous whole life insurance issued to (50) with mortality following DeMoivre's law (ω=100), with death benefit b_t = 1000-0.1t² and with simple interest of 1%.*.

SOLUTION: (*Sketch*) The derivation in table 6.2 is *modified* in the *fifth line*: The present value function is simple interest instead of the usual compound interest and the benefit function b_t has been *picked out of a hat* in such a way that the resulting integrand can be integrated.

Example 6.2 is useful in clarifying the utility and nature of *writing* problems. PROBLEM 150-9-5 is not intrinsically difficult but it WILL stump the *formula student*. The problem whether presented in class, homework or on exams encourages students in developing the skills of *analysis into component parts* and *the capacity to generalize component parts by replacing them with similar parts.*

REMARK: We suggest that the above analysis may give definitional insight into the difference between *rote* learning vs. *conceptual* learning:

When using the *formula method* of instruction for each "formula" learned there can be basically only one 'type' of problem with that formula (in the *plug*, or *solve*, or *decide*...mode--the point being that we regard two problems as being of the same 'type' if they use the same formula). Thus the ratio between " 'types' of problems that can be solved" and "formulae learned" is 1. We therefore term this type of learning as *rote* since, as the one to one ratio suggests, you can only solve what you learned..

On the other hand suppose each line in, say, table 6.1a can be *modified* in only two ways (independent of each other). There are five lines: Hence since each line can be modified in two ways there are ten formulae to learn. However, the total number of problems that can be made (one for each distinct derivation) is $2^5 = 32$. Thus the ratio of problems that can be solved to formulae learned is 3.2. If lines could be *broken* in more than 1 way (or if we had started with a derivation that had more than five lines) then this ratio increases further.

We therefore term this type of learning as *conceptual* since more can be solved than was learned. In conclusion we see that the ability to compose and solve *Writing* problems--particularly when derivations are based on *Intuitive* formulae and the totality of *Comprehensive* techniques available (which maximizes the degree by which derivations can be modified)--- enhances student *iNteractiveness* with new problems.

SECTION 7: *FURTHER EXAMPLES*--*Useful classroom ideas/problem lists*

In the preceeding six sections we have

--reviewed the educational process,
--carefully defined traditional modes of presentation,
--introduced three further problem modes--*WINC*,
--presented clarifying examples for *WINC*,
--argued that WINC enhances increased *interactiveness*,
--suggested a definition of *interactiveness*.

The primary purpose of this section is to show how to implement the introduction of the *WINC* presentation modes in the upper level sequence 100 actuarial courses---courses 150, 151, 160 and 165. The section also contains problem collections of use to both students as well as instructors who thru skillful, classroom, problem selection can enrich subject appreciation by suggesting a more refined development of the syllabus.

For reasons of space we have mostly restricted ourselves to illustrations of the *intuitive-writing* mode. For purposes of completeness we include some examples of *comprehensiveness*.

These examples of *writing* will be presented using *context free grammars* and *formal derivations* as illustrated in tables 4.2a and 4.2b. Table 7.1 below repeats and further develops example 4.2 and will be used to review and illustrate our notation.

TABLE USAGE: (a) *Grammar interpretation* (4.2a): *Root level* equations are presented in **bold**; level one equations (the next level down) are in **bold**==>*italics*; level two equations are in *italics*===>non italics. Tables 4.2a and b illustrate the method of transforming *context free grammars* into *formal derivations*.

TABLE USAGE:(b) *Curriculum design and development*. By checking the course 150 text for the table 7.1 topic, we see that reserve formulae are developed in chapter 7 (basic formula), sections 14.2(general expense theory) and 14.3--14.4(types of expenses). This illustrates the curriculum technique of *spiral design*.

TABLE USAGE: (c) *Problems*: Problems may involve (c1) the plug mode-- *based on duration of reserve, payment years, duration of insurance, survival model distribution etc.* (problems 150-13-28, 10-23,8-13, 12-28), (c2) the iterative mode *using recursive formulae* [Bowers et al 7.8](problems 150-8-14, 13-5, 11-13, 14-17), (c3) the solve mode, *using the seven types of reserve formulae-- retrospective, prospective--etc.* [Bowers et al, exercise 7.5]), (problems 150-8-15, 10-13, 14-10, 11-23, 13-16, 11-3 14-30, 12-8, 9-20) (c4) total grammar development, *using expense reserves* (10-15, 11-2, 10-9, 8-26, 9-26, 14-6, 13-9--note: these problems are typically premium, not reserve, but were included here since the principles are related). Note that further augmentation of both the grammar and problems could be done by e.g. including modified reserve techniques (150-12-20, 8-25, 13-25, 11-21, 10-25, 14-26, 13-7, 9-28, 8-24, 12-10, 11-15).

EXAMPLE 7.1:

****

RESERVE PROBLEMS:--*COURSE 150*

RESERVES ======> RESOURCES, ACTUARIALLY EQUALS, REQUIREMENTS

RESOURCES =============> *payments, (last year) reserves, loadings*
ACTUARIALLY EQUALS ====> *with respect to: interest, mortality*
REQUIREMENTS ==========> *benefits, (this year) reserves, expenses*

Expenses =========> Per premium dollar, Per policy, Per year, Fixed

****

TABLE 7.1: *Context free grammar for reserve problems in course 150. Some good problems are (reserve definitions)150-13-28, 10-23, 8-13, 12-28, (recursive formulae) 8-14, 13-5, 11-13,14-17, (retrospective-prospective) 8-15, 10-13, 14-10, 11-23, 13-16, 11-3, 14-30, 12-8, 9-20 (expense reserves/premiums)(10-15,11-2,10- 9,8-26,9-26, 14-6,13-9).*

EXAMPLE 7.2:

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VERBAL PREMIUM PROBLEMS:---*COURSE 150*

EQUIVALANCE PRINCIPLE ======> PAYMENTS, BENEFITS

PAYMENTS =========> *Single, stream, continuous, discrete*
BENEFITS =========> *benefits, premium-return, expenses*

Premium-return ====> With Interest, Without Interest
Expenses ======> Per Premium Dollar, per Policy, per Year, Fixed

****

TABLE 7.2: *Context free grammar for verbal premium problems in course 150. Some good problems are 150-11-30, 14-4, 12-1, 8-11,12-24,8-12,11-2, 10-9, 8-26, 9-26, 14-6, 13-9 & 14-18.Note that these problems are classified in the verbal-writing-intuitive mode.*

EXAMPLE 7.3:

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EXPECTED VALUE AND VARIANCE OF (AGGREGATE CLAIMS):*COURSE 151*

AGGREGATE AVERAGE/VARIANCE ====> Function of: **CLASS, n, q, DISTRIBUTION**

DISTRIBUTION =====================> *Known, Discrete*

Known ===========================> μ and σ from formula involving parameters.
Discrete ========================> μ and σ from {q,b}

****

TABLE 7.3: *Context free grammar for aggregate average/variance problems. Note that the above derivation includes the following four situations: Aggregate Average equals (i) nqb, (ii) Σnqb,(the sum being over all classes), (iii) $\Sigma nq\mu$ where μ is given and (iv) $\Sigma nq\mu$, where μ must be computed by formula from parameters. Similar remarks can be made for the variance. Some good problems are 151-6-4, 6-5, 4-6, 1-4, 3-8(part I), 7-8, 5-7, 8-7, 7-9, 7-4, 6-8, 5-9, 4-7 (parts I and II), 4-9, 2-8, 1-10, 1-7, 6-7, 8-8 (parts I and II). Many of these problems are in the solve mode--e.g. they may involve 'solving' a generating function for its parameters.*

EXAMPLE 7.4:

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MORTALITY ESTIMATION: *COURSE 160*

SURVIVAL EQUATION =====> By: **MODEL, DISTRIBUTION, DATA , DECREMENTS**

BY MODEL ======================> *Moment model, maximum likelihood*
BY DISTRIBUTION ASSUMPTIONS ====> *Type A,B,C (linear, hyperbolic...)*
BY DATA INFO ===========> *Complete, incomplete*
BY DECREMENT TYPES ============> *Single, Double*

Moment model ===============> Actuarial, Hoem, Schwartz Lazar

****

TABLE 7.4: *Context free grammar for estimation of mortality probabilities. Note that the grammar is (deliberately incomplete)--further items could be included--for example, the Kaplan-Meier or the Nelson-Aalen estimates About 27% of the recent exam problems in [ACTEX 160] are mortality estimation. Some good problems are (ordinary moment--entry, death, withdrawal, enders--analyzed by the three moment methods--moment, actuarial, Hoem) 160-5-8, 2-11, 5-19, 6-17, 3-7, 3-11 (ordinary moment--using calendar dates vs. age) 6-10, 6-9, 2-14, 2-13, 1-1, 7-9, 3-6, 4-13, 7-17 (variance) 7-6, 3-10, 5-7, 1-8, 3-8, 5-6, 6-6, 7-5 (bias) 5-11, 2-12, 1-9 (distibution assumptions) 4-2, 7-10, 4-1, 4-4.*

EXAMPLE 7.5:

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PROFIT/LOSS PROBLEMS: *COURSE 151*

TYPICAL PROBLEM: *Find the security loading yielding a 5% chance of loss.*
MODEL SOLUTION:
P(Loss(Premium)) = 0.05... *Problem statement*

P(Observed > Premium) =0..05.... *Loss Observed Charged*
claims charged

P(S > Expected claim)=0.05.... *Premium Charged expected,security (Or loading)*
with security

P(S > (1+ θ)E(S)) = 0.05.... *Loaded Premium = (1+ θ)E(S)*

P(**Z** > θ E(S)/σ) =0.05.... *Normalization*

1 - Φ(θ E(S)/σ)) =0.05.... *Normal curve function*

θ = 1.645 σ/E(S).... *Algebra*

****

TABLE 7.5: *Formal derivation for the "security loading for profit" problem described above. This problem spans over 10% of recent [ACTEX151] problems. Most problems ask to compute a security loading; however several problems use the solve mode and seek computation of benefits or the number of members. Some good problems are 151-1-5,2-6, 4-4, 6-6, 8-4, 3-10, 8-5, 3-7, 7-6, 5-4, 3-4, 4-5, 5-6, 2-4, 7-5, 3-6.Problem wordings can equivalently ask for obtaining profit, avoiding loss, etc. ...*

EXAMPLE 7.6: *INSURANCE-ANNUITY PROBLEMS:* For purposes of completeness we give two examples of *comprehensiveness*. Expanding on example 5.2 we give the following comprehensive list of nine *insurance-annuity-premium identities*: The standard identities between *(i) insurances & annuities (ii) whole & term, (iii) recursive formulae, (iv) initial values, (v) pure endowment & endowment, (vi) continuous & discrete, (vii)present & accumulated value, (viii) fractional period & annual period, (ix) premium-insurance-annuity.* Some good problems are 150-10-3, 11-29, 13-13, 10-27, 13-10, 11-5, 8-9, 13-23, 12-30, 14-1, 14-28. Note that formulae from other problem domains,such as those relating d-i-δ, may be needed.

EXAMPLE 7.7: *MAXIMUM LIKELIHOOD PROBLEMS*: We present sample problems for the maximum likelihood verbal problems mentioned in example 5.3. The interested reader is thereby afforded the opportunity to assess the usefullness of using *function-verbal dictionaries* (table 5.3) in verbal problems. Problems come from two courses:*(Traditional MLE) 160-3-16, 4-11, 2-20, 3-20, 7-15, 4-3, 1-14, 1-2, and 165-1-10, 165-3-10, 165-8-10, 165-6-11, 165-2-10, (with constant force assumption) 160-1-1, 160-6-13, 160-3-18,1-4,2-18,7-14,3-13,(uniform and other distribution survival mortality assumptions) 6-12, 7-11,5-15, 4-12,2-16, 2-15, 7-12.*

EXAMPLES 7.8: *MISCELLANEOUS*: We close with a collection of problem classes whose underlying *grammars* are one level (besides the *root*).For example the moment generating function of the loss variable can be stated in terms of the "first-loss" variable (*root level*) or the "claims" variable (*level one*). These *writing* problems are best developed thru *spiral curriculum* and offer a moderate degree of challenge to students encountering *writing* for the first time.

LINEAR REGRESSION: **Ordinary** (root),& *Logarithmic* (level one)-COURSE 160
UTILITY OF PREMIUMS:**Without reimbursement**, *with reimbursement*- COURSE 151
RUIN:As a function of **adjustment coeficient**; *As a function of security loading*- COURSE 151
FORFEITURE: **Without loan and pure endowment** and *with them*---COURSE 150
ADJUSTMENT COEFFICIENT EQUATION: **Without reinsurance** and *with*--COURSE 151
CLOSENESS: **M = A function of F,h,s**; *M=Function of w,v,u,the Δ operator,powers*-COURSE 160
MGF OF L: **As a function of L_1**; *As a function of X*---COURSE 151

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