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Integer Functions, UDDYA, and Annuity Coefficients

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1. Integer Functions

For a real number t, let $\lceil t \rceil$ denote the *ceiling* of t, which is the least integer greater than or equal to t, and let $\lfloor t \rfloor$ denote the *floor* of t, which is the greatest integer less than or equal to t. Some authors write $\lfloor t \rfloor$ as [t]. If T = T(x) denotes the random variable of the future lifetime or the time until death of a life now aged x, then $\lfloor T \rfloor$ = K, the curtate-future-lifetime of (x), and $\lceil T \rceil$ is the time until the end of the year of death of (x). Because 12T is the time, measured in months, until the death of (x), we see that $\lceil 12T \rceil$ is the time, measured in months, until the end of the month of death of (x), and hence $\frac{1}{12}\lceil 12T \rceil$ is the time, measured in years, until the end of the month of death of (x). Similarly, $\lceil 52T \rceil/52$ gives the time, measured in years, until the end of the week of death of (x), and so on. Thus we have, for each positive integer m,

$$A_x^{(m)} = E\left[v^{\frac{\lceil mT \rceil}{m}}\right], \tag{1.1}$$

$$\ddot{a}_x^{(m)} = E\left[\ddot{a}^{(m)}_{\overline{\lceil mT \rceil / m|}}\right], \tag{1.2}$$

and

$$a_x^{(m)} = E\left[a^{(m)}_{\overline{\lfloor mT \rfloor / m|}}\right]. \tag{1.3}$$

For two positive numbers s and t, we define

$$t \bmod s = t - s\lfloor t/s \rfloor \tag{1.4}$$

and

$$t \text{ pad } s \;=\; s\lceil t/s\rceil \;-\; t. \tag{1.5}$$

The quantity "t mod s" is the (nonnegative) remainder when t is divided by s, while "t pad s" is the least nonnegative addition to t so that the result is divisible by s. The term *mod*, short for *modulo*, is standard mathematical usage. In defining *pad*, we are borrowing from computer science, in which the term *padding* means the adding of blanks or non-significant characters to the end of a block or record in order to bring it up to a certain fixed size.

Actuarial Mathematics [formula (3.6.1)] defines the random variable

$$S \;=\; T \;-\; K \;=\; T \;-\; \lfloor T\rfloor \;=\; T \text{ mod } 1. \tag{1.6}$$

Since S is the fractional part of T, we have $0 \le S < 1$. We make the assumption that

$$\Pr(S = 0) \;=\; 0; \tag{1.7}$$

hence

$$\lceil S\rceil \;=\; 1 \tag{1.8}$$

with certainty.

2. The UDDYA Assumption

The assumption of a *uniform distribution of deaths throughout each year of age* (or each policy year) may be characterized as follows: for each positive number t, t not an integer,

$$\begin{aligned} \ell_{x+t} \;&=\; (\lceil t\rceil - t)\ell_{x+\lfloor t\rfloor} \;+\; (t - \lfloor t\rfloor)\ell_{x+\lceil t\rceil} \\ &=\; (t \text{ pad } 1)\ell_{x+\lfloor t\rfloor} \;+\; (t \text{ mod } 1)\ell_{x+\lceil t\rceil}. \end{aligned} \tag{2.1}$$

Dividing (2.1) by ℓ_x and rearranging yields

$${}_tp_x \;=\; {}_{\lfloor t\rfloor}p_x \;-\; \big(t - \lfloor t\rfloor\big)\,{}_{\lfloor t\rfloor|}q_x. \tag{2.2}$$

Differentiating (2.2) with respect to t, we obtain

$$\frac{d}{dt}\,{}_tq_x = {}_{\lfloor t \rfloor}q_x, \tag{2.3}$$

or

$$d\,{}_tq_x = {}_{\lfloor t \rfloor}q_x\,dt. \tag{2.4}$$

Let $I(\cdot)$ denote the indicator function:

$$I(\mathcal{E}) = \begin{cases} 1 & \text{if } \mathcal{E} \text{ is true} \\ 0 & \text{if } \mathcal{E} \text{ is false} \end{cases}.$$

The proof of the following lemma will be given in the next section.

Factorization Lemma Let g be a periodic function with period 1, and let

$$f(t) = \sum_j a_j I(j < t \le j+1), \tag{2.5}$$

where $\{a_j\}$ are constants. It follows from the UDDYA assumption that

$$E[g(T)\,f(T)] = E[g(S)]\,E[f(T)] = [\int_0^1 g(s)\,ds]\,E[f(T)]. \tag{2.6}$$

In defining the function f in (2.5), the inequality "<" in the indicator function can be replaced by "≤" and "≤" can be replaced by "<". The factorization formula (2.6) holds as long as f is a step function with step size = 1. In the context of life contingencies, we usually consider step functions of the form

$$f(t) = f(\lfloor t \rfloor), \qquad t \ge 0,$$

or

$$f(t) = f(\lceil t \rceil), \qquad t \ge 0.$$

Under UDDYA, because

$$E[g(S)] = \int_0^1 g(s)\,ds, \tag{2.7}$$

the expectation $E[g(S)]$ is the *average* of $g(s)$, $s \in [0, 1]$. See Figure 1 for a graph of the periodic function $\lceil t \rceil - \frac{\lceil mt \rceil}{m}$ with $m = 4$.

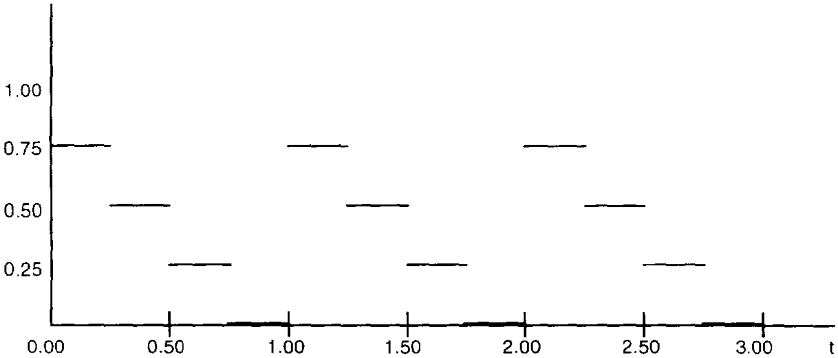

Figure 1. Graph of the periodic function $\lceil t \rceil - \frac{\lceil 4t \rceil}{4}$

Consider an insurance policy with death benefit depending only on the policy year of death (or the age last birthday at death). That is, there is a sequence of nonnegative numbers $b_1, b_2, b_3, \ldots$ such that the death benefit is b_j if death occurs in policy year j (of if the age at death is $x+j-1$). Moreover, assume that the death benefit is payable at the end of the m-thly time interval in which death occurs. Then the insurance's present value random variable is

$$Z = b_{\lceil T \rceil} \, v^{\frac{\lceil mT \rceil}{m}}. \tag{2.8}$$

We define the single premiums

$$A^{(m)} = E[\, b_{\lceil T \rceil} \, v^{\frac{\lceil mT \rceil}{m}} \,], \tag{2.9}$$

and

$$A = E[\, b_{\lceil T \rceil} \, v^{\lceil T \rceil} \,]. \tag{2.10}$$

Consider the step function

$$\begin{aligned} f(t) &= b_{\lceil t \rceil} \, v^{\lceil t \rceil} \\ &= \sum_{j \ge 0} b_{j+1} v^{j+1} \, I(j < t \le j+1), \end{aligned} \tag{2.11}$$

and the periodic function

$$g(t) = (1+i)^{1-\frac{\lceil mt \rceil}{m}}.$$

Then

$$E[f(T)] = A,$$

$$E[g(T)\, f(T)] = E[\, b_{\lceil T \rceil}\, v^{\frac{\lceil mT \rceil}{m}} \,] = A^{(m)},$$

and

$$E[g(S)] = E\left[(1+i)^{1-\frac{\lceil mS \rceil}{m}}\right].$$

Now under UDDYA, S is a uniform random variable on [0, 1]; hence we have

$$\begin{aligned}
E[g(S)] &= \int_0^1 (1+i)^{1-\frac{\lceil ms \rceil}{m}} ds \\
&= \int_0^{\frac{1}{m}} + \int_{\frac{1}{m}}^{\frac{2}{m}} + \cdots + \int_{\frac{m-1}{m}}^{1} \\
&= \int_0^{\frac{1}{m}} (1+i)^{1-\frac{1}{m}} ds + \int_{\frac{1}{m}}^{\frac{2}{m}} (1+i)^{1-\frac{2}{m}} ds + \cdots + \int_{\frac{m-1}{m}}^{1} (1+i)^{1-1} ds \\
&= \sum_{j=0}^{m-1} (1+i)^{1-\frac{j+1}{m}} \frac{1}{m} \\
&= s_{\overline{1|}}^{(m)} \qquad (2.12) \\
&= \frac{i}{i^{(m)}}. \qquad (2.13)
\end{aligned}$$

Hence, by the Factorization Lemma, we have

$$A^{(m)} = \frac{i}{i^{(m)}} A, \qquad (2.14)$$

which is (4.4.6) of *Actuarial Mathematics.*

3. Average Value Theorem

Let k be an integrable periodic function with period p. Then the value of the integral

$$\int_y^{y+p} k(t)\, dt$$

is independent of the lower limit y. The *average value* of the function k is

$$\text{Avg}(k) = \frac{1}{p}\int_y^{y+p} k(t)\, dt. \tag{3.1}$$

The following theorem is reminiscent of the weighted mean value theorem for integrals in elementary calculus.

Average Value Theorem Let k be an integrable periodic function with period p, and let

$$h(t) = \sum_j c_j I(d_j < t \leq d_j + p), \tag{3.2}$$

where $\{c_j\}$ and $\{d_j\}$ are two sequences of constants. Then

$$\int_{-\infty}^{\infty} k(t)\, h(t)\, dt = \text{Avg}(k) \int_{-\infty}^{\infty} h(t)\, dt. \tag{3.3}$$

Proof The left-hand side of (3.3) is

$$\begin{aligned}
\int_{-\infty}^{\infty}\left[k(t) \sum_j c_j I(d_j < t \leq d_j + p)\right] dt &= \sum_j c_j\left[\int_{-\infty}^{\infty} k(t)\, I(d_j < t \leq d_j + p)\, dt\right] \\
&= \sum_j c_j\left[\int_{d_j}^{d_j+p} k(t)dt\right] \\
&= \sum_j c_j\, [p\, \text{Avg}(k)] \\
&= \text{Avg}(k) \sum_j c_j p \\
&= \text{Avg}(k) \sum_j c_j \left[\int_{-\infty}^{\infty} I(d_j < t \leq d_j + p)\, dt\right],
\end{aligned}$$

which is the right-hand side of (3.3). |||||

To apply the Average Value Theorem to prove the Factorization Lemma, we consider

$$k(t) = g(t), \tag{3.4}$$

which is a periodic function with period = 1, and

$$\begin{aligned} h(t) &= f(t)\ {}_{\lfloor t \rfloor|}q_x \\ &= \sum_{j\geq 0} c_j\ {}_{j|}q_x I(j < t \leq j+1), \end{aligned} \tag{3.5}$$

which is a step function with step size = 1. Then

$$\int_{-\infty}^{\infty} k(t)\, h(t)\, dt = E[g(T)\, f(T)],$$

$$\mathrm{Avg}(k) = E[g(S)],$$

and

$$\int_{-\infty}^{\infty} h(t)\, dt = E[f(T)].$$

4. Increasing Insurances

Let n be a positive integer. Then

$$A^{(m)}_{\overset{1}{x}:\overline{n}|} = E\left[I(T \leq n)\, v^{\frac{\lceil mT \rceil}{m}}\right], \tag{4.1}$$

and it follows from (2.14) that

$$A^{(m)}_{\overset{1}{x}:\overline{n}|} = \frac{i}{i^{(m)}} A_{\overset{1}{x}:\overline{n}|}. \tag{4.2}$$

Interchanging the order of differentiation and expectation, we have

$$\begin{aligned} -\frac{\partial}{\partial \delta} A^{(m)}_{\overset{1}{x}:\overline{n}|} &= E\left[I(T \leq n) \frac{\lceil mT \rceil}{m}\, v^{\frac{\lceil mT \rceil}{m}}\right] \\ &= \left(I^{(m)}A\right)^{(m)}_{\overset{1}{x}:\overline{n}|}. \end{aligned} \tag{4.3}$$

Thus differentiating (4.2) with respect to δ yields

$$-\left(I^{(m)}A\right)^{(m)}_{\overset{1}{x:\overline{n}|}} = \frac{i}{i^{(m)}}\frac{\partial}{\partial\delta}A^{\,1}_{x:\overline{n}|} + \left(\frac{\partial}{\partial\delta}\frac{i}{i^{(m)}}\right)A^{\,1}_{x:\overline{n}|}$$

$$= -\frac{i}{i^{(m)}}(IA)^{\,1}_{x:\overline{n}|} + \left(\frac{\partial}{\partial\delta}\frac{i}{i^{(m)}}\right)A^{\,1}_{x:\overline{n}|}.$$

Since

$$1 + \frac{i^{(k)}}{k} = e^{\delta/k},$$

we have

$$\frac{\partial}{\partial\delta}i^{(k)} = e^{\delta/k} = \frac{i^{(k)}}{d^{(k)}},$$

or

$$\frac{\partial}{\partial\delta}\ln[i^{(k)}] = \frac{1}{d^{(k)}}.$$

Hence

$$\frac{\partial}{\partial\delta}\ln[\frac{i^{(k)}}{i^{(m)}}] = \frac{1}{d^{(k)}} - \frac{1}{d^{(m)}},$$

from which it follows that

$$\left(I^{(m)}A\right)^{(m)}_{\overset{1}{x:\overline{n}|}} = \frac{i}{i^{(m)}}\left[(IA)^{\,1}_{x:\overline{n}|} - \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right)A^{\,1}_{x:\overline{n}|}\right]. \tag{4.4}$$

With $m = \infty$ and $n = \infty$, the left-hand side of (4.4) becomes $\left(\overline{IA}\right)_x$; this gives the last formula in Section 4.4 of *Actuarial Mathematics* (p.124).

For positive integers k and m, to evaluate

$$(I^{(k)}A)^{(m)}_x = E\left[\frac{\lceil kT\rceil}{k}v^{\frac{\lceil mT\rceil}{m}}\right], \tag{4.5}$$

we use the identity

$$\frac{\lceil kT\rceil}{k}v^{\frac{\lceil mT\rceil}{m}} - \frac{\lceil mT\rceil}{m}v^{\frac{\lceil mT\rceil}{m}} = \left(\frac{\lceil kT\rceil}{k} - \frac{\lceil mT\rceil}{m}\right)v^{\frac{\lceil mT\rceil}{m}}$$

$$= \left(\frac{\lceil kT\rceil}{k} - \frac{\lceil mT\rceil}{m}\right)(1+i)^{\lceil T\rceil-\frac{\lceil mT\rceil}{m}}\,v^{\lceil T\rceil}. \tag{4.6}$$

Taking expectations and applying the Factorization Lemma yields

$$(I^{(k)}A)^{(m)}_x - (I^{(m)}A)^{(m)}_x = E\left[\left(\frac{\lceil kS\rceil}{k} - \frac{\lceil mS\rceil}{m}\right)(1+i)^{1-\frac{\lceil mS\rceil}{m}}\right]A_x. \tag{4.7}$$

The term $(I^{(m)}A)^{(m)}_x$ can be evaluated using (4.4) with $n = \infty$.

5. Annuities-Due

Because

$$E\left[\ddot{a}^{(m)}_{\overline{\lceil mT\rceil/m}|}\right] = \ddot{a}^{(m)}_x,$$

$$E\left[\ddot{a}^{(m)}_{\overline{\lceil T\rceil}|}\right] = E\left[\ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_{\overline{\lceil T\rceil}|}\right]$$

$$= \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x, \tag{5.1}$$

and

$$\ddot{a}^{(m)}_{\overline{\lceil T\rceil}|} - \ddot{a}^{(m)}_{\overline{\lceil mT\rceil/m}|} = v^{\lceil T\rceil}\ddot{s}^{(m)}_{\overline{\lceil T\rceil - \frac{\lceil mT\rceil}{m}}|}, \tag{5.2}$$

we have

$$\ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x - \ddot{a}^{(m)}_x = E\left[v^{\lceil T\rceil}\ddot{s}^{(m)}_{\overline{\lceil T\rceil - \frac{\lceil mT\rceil}{m}}|}\right]$$

$$= E\left[v^{\lceil T\rceil}\right]E\left[\ddot{s}^{(m)}_{\overline{\lceil S\rceil - \frac{\lceil mS\rceil}{m}}|}\right]$$

$$= A_x E\left[\ddot{s}^{(m)}_{\overline{1-\frac{\lceil mS\rceil}{m}}|}\right] \tag{5.3}$$

by the Factorization Lemma and (1.8). Now under UDDYA, it follows from (2.12) that

$$E\left[\ddot{s}^{(m)}_{\overline{1-\frac{\lceil mS\rceil}{m}}|}\right] = E\left[\frac{(1+i)^{1-\frac{\lceil mS\rceil}{m}} - 1}{d^{(m)}}\right]$$

$$= \frac{s^{(m)}_{\overline{1}|} - 1}{d^{(m)}}$$

$$= \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}$$

$$= \beta(m), \tag{5.4}$$

which is (5.4.13) of *Actuarial Mathematics*. Hence (5.3) can be written as

$$\ddot{a}^{(m)}_x = \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x - \beta(m)A_x, \tag{5.5}$$

which is (5.4.14) of *Actuarial Mathematics* .

The quantity

$$\beta(m) = E\left[\ddot{s}^{(m)}_{\overline{1-\frac{\lceil mS\rceil}{m}}|}\right]$$

can be interpreted as follows. Prior to the year of death, the annuity-due pays m payments of 1/m each year. In the year of death, between one and m payments of 1/m are made, depending on the date of death. Accumulate with interest the "non-payments" forward to the end of the year. Then the expectation of the sum of the accumulated values is $\beta(m)$.

Under UDDYA, $\beta(m) = \int_0^1 \ddot{s}^{(m)}_{\overline{1-\frac{\lceil ms \rceil}{m}}|} ds$. Integrating by parts yields

$$\beta(m) = s\, \ddot{s}^{(m)}_{\overline{1-\frac{\lceil ms \rceil}{m}}|} \Bigg|_{s=0}^{s=1} - \int_0^1 s\, d\, \ddot{s}^{(m)}_{\overline{1-\frac{\lceil ms \rceil}{m}}|}$$

$$= -\int_0^1 s\, d\, \ddot{s}^{(m)}_{\overline{1-\frac{\lceil ms \rceil}{m}}|}, \tag{5.6}$$

because $\ddot{s}^{(m)}_{\overline{0}|} = 0$. The right-hand side of (5.6) is a Riemann-Stieltjes integral. Since

$$\ddot{s}^{(m)}_{\overline{1-\frac{\lceil ms \rceil}{m}}|} = \sum_{j=\lceil ms \rceil}^{m-1} \frac{1}{m}(1+i)^{1-\frac{j}{m}}, \qquad 0 < s \le 1 - \frac{1}{m}, \tag{5.7}$$

we have

$$\beta(m) = \sum_{j=1}^{m-1} \frac{j}{m^2}(1+i)^{1-\frac{j}{m}}$$

$$= \left(I^{(m)}\ddot{s}\right)^{(m)}_{\overline{1-\frac{1}{m}}|}. \tag{5.8}$$

To reconcile this formula with (5.4), we can use the compound-interest identity

$$\left(I^{(q)}\ddot{s}\right)^{(m)}_{\overline{n}|} = \frac{\ddot{s}^{(q)}_{\overline{n}|} - n}{d^{(m)}}. \tag{5.9}$$

6. Annuities-Immediate

The analysis in the last section can be extended to annuities-immediate. Consider

$$a^{(m)}_{\overline{\lfloor mT \rfloor / m}|} = a^{(m)}_{\overline{\lfloor T \rfloor}|} + v^{\lfloor T \rfloor}\, a^{(m)}_{\overline{\frac{\lfloor mT \rfloor}{m} - \lfloor T \rfloor}|}$$

$$= s^{(m)}_{\overline{1}|} a_{\overline{\lfloor T \rfloor}|} + v^{\lfloor T \rfloor}\, a^{(m)}_{\overline{\frac{\lfloor mT \rfloor}{m} - \lfloor T \rfloor}|}. \tag{6.1}$$

Taking expectations and applying the Factorization Lemma yields

$$a_x^{(m)} = s^{(m)}_{\overline{1}|}\, a_x + E\left[v^{\lfloor T \rfloor}\right] E\left[a^{(m)}_{\overline{\frac{\lfloor mS \rfloor}{m} - \lfloor S \rfloor}|}\right]$$

$$= s^{(m)}_{\overline{1}|}\, a_x + (1+i)\, A_x\, E\left[a^{(m)}_{\overline{\frac{\lfloor mS \rfloor}{m}}|}\right]. \tag{6.2}$$

With the definition

$$\gamma(m) = E\left[a^{(m)}_{\overline{\frac{\lfloor mS \rfloor}{m}}|}\right], \tag{6.3}$$

we can rewrite (6.2) as

$$a_x^{(m)} = s^{(m)}_{\overline{1}|}\, a_x + (1 + i)\, A_x\, \gamma(m). \tag{6.4}$$

The quantity $\gamma(m)$ can be interpreted as follows. Prior to the year of death, the annuity-immediate pays m payments of 1/m each year. In the year of death, between 0 and m−1 payments of 1/m are made, depending on the date of death. Discount with interest the payments made, back to the beginning of the year of death. Then the expectation of the sum of the discounted values is $\gamma(m)$.

Under UDDYA, $\gamma(m) = \int_0^1 a^{(m)}_{\overline{\lfloor ms \rfloor/m}|}\, ds$, which can be evaluated by an integration by parts as follows:

$$\begin{aligned}
\gamma(m) &= -\int_0^1 a^{(m)}_{\overline{\lfloor ms \rfloor/m}|}\, d(1 - s) \\
&= -(1 - s)\, a^{(m)}_{\overline{\lfloor ms \rfloor/m}|}\Big|_{s=0}^{s=1} + \int_0^1 (1 - s)\, d\, a^{(m)}_{\overline{\lfloor ms \rfloor/m}|} \\
&= \int_0^1 (1 - s)\, d\, a^{(m)}_{\overline{\lfloor ms \rfloor/m}|},
\end{aligned} \tag{6.5}$$

because $a^{(m)}_{\overline{0}|} = 0$. To evaluate the Riemann-Stieltjes integral on the right-hand side of (6.5), we note that

$$a^{(m)}_{\overline{\lfloor ms \rfloor/m}|} = \sum_{j=1}^{\lfloor ms \rfloor} \frac{1}{m} v^{\frac{j}{m}}, \qquad \frac{1}{m} \le s < 1. \tag{6.6}$$

Hence

$$\begin{aligned}
\gamma(m) &= \sum_{j=1}^{m-1} \left[1 - \frac{j}{m}\right] \frac{1}{m} v^{\frac{j}{m}} \\
&= \left(D^{(m)}a\right)^{(m)}_{\overline{1-\frac{1}{m}}|}.
\end{aligned} \tag{6.7}$$

It follows from the formula

$$(D^{(q)}a)^{(m)}_{\overline{n}|} = \frac{n - a^{(q)}_{\overline{n}|}}{i^{(m)}}$$

that

$$\gamma(m) = \frac{1-\ddot{a}^{(m)}_{\overline{1}|}}{i^{(m)}}$$

$$= \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}. \tag{6.8}$$

Therefore, Exercise 5.18 in *Actuarial Mathematics* follows from (6.4). Formula (6.7) shows that $\gamma(m)$ can also be interpreted as the discounted value of m – 1 payments. The sum on the right-hand side of (6.7) is the discounted value at time k, of m – 1 payments with the amount $\frac{m-j}{m^2}$ paid at time $k+\frac{j}{m}$, j = 1, 2, 3, ..., m–1. See the right half of Figure 2.

7. $\alpha(m)$

We can use the relationship

$$\alpha(m) = \frac{1}{m} + \beta(m) + \gamma(m) \tag{7.1}$$

to show that

$$\ddot{a}^{(m)}_x = \alpha(m)\,\ddot{a}_x - \beta(m) \tag{7.2}$$

and

$$a^{(m)}_x = \alpha(m)\,a_x + \gamma(m). \tag{7.3}$$

(cf. Exercise 5.19 in *Actuarial Mathematics*). It follows from (7.1), (5.8) and (6.7) that $\alpha(m)$ can be interpreted as the value of 2m – 1 cash flows, with valuation date being the mid-point of the occurrences of the cash flows. In Figure 2, the valuation date is k. There are m – 1 increasing cash flows from the year before time k, their accumulated value at time k being $\beta(m)$. There is one cash flow of amount 1/m at time k. There are m – 1 decreasing cash flows from the year after time k, their discounted value at time k being $\gamma(m)$. See also (7.11) below.

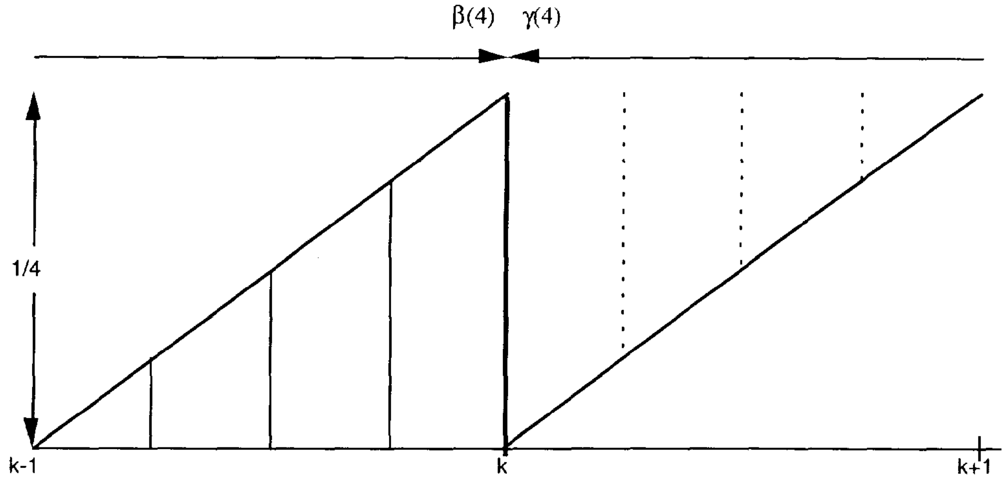

Figure 2. The 7 cash flows contributing to $\alpha(4)$

Let n be a positive integer. Consider an n-year annuity-certain with payments of 1/m in each m-th of a year. Between successive policy anniversary dates, there are exactly m–1 payments of 1/m. Under the UDDYA assumption, we can partition these m–1 payments into a stream of m–1 increasing cash flows (with accumulated value of $\beta(m)$ at the later anniversary date) and a stream of m–1 decreasing cash flows (with discounted value of $\gamma(m)$ at the earlier anniversary date). Hence, for an n-year annuity-due, we have

$$\ddot{a}^{(m)}_{\overline{n}|} = \alpha(m)\,\ddot{a}_{\overline{n}|} - \beta(m)(1 - v^n), \tag{7.4}$$

and, for an n-year annuity-immediate,

$$a^{(m)}_{\overline{n}|} = \alpha(m)\,a_{\overline{n}|} + \gamma(m)(1 - v^n). \tag{7.5}$$

Each of $\beta(m)(1 - v^n)$ and $\gamma(m)(1 - v^n)$ is an adjustment term for both year 1 and year n.

Putting $n = \lceil T \rceil$ in (7.4) and taking expectations, we have

$$E\left[\ddot{a}^{(m)}_{\overline{\lceil T \rceil}|}\right] = \alpha(m)\,\ddot{a}_x - \beta(m)(1 - A_x). \tag{7.6}$$

By (5.1), the left-hand side of (7.6) is $\ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x$. It follows from (5.5) and (7.6) that

$$\ddot{a}^{(m)}_x = \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x - \beta(m)A_x = \alpha(m)\,\ddot{a}_x - \beta(m), \tag{7.7}$$

which is (7.2). Note that on the right-hand side of (7.2) [or of (7.7)] there is no adjustment term for the "loss" of annuity payments in the year of death. Under the UDDYA assumption, the increasing stream of cash flows in the previous paragraph is exactly the stream of *expected* cash flows to be lost in the year of death. However, there is an adjustment term, $-\beta(m)$, for year 1. Also note that, in (5.5) [or in the middle expression of (7.7)], there is an adjustment term, $-\beta(m)A_x$, for the year of death, but none for year 1.

Similarly, putting $n = \lfloor T \rfloor$ in (7.5) and taking expectations yields

$$E\left[a^{(m)}_{\overline{\lfloor T \rfloor|}}\right] = \alpha(m)a_x + \gamma(m)[1 - (1+i)A_x]. \tag{7.8}$$

Since the left-hand side of (7.8) is $s^{(m)}_{\overline{1|}}\, a_x$, it follows from (6.4) and (7.8) that

$$\begin{aligned} a^{(m)}_x &= s^{(m)}_{\overline{1|}}\, a_x + \gamma(m)(1+i)A_x \\ &= \alpha(m)a_x + \gamma(m), \end{aligned}$$

which is (7.3).

With n = 1, (7.4) becomes

$$\ddot{a}^{(m)}_{\overline{1|}} = \alpha(m) - \beta(m)(1-v) = \alpha(m) - d\beta(m) \tag{7.9}$$

which is Exercise 5.49 in *Actuarial Mathematics*. Formula (7.9) can be also deduced by substituting $A_x = 1 - d\ddot{a}_x$ into the middle expression of (7.7).

With n = 1, (7.5) becomes

$$a^{(m)}_{\overline{1|}} = \alpha(m)v + \gamma(m)(1-v),$$

or

$$s^{(m)}_{\overline{1|}} = \alpha(m) + i\,\gamma(m). \tag{7.10}$$

Both (7.9) and (7.10) can be explained in terms of cash-flow decomposition.

Note that

$$\begin{aligned} \alpha(m) &= \frac{1}{m} + \beta(m) + \gamma(m) \\ &= \left(I^{(m)}\ddot{s}\right)^{(m)}_{\overline{1-\frac{1}{m}|}} + \left(D^{(m)}\ddot{a}\right)^{(m)}_{\overline{1|}} \\ &= \left(I^{(m)}s\right)^{(m)}_{\overline{1|}} + \left(D^{(m)}a\right)^{(m)}_{\overline{1-\frac{1}{m}|}}. \end{aligned} \tag{7.11}$$

Also, it follows from (5.4), (6.8) and the formula

$$\frac{1}{m} = \frac{1}{d^{(m)}} - \frac{1}{i^{(m)}}$$

that

$$\alpha(m) = \frac{i-d}{i^{(m)}d^{(m)}} = \frac{id}{i^{(m)}d^{(m)}}, \tag{7.12}$$

which is (5.4.12) of *Actuarial Mathematics*.

8. Apportionable Annuity-Due

Let s be a positive number, not necessarily an integer; we define

$$\ddot{a}_{\overline{s}|}^{(m)} = \frac{1 - v^s}{d^{(m)}}. \tag{8.1}$$

This definition extends the usual definition for $\ddot{a}_{\overline{s}|}^{(m)}$, where s is a positive integer. Then

$$E\left[\ddot{a}_{\overline{T}|}^{(m)}\right] = \ddot{a}_x^{\{m\}}, \tag{8.2}$$

which is the single premium for an *apportionable life annuity-due* of 1 per year payable in installments of 1/m at the beginning of each m-th of a year while (x) survives (cf. Section 5.5 of *Actuarial Mathematics*). It follows from (8.1) that

$$\ddot{a}_{\overline{\lceil mT \rceil/m}|}^{(m)} - \ddot{a}_{\overline{T}|}^{(m)} = v^T \, \ddot{a}_{\overline{\frac{\lceil mT \rceil}{m} - T}|}^{(m)}. \tag{8.3}$$

Taking expectations and applying (1.2) and (8.2) yields

$$\ddot{a}_x^{(m)} - \ddot{a}_x^{\{m\}} = E\left[v^T \, \ddot{a}_{\overline{\frac{\lceil mT \rceil}{m} - T}|}^{(m)}\right]. \tag{8.4}$$

The amount of refund at T, the time of death of (x), is

$$\ddot{a}_{\overline{\frac{\lceil mT \rceil}{m} - T}|}^{(m)}. \tag{8.5}$$

Note that

$$\frac{\lceil mT \rceil}{m} - T = T \text{ pad } \frac{1}{m} \tag{8.6}$$

is the time between death and the next payment date.

It follows from (8.1) that

$$\ddot{a}^{(m)}_{\overline{s}|} = \frac{\delta}{d^{(m)}} \bar{a}_{\overline{s}|} \tag{8.7}$$

$$= \frac{1}{m\bar{a}_{\overline{1/m}|}} \bar{a}_{\overline{s}|}. \tag{8.8}$$

Hence expression (8.5), the amount of refund at the time of death, can be rewritten as

$$\frac{1}{m\bar{a}_{\overline{1/m}|}} \bar{a}_{\overline{\frac{\lceil mT \rceil}{m} - T}|}, \tag{8.9}$$

which is an expression that can be found in (5.5.3) of *Actuarial Mathematics*.

9. Endowment Insurance and Temporary Life Annuities

For two real numbers s and t, let $s \wedge t$ denote the minimum of s and t. Replacing $\frac{\lceil mT \rceil}{m}$ by $\frac{\lceil mT \rceil}{m} \wedge n$ and T by $T \wedge n$, we can extend much of the analysis above from whole life insurance to n-year endowment insurance. For example, in place of (1.1), (1.2), and (1.3), we have

$$A^{(m)}_{x:\overline{n}|} = E\left[v^{\frac{\lceil mT \rceil}{m} \wedge n}\right], \tag{9.1}$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = E\left[\ddot{a}^{(m)}_{\overline{\frac{\lceil mT \rceil}{m} \wedge n}|}\right], \tag{9.2}$$

and

$$a^{(m)}_{x:\overline{n}|} = E\left[a^{(m)}_{\overline{\frac{\lfloor mT \rfloor}{m} \wedge n}|}\right], \tag{9.3}$$

respectively.

10. Complete Annuities-Immediate

Parallel to the notion of the apportionable annuity-due is that of the *complete annuity-immediate;* see Section 5.5 of *Actuarial Mathematics*. Let t be a positive number, not necessarily an integer; we define

$$a^{(m)}_{\overline{t}|} = \frac{1 - v^t}{i^{(m)}} \tag{10.1}$$

and

$$s_{\overline{t}|}^{(m)} = \frac{(1+i)^t - 1}{i^{(m)}}. \tag{10.2}$$

Then

$$\overset{\circ}{a}{}_x^{(m)} = E\left[a_{\overline{T}|}^{(m)} \right]. \tag{10.3}$$

It follows from (10.1) and (10.2) that

$$a_{\overline{T}|}^{(m)} - a_{\overline{\lfloor mT \rfloor / m}|}^{(m)} = v^T s_{\overline{T - \frac{\lfloor mT \rfloor}{m}}|}^{(m)}. \tag{10.4}$$

Taking expectations and applying (10.3) and (1.3) yields

$$\overset{\circ}{a}{}_x^{(m)} - a_x^{(m)} = E\left[v^T s_{\overline{T - \frac{\lfloor mT \rfloor}{m}}|}^{(m)} \right]. \tag{10.5}$$

The adjustment payment at time T is $s_{\overline{T - \frac{\lfloor mT \rfloor}{m}}|}^{(m)}$. Note that

$$T - \frac{\lfloor mT \rfloor}{m} = T \bmod \frac{1}{m} \tag{10.6}$$

is the time between the last payment date before death and the date of death.

Let n be a positive number divisible by 1/m, or n mod 1/m = 0. Replacing T by $T \wedge n$ and $\frac{\lfloor mT \rfloor}{m}$ by $\frac{\lfloor mT \rfloor}{m} \wedge n$ in (10.3) and (10.4), we have

$$\overset{\circ}{a}{}_{x:\overline{n}|}^{(m)} = E\left[a_{\overline{T \wedge n}|}^{(m)} \right] \tag{10.7}$$

and

$$a_{\overline{T \wedge n}|}^{(m)} - a_{\overline{\frac{\lfloor mT \rfloor}{m} \wedge n}|}^{(m)} = v^{T \wedge n} s_{\overline{(T \wedge n) - \left(\frac{\lfloor mT \rfloor}{m} \wedge n\right)}|}^{(m)}. \tag{10.8}$$

Observe that

$$(T \wedge n) - \left(\tfrac{\lfloor mT \rfloor}{m} \wedge n\right) = \begin{cases} T - \frac{\lfloor mT \rfloor}{m} & \text{if } T < n \\ n - n = 0 & \text{if } T \geq n \end{cases}.$$

Since $s_{\overline{0}|}^{(m)} = 0$, the right-hand side of (10.8) can be simplified as

$$v^T s_{\overline{T - \frac{\lfloor mT \rfloor}{m}}|}^{(m)} I(T < n).$$

Hence it follows from (10.8), (10.7) and (9.3) that

$$\overset{\circ}{a}{}_{x:\overline{n}|}^{(m)} - a_{x:\overline{n}|}^{(m)} = E\left[v^T s_{\overline{T - \frac{\lfloor mT \rfloor}{m}}|}^{(m)} I(T < n) \right]. \tag{10.9}$$

Also note that

$$\ddot{a}^{(m)}_{x:\overline{n}|} / \mathring{a}^{(m)}_{x:\overline{n}|} = i^{(m)} / d^{(m)}$$

$$= (1 + i)^{1/m}, \qquad (10.10)$$

which is Exercise 5.28.e of *Actuarial Mathematics*.

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