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Stepwise recursions for a class of compound Lagrangian Distributions

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ABSTRACT

In this paper, Lagrangian distributions are reviewed and some of their properties are studied. Recursive methods are proposed to calculate compound Lagrange distributions. Some examples for a sub-family of Lagrangian Distributions and its compound distributions are studied.

Key words: *Lagrange expansion, Modified Power Series Distributions, Generalized Power Series Distributions.*

1 Introduction

Consul and Shenton (1972, 1973) introduced a class of discrete probability distributions and called it the class of Lagrangian distributions. The particular name originates from the fact that these probability functions are derived from the Lagrange expansion of a function $f(t)$ as a power series in u where $ug(t) = t$ and, $f(t)$ and $g(t)$ are both pgf's of certain discrete distributions. From a mathematical functional point of view, the Lagrange expansion is a generalization of Taylor's expansion. For $g(t) = 1$, a Lagrange expansion becomes identical with a Taylor's expansion. This class consists of many families; for example: generalized Poison, generalized negative binomial, Borel-Tanner and, Haight distributions. Kling and Goovaerts (1993), and Sharif and Panjer (1994) have derived three step recursions for compound generalized power series distributions(GPSD). Goovaerts and Kaas (1991) defined a two step recursion to calculate compound generalized Poisson probability functions. Goovaerts and Kaas (1991) used the fact that generalized Poisson distributions is a compound Poisson distribution with the Borel distribution. The Borel distribution was derived from another Poisson distributions pgf using the Lagrange expansion. The main motivation of this work came from Goovaerts and Kaas (1991). Their idea is explored and generalized for a wider class of distributions which includes generalized Poisson distributions as a particular case. Detail derivation and discussions are given in an example at the end of this article.

In this article, we first investigate the definition of a class of Lagrangian distributions and then introduce a recursive way of calculating their probability functions and their compound probability functions for a suitable subclass, namely the $(\boldsymbol{a};\boldsymbol{b})$ subclass (defined in section 3.3) of the Lagrangian family.

2 Definitions of the Lagrange distribution

In this section we will define, as in Consul (1973), two different kinds of Lagrangian probability distributions (LPD), namely (i) a basic LPD, and (ii) a general LPD.

2.1 A basic LPD

Definition 1 *Basic LPD : Let $g(t)$ be a pgf of a discrete random variable such that $g(0) \neq 0$, then the transformation*

$$t = ug(t)$$

defines, for the smallest non-zero root of t, a new pgf $t = \phi(u)$ whose expansion in powers of u is given by the Lagrange's expansion as

$$t = \phi(u)$$

$$= \sum_{x=1}^{\infty} \frac{u^x}{x!} \left[\left(\frac{\partial}{\partial t} \right)^{x-1} (g(t))^x \right]_{t=0} \tag{2.1}$$

The above pgf $\phi(u)$ will be referred to as the basic Lagrangian pgf and the discrete distribution represented by it, namely,

$$Pr(X = x) = \begin{cases} \frac{1}{x!}(\frac{\partial}{\partial t})^{x-1}(g(t))^x \,|_{t=0} & x \in N \\ 0 & for \;\; x = 0 \end{cases} \tag{2.2}$$

as the basic Lagrangian probability distribution (basic LPD) defined on N, the set of positive integers. Equivalently, we can write

$$Pr(X = x) = \begin{cases} \frac{1}{x} g_{x-1}^{*x} & x \in N \\ 0 & for \;\; x = 0 \end{cases}$$

where $g(t) = \sum_{j=0}^{\infty} g_j t^j$ and g_j^{*n} is the n-th convolution of g_j.

Note that, by definition, all basic LPD's must have tails extending to infinity, and $Pr(X = 0) = 0$.

Examples:

(i) The Borel distribution is a basic LPD generated by the Poisson distribution given by $g(t) = e^{\lambda(t-1)}$, $m > 0$. The pf of Borel distribution, as derived from (3.2), is

$$\frac{e^{-\lambda x}(\lambda x)^{x-1}}{x!}.$$

For detail derivation, see Borel (1942). A practical and easier derivation is in Tanner (1961). ■

(ii) The Haight distribution is the basic LPD generated by the geometric distribution given by $g(t) = q(1-pt)^{-1}$ where $p + q = 1$. A generalization of this results could be found in Haight (1961) and is shown in table 2. ■

(iii) The geometric distribution itself is a member of basic LPD and is generated by the simple Bernoulli distribution given by $g(t) = p + qt$, where $p + q = 1$. Note that $t = ug(t) = u(p + qt)$ gives $t = up(1 - qu)^{-1}$ which is a pgf for a geometric distribution. ■

(iv) The Consul distribution:

$$Pr(X = x) = \frac{1}{x}\binom{mx}{x-1}\left(\frac{\theta}{1-\theta}\right)^{x-1}(1-\theta)^{mx}, \; x \in N \qquad (2.3)$$

where $m \geq 1$ and integer and $0 < \theta < 1$, is a member of basic LPD generated by the binomial distribution with pgf $g(t) = (1 - \theta + \theta t)^m$. The pf in (3.3) is derived using (3.2) by taking $g(t)$ as the pgf of a binomial distribution. For detail derivation, see Islam and Consul (1992). It can also be derived by taking $g(t) = (1 + \theta - \theta t)^{-m}$ which is the pgf of a negative binomial distribution. Note that for $m = 1$, a Consul distribution is a geometric distribution. ■

(v) Another important distribution is the pf of the first visit to +1 (i.e. first passage through 1) in n−th step in a random walk started at the origin. It is a basic LPD generated by $g(t) = p + qt^2$, where $p + q = 1$. The substitution, $t = ug(t)$, provides the pgf

$$t(u) = \frac{1 - \sqrt{(1 - 4u^2pq)}}{2qu}$$

which is same as (3.6) in Feller (1968, page 272). ■

Having defined the basic LPD, the idea of generating new distributions using the Lagrange expansion could be taken one step further. Let $g(t)$ be a pgf of a nonnegative integer valued discrete random variable such that $g(0) \neq 0$. As in the basic LPD, consider the transformation

$$t = ug(t)$$

so that $u = 0$ for $t = 0$ and $u = 1$ for $t = 1$. We can expand t as a series in powers of u with the help of the Lagrange expansion. Hence for any well-behaved function which is analytic on and within a given contour of t on (u,t) plane, $f(t)$ can be expressed as a series in powers of u. Therefore we have the following definition in the next subsection.

2.2 A general LPD

Definition 2 *Let $f(t)$ and $g(t)$ be pgf's of discrete random variables defined on nonnegative integers, such that $g(0) \neq 0$, then $f(t)$, with the transformation $t = ug(t)$, could be expanded in series in powers of u and is given by the Lagrange's expansion as*

$$\begin{aligned} \phi_*(u) &= f(t(u)) \qquad (2.4) \\ &= f(0) + \sum_{x=1}^{\infty} \frac{u^x}{x!} \left(\frac{\partial}{\partial t}\right)^{x-1} \left[(g(t))^x \frac{\partial}{\partial t} f(t)\right]_{t=0} \end{aligned}$$

The above pgf $\phi_*(u)$ is defined as the general Lagrangian pgf and the discrete distribution represented by it, namely,

$$\begin{aligned} Pr(X=0) &= f(0) \qquad (2.5) \\ \text{and} \quad Pr(X=x) &= \frac{1}{x!} \left(\frac{\partial}{\partial t}\right)^{x-1} \left[(g(t))^x \frac{\partial}{\partial t} f(t)\right]_{t=0} \quad x \in N \end{aligned}$$

as the general Lagrangian probability distribution (general LPD) defined on N, the set of positive integers.

Note that, by definition, $t(u)$ is a pgf of a basic LPD and hence $\phi_*(u)$ is a pgf of a compound distribution. For $f(t) = t$ (equivalently having total mass at point one), general LPD corresponds to a basic LPD; for $g(t) = 1$ (having total mass at point zero), u becomes identically equal to t and hence $\phi(u)$ identically matches with parent pgf $f(t)$. For $f(t) = t^n$ (equivalently having total mass at point n), the general LPD becomes identical with the n−th fold convolution of basic LPD.

Table 1: **The pf of some basic Lagrange Distributions**

Sl. No.	Description	$g(t)$	$f(t)$	pf of some basic LPD
1.	Borel	$e^{\lambda(t-1)}$	t	$\frac{e^{-\lambda x}(\lambda x)^{x-1}}{x!}$
2.	Consul	$(1-\theta+\theta t)^m$	t	$\binom{mx}{x-1}\frac{(1-\theta)^{mx+1-x}\theta^{x-1}}{x}$
3.	Bernouli.delta (Geometric)	$(1-\theta+\theta t)$	t	$(1-\theta)\theta^{x-1}$
4.	Neg.bin-delta (Haight)	$\left(\frac{1-p}{1-pt}\right)^r$	t	$\frac{\Gamma(rx+x-1)}{x!\Gamma(rx)}(1-p)^{rx}p^{x-1}$
5.	Geom.delta	$\left(\frac{1-p}{1-pt}\right)$	t	$\frac{\Gamma(2x-1)}{x!\Gamma(x)}(1-p)^{x}p^{x-1}$

Examples:

(i) The generalized Poisson distribution(GPD) is introduced by Consul and Jain (1970). For extensive study and literature on GPD, see Consul (1989). GPD is a general LPD generated by the Poisson distribution given by the pgf $g(t) = e^{\lambda(t-1)}$, $\lambda > 0$, and the Poisson distribution given by the pgf $f(t) = e^{\theta(t-1)}$, $\theta > 0$. ■

(ii) The generalized negative binomial (GNB) distribution is introduced by Jain and Consul (1971). For its characterization see Consul and Gupta (1980). A GNB distribution is a general LPD generated by the binomial distributions given by the pgf's $g(t) = (q+pt)^m$ and $f(t) = (q+pt)^n$ where $p+q=1$, $p>0$, and m and n are positive integers. ■

Table 2: **The pf of n-th convolution of some basic Lagrange Distributions**

Sl. No.	Description	$g(t)$	$f(t)$	pf of n-th convolution of some basic LPD
1.	Borel-Tanner	$e^{\lambda(t-1)}$	t^n	$\frac{ne^{-\lambda x}(\lambda x)^{x-n}}{x(x-n)!}$
2.	Binom.delta (Consul)	$(q+pt)^m$	t^n	$\binom{mx}{x-n}\frac{n(q)^{mx+n-x}p^{x-n}}{x}$
3.	Bernouli.delta (Geometric)	$(q+pt)$	t^n	$\binom{x}{x-n}\frac{n(q)^n p^{x-n}}{x}$
4.	Neg.bin-delta (Haight)	$\left(\frac{1-p}{1-pt}\right)^r$	t^n	$\frac{n\Gamma(rx+x-n)}{x(x-n)!\Gamma(rx)}(1-p)^{rx}p^{x-n}$
5.	Geom.delta	$\left(\frac{1-p}{1-pt}\right)$	t^n	$\frac{n\Gamma(2x-n)}{x(x-n)!\Gamma(x)}(1-p)^x p^{x-n}$

Table 3: **The pf of some general Lagrange Distributions**

$g(t)$; $f(t)$	pf of some general LPD
1. $e^{\lambda(t-1)}$; $e^{\theta(t-1)}$	$\frac{\theta(\theta+\lambda x)^{x-1}e^{-(\theta+\lambda x)}}{x!}$
2. $e^{\lambda(t-1)}$; $(q+pt)^n$	$\begin{cases} \frac{(\lambda x)^{x-1}}{x!}e^{-\lambda x}npq^{n-1}{}_2F_0(1-x,1-n;\frac{p}{\lambda q x}) \\ q^n \quad \text{for} \quad x=0 \end{cases}$
3. $e^{\lambda(t-1)}$; $\left(\frac{1-p}{1-pt}\right)^k$	$\begin{cases} \frac{(\lambda x)^{x-1}}{x!}e^{-\lambda x}kpq^{k+2}{}_2F_0(1-x,1+k;-\frac{p}{\lambda x}) \\ q^k \quad \text{for} \quad x=0 \end{cases}$
4. $(q+pt)^m$; $e^{M(t-1)}$	$e^{-M}\frac{(M^x q^{mx})}{x!}{}_2F_0(1-x,-mx;\frac{p}{Mq})$
5. $(q+pt)^m$; $(q+pt)^n$	$\frac{n}{n+mx}\binom{n+mx}{x}p^x q^{n+mx-x}$
6. $(q+pt)^m$; $\left(\frac{1-p}{1-pt}\right)^k$	$\frac{\Gamma(k+x)}{x!\Gamma(k)}p^x q^{mx+k}{}_2F_1(1-x,-mx;1-x-k;-\frac{1}{q})$
7. $\left(\frac{1-p}{1-pt}\right)^k$; $e^{M(t-1)}$	$\frac{e^{-M}M^x}{x!}q^{kx}{}_2F_0(1-x,kx;-\frac{p}{M})$
8. $\left(\frac{1-p}{1-pt}\right)^k$; $(q+pt)^n$	$\begin{cases} \frac{\Gamma(kx+x-1)}{x!\Gamma(kx)}np^x q^{kx+n-1}{}_2F_1(1-x,1-n;2-k-kx;-\frac{1}{q}) \\ q^n \quad \text{for} \quad x=0 \end{cases}$
9. $\left(\frac{1-p}{1-pt}\right)^k$; $\left(\frac{1-p}{1-pt}\right)^M$	$\frac{M}{M+kx+x}\frac{\Gamma(kx+M+x+1)}{x!\Gamma(M+kx+1)}p^x q^{M+kx}$

(iii) The Borel-Tanner distribution is a general LPD generated by the Poisson distribution given by the pgf $g(t) = e^{\lambda(t-1)}$, $\lambda > 0$, and the degenerate distribution given by the pgf $f(t) = t^n$, n is positive integer. For further detail, see Haight and Brener (1960). ■

Some examples of the basic LPD are shown in table 1. More examples of basic LPD are shown in table 2. Taking $g(t)$ and $f(t)$ to be the pgf of one of the common power series distributions, namely Poisson, binomial, or negative binomial, we have nine possible combinations for general LPD. Their corresponding pf as derived by Consul and Shenton (1972) by the Lagrange expansion is shown in the table 3. Some results in table 3 are expressed in hypergeometric function whose definitions are given below.

Definition 3 *The so-called generalized hypergeometric function, denotes by* ${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z)$, *is defined as the series sum*

$$\sum_{k=0}^{\infty} \frac{a_1^{[k]} \cdots a_p^{[k]}}{b_1^{[k]} \cdots b_q^{[k]}} \frac{z^k}{k!}$$

where $a^{[k]}$ *is the ascending factorials given by*

$$a^{[k]} = a(a+1) \cdots (a+k-1).$$

Definition 4 *The ordinary hypergeometric function, denoted by* ${}_2F_1(a, b; c; z)$, *is defined as the series sum*

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots$$

For more information on hypergeometric function, see Slater (1966).

More examples of the LPD pf can be found in Consul and Shenton (1972). Even though the pf of some LPD, as in table 3, are complicated by involving hypergeometric functions, we will see at the end of this article that they satisfy a simple recursion relation. And hence their compound LPD can be calculated recursively.

2.2.1 Link of LPD to queues and insurance

Consul (1989) described several chance mechanisms generating GP distributions. Similar reasoning can be used to show a link of a LPD to some practical chance mechanism that produces some real data, in particular some insurance data. The most important two mechanisms mentioned in Consul (1989) are (i) Galton-Watson branching process and (ii) the queueing process. Consul (1990) used Galton-Watson branching process arguments and modeled the distribution of injuries in auto-accidents by GP distribution which is a member of general LPD. Islam and Consul (1992) used similar logic to model automobile claims by a member of LPD, namely Consul distribution. Similar reasoning can be used for some other members of LPD, namely GNB distribution, for modeling purposes.

Consul and Shenton (1973) have shown that the number of customers served in any busy period of a counter will be a random variable having a LPD. In an actuarial application, Gerber (1990) used this result and linked the GP distribution to the ruin model.

3 Lagrange $(a; b)$ family

3.1 Definition of $(a; b)$ family

We have defined the class of Lagrange distribution. This class contains a large number of distributions. Of which, only a few are shown in table 1, table 2, and table 3. Our intention is to study only a subclass which was determined by $(a; b)$ family.

Panjer (1981) introduced a class of discrete claim frequency distributions whose pf satisfy the following first order difference equation:

$$p_n = (a + \frac{b}{n})\, p_{n-1} \quad \text{for} \quad n = 1, 2, \ldots, \infty. \tag{3.6}$$

Sundt and Jewell (1981) extended the class to a larger class whose pf satisfies the following first order difference equation.

$$p_n = (a + \frac{b}{n})\, p_{n-1} \quad \text{for} \quad n = r+1, r+2, \ldots, \infty \tag{3.7}$$

where r is a non-negative integer.

Hence we have the following definition of a family of discrete distribution whose pf satisfies the above first order difference equation.

Definition 5 *The class of discrete distribution whose pf satisfies the first order difference equation (3.7) will be called the (a, b) family, where a and b are both suitable constants.*

In the literature, this family with $r = 1$ is popularly known as "(a, b) class". For arbitrary r, we will call it "general (a, b) class". The family of frequencies given by Panjer (1981) is a subclass of the (a, b) class with $r = 0$ and is called, in the actuarial literature, the $(a, b, 0)$ subclass. The subfamily with $r = 1$ and $p_0 = 0$ is called, in the literature, the $(a, b, 1)$ subclass.

Sundt and Jewell (1981) showed that the only non-degenerate members of $(a, b, 0)$ subclass are Poisson, binomial, geometric and negative binomial. The pf, pgf, and possible values of a and b in terms of the parameters of the distributions of this subclass are shown in the table 4. The members of $(a, b, 1)$ subclass include logarithmic distributions, extended truncated negative binomial (ETNB) distributions, truncated Poisson, truncated binomial, truncated geometric, and truncated negative binomial and zero-modified mass of these. In addition $(a, b, 1)$ subclass includes another family of distribution with pgf, $p(z) = 1 - (1 - z)^\alpha,\ \ 0 < \alpha < 1$. Since this class has an infinite mean and, in fact, infinite moments of all orders, this distribution is not of much interest in insurance contexts (see Panjer and Willmot (1992) p-250). The pgf and pf of the $(a, b, 0)$ subclass are shown in table 4 and will be used later in this article to find pgf and pf for a basic LPD.

Table 4: **The pgf of $(a, b, 0)$ class of claim frequency**

Name	range	p_n	pgf	a	b
Poisson	$[0, \infty)$	$\frac{\lambda^n}{n!}e^{-\lambda}$	$e^{\lambda(t-1)}$	0	λ
Binomial	$[0, N]$	$\binom{N}{n}\theta^n(1-\theta)^{N-n}$	$(1-\theta+\theta t)^N$	$\frac{\theta}{\theta-1}$	$\frac{(N+1)\theta}{1-\theta}$
Geometric	$[0, \infty)$	$(1-p)^r p^n$	$\frac{1-p}{1-pt}$	p	0
Negative Binomial	$[0, \infty)$	$\frac{\Gamma(n+r)}{\Gamma(r)n!}(1-p)^r p^n$	$(\frac{1-p}{1-pt})^r$	p	$(r-1)p$

Sundt (1992) introduced a class of discrete claim frequency distribution whose pf satisfies the following higher order difference equation.

$$p_n = \sum_{i=1}^{m}(a_i + \frac{b_i}{n})\, p_{n-i} \quad \text{for } n = r+1, r+2, \ldots, \infty \tag{3.8}$$

where r is a non-negative integer while m is a strictly positive integer. With the convention that p_n equals zero for $n < 0$. Hence we have the following definition.

Definition 6 *The class of discrete distribution whose pf satisfies the $m-th$ order difference equation (3.8) will be called Sundt's $(\mathbf{a};\mathbf{b})$ family where $\mathbf{a}$ and $\mathbf{b}$ are both suitable vectors of m constants. To emphasize the value of r sometimes this family will also be called $(\mathbf{a};\mathbf{b};r)$ family.*

By definition, two pgf $f(t)$ and $g(t)$ with $g(0) \neq 0$ are needed to generate a Lagrange PDF. If pgf $f(t)$ and $g(t)$ both belong to a particular family of distributions, they will define a subfamily of Lagrange PDF. Hence we have the following definition.

Definition 7 *Let the pgf's $f(t)$ and $g(t)$ belong to Sundt's $(\boldsymbol{a};\boldsymbol{b})$ family with $g(0) \neq 0$. We will call the family of Lagrange PDF generated by them as the* **Lagrange $(\boldsymbol{a};\boldsymbol{b})$ family**

Note that the $(\boldsymbol{a};\boldsymbol{b})$ family contains the (a,b) family since the second is a particular case of the first for $m = 1$.

3.2 Probability generating function of (a, b) family

The probability generating function of the (a, b) class is given by the following theorem.

Theorem 3.1 *For members of the general (a, b) class, the pgf, $P(z)$, is given by the differential equation*

$$P'(z) - \frac{a+b}{1-az}P(z) = \frac{1}{1-az}\sum_{n=1}^{r} n\left[p_n - (a+\frac{b}{n})p_{n-1}\right]z^{n-1}. \tag{3.9}$$

Proof : First rewrite (3.7) in the form

$$np_n = [a(n-1) + (a+b)]\,p_{n-1}$$

Then multiply both sides by z^{n-1} and sum over $n = r+1, r+2, \ldots \infty$. On simplification we get the result. □

Note that, from the general results, we can easily get the pgf for (a, b) class and for its two subclasses. Taking $r = 1$, we have the pgf for (a, b) class as

$$P(z) = \begin{cases} \left(\frac{1-a}{1-az}\right)^{\frac{a+b}{a}} - \left[\frac{p_1-(a+b)p_0}{a+b}\right]\left[1 - \left(\frac{1-a}{1-az}\right)^{\frac{a+b}{a}}\right] & \\ e^{b(z-1)} - \left[\frac{p_1-bp_0}{b}\right]\left[1 - e^{b(z-1)}\right] & \text{for} \quad a = 0 \end{cases} \tag{3.10}$$

For $(a, b, 1)$ subclass with $p_0 = 0$, the pgf is given by

$$P(z) = \begin{cases} \left(\frac{1-a}{1-az}\right)^{\frac{a+b}{a}} \left(1+\frac{p_1}{a+b}\right) - \frac{p_1}{a+b} & \\ e^{b(z-1)}\left(1+\frac{p_1}{b}\right) - \frac{p_1}{b} & \text{for} \quad a = 0 \end{cases} \tag{3.11}$$

For $r = 0$, $P(z)$ satisfies the simple differential equation

$$P'(z) = \frac{a+b}{1-az} P(z) \tag{3.12}$$

and hence the pgf for the $(a, b, 0)$ subclass is given by

$$P(z) = \begin{cases} \left(\frac{1-a}{1-az}\right)^{\frac{a+b}{a}} & \text{for} \quad a \neq 0 \\ e^{b(z-1)} & \text{for} \quad a = 0 \end{cases} \tag{3.13}$$

For $a < 0$, $a = 0$, and $0 < a < 1$, $P(z)$ becomes pgf of binomial, Poisson, and negative binomial distributions respectively. Note that for the $(a, b, 0)$ subclass,

$$p_0 = \begin{cases} (1-a)^{\frac{a+b}{a}} & \\ e^{-b} & \text{for} \quad a = 0 \end{cases}$$

and

$$p_1 = \begin{cases} (a+b)(1-a)^{\frac{a+b}{a}} & \\ be^{-b} & \text{for} \quad a = 0 \end{cases}$$

while for the $(a, b, 1)$ subclass,

$$p_0 = 0$$

$$\text{and} \qquad p_1 = \begin{cases} \left(\frac{(a+b)(1-a)^{\frac{a+b}{a}}}{1-(1-a)^{\frac{a+b}{a}}} \right) & \\ \\ \left\{ \frac{be^{-b}}{1-e^{-b}} \right\} & \text{for} \quad a = 0 \end{cases}$$

and p_n is given by (3.7) for all $n > 1$. The above initial values will be required later in this article when we generate recursions for LPD.

Obviously all members in the $(a, b, 0)$ subclass which are truncated at zero are members in the $(a, b, 1)$ subclass. Since the $(a, b, 0)$ subclass is the simplest, the subsequent results will, at first, be derived for that class and comment will follow, if necessary, for the other subclasses.

Katz (1963) considered a family of discrete distributions, known as *Katz family*, which has pgf

$$P(z) = \left(\frac{1-\alpha}{1-\alpha z} \right)^{\frac{\beta}{\alpha}} \tag{3.14}$$

It is obvious that $\alpha < 0$, $\alpha = 0$, and $0 < \alpha < 1$ give rise to binomial, Poisson and negative binomial distributions. In fact, the Katz family is identical with our $(a, b, 0)$ subclass. In the rest of this article, we will find suitable recursions for compound Lagrange distribution for the $(\boldsymbol{a}; \boldsymbol{b}; 0)$ subfamily.

3.3 Probability generating function of Sundt's $(a; b)$ family.

Having previously defined Sundt's $(\boldsymbol{a}; \boldsymbol{b})$ family, we have the following general theorem for its pgf.

Theorem 3.2 *For members of Sundt's* $(\boldsymbol{a}; \boldsymbol{b})$ *class, the pgf.* $P(z)$, *satisfies the differential equation*

$$P'(z)(1 - \sum_{i=1}^{m} a_i z^i)$$

$$= \sum_{i=1}^{m}(ia_i + b_i)z^{i-1}P(z) + \sum_{n=1}^{r} n\left\{p_n - \sum_{i=1}^{m}(a_i + \frac{b_i}{n})\, p_{n-i}\right\} z^{n-1}$$

Proof:

First multiply both sides of (3.8) by nz^{n-1}. Then sum over n and simplify, and the results follows by rearrangement. □

The pgf is given by simple integration. In particular, if $(1 - \sum_{i=1}^{m} a_i z^i)$ has m distinct non-zero roots whose reciprocals are $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$(1 - \sum_{i=1}^{m} a_i z^i) = \prod_{i=1}^{m}(1 - \alpha_i z)$$

$$\text{and} \quad \frac{\sum_{i=1}^{m} b_i z^{i-1}}{(1 - \sum_{i=1}^{m} a_i z^i)} = \sum_{i=1}^{m} \frac{\beta_i}{(1 - \alpha_i z)} \tag{3.15}$$

then for $r = 0$, the pgf satisfies the differential equation

$$\frac{P'(z)}{P(z)} = \frac{\sum_{i=1}^{m}(ia_i + b_i)z^{i-1}}{(1 - \sum_{i=1}^{m} a_i z^i)} \tag{3.16}$$

and hence the pgf is given by

$$P(z) = \begin{cases} \prod_{i=1}^{m} \left(\frac{1-\alpha_i}{1-\alpha_i z}\right)^{\frac{\alpha_i + \beta_i}{\alpha_i}} & \\ e^{\left\{\sum_{i=1}^{m} \frac{b_i}{i}(z^i - 1)\right\}} & \text{when } a_i = 0 \text{ for all } i \end{cases}$$

Note that for $m = 1$, $\alpha_i \equiv a$ and $\beta_i \equiv b$, and then $P(z)$ becomes identical with the pgf of $(a, b, 0)$ family, the results derived earlier.

3.4 Probability generating function and pf of the Lagrange $(a; b)$ family

3.4.1 PGF for the basic LPD $(a; b)$ family

Let $g(t)$ be a pgf of a member of Sundt's $(\boldsymbol{a};\boldsymbol{b})$ family. From the definition of the basic LPD, we need $g_0 = g(0) \neq 0$. It will be clear in the following theorem that to get a suitable explicit functional equation of a pgf of a basic LPD, we must need that the ratio of $g'(t)$ and $g(t)$ be free of $g(t)$ and must be the ratio of polynomials in t only. This is possible when $g(t)$ is a pgf of a member of Sundt's $(\boldsymbol{a};\boldsymbol{b})$ family. Hence we have the following theorem.

Theorem 3.3 *Let $g(t)$ be the pgf of a discrete pf of Sundt's $(\boldsymbol{a};\boldsymbol{b})$ family with $g_0 > 0$, then the pgf of basic LPD, $\phi(u)$, satisfies the following first order ordinary differential equation:*

$$u\phi'(u)\left(1-\sum_{i=1}^{m}[(i+1)a_i+b_i]\phi(u)^i\right) = \phi(u)\left(1-\sum_{i=1}^{m}a_i\phi(u)^i\right) \quad (3.17)$$

Proof : Since $g(t)$ belongs to the $(\boldsymbol{a};\boldsymbol{b};0)$ family, we have from (3.17)

$$\frac{g'(t)}{g(t)} = \frac{\sum_{i=1}^{m}(ia_i+b_i)\,t^{i-1}}{1-\sum_{i=1}^{m}a_i t^i}$$

and by definition we have

$$\phi(u) = ug(\phi(u))$$

Taking logarithm of both sides and differentiating with respect to u we have

$$u\frac{\phi'(u)}{\phi(u)} - \frac{g'(t)}{g(t)}u\phi'(u) = 1$$

$$\text{or} \quad u\frac{\phi'(u)}{\phi(u)} - \frac{\sum_{i=1}^{m}(ia_i+b_i)\,t^{i-1}}{(1-\sum_{i=1}^{m}a_i t^i)}u\phi'(u) = 1$$

where $t = \phi(u)$. Hence cross multiplying and simplifying we have the desired results. □

Corollary 3.1 *For a basic LPD generated by a member of the* $(\boldsymbol{a};\boldsymbol{b})$ *class, the pf satisfies the following recursion.*

$$p_0 = 0$$

$$p_1 = C \text{ (the constant of integration in (3.18))} \quad (3.18)$$

$$= \left\{ \begin{array}{ll} \prod_{i=1}^{m}(1-\alpha_i)^{\frac{\alpha_i+\beta_i}{\alpha_i}} & \text{where } \alpha_i, \beta_i\text{'s are given by (3.17)} \\ \\ e^{-\left(\sum_{i=1}^{m}\frac{b_i}{i}\right)} & \text{for all} \quad a_i = 0 \end{array} \right\} \quad (3.19)$$

$$p_n = \frac{1}{n-1}\left\{\sum_{i=1}^{m}[(i+1)a_i + b_i]\sum_{y=1}^{n-1} y p_y p_{n-y}^{*i} - \sum_{i=1}^{m} a_i p_n^{*(i+1)}\right\} \quad (3.20)$$

$$\text{for } n = 2,\ 3,\ \ldots, \infty$$

where

$$p_n^{*(i+1)} = \sum_{y=1}^{n-1} p_y p_{n-y}^{*i} \quad \text{for } i = 1,\ 2,\ \ldots, m$$

Proof : Equate coefficients in (3.18) and rearrange to get (3.21). The initial values follow from the pgf $P(z)$ derived earlier. □

Note that the calculation of $p_n^{*(i+1)}$ does not need p_n for $i \geq 1$, since $p_0 = 0$. Obviously, for large m, the scheme loses its practicality. But for $m = 1$, the recursions is simple as is shown in the following corollary, and it works for some popular distributions as mentioned in the examples followed.

Corollary 3.2 *For a basic LPD generated by a member of* $(a, b, 0)$, *the pf satisfies the following recursion.*

$$p_0 = 0$$

$$p_1 = \left\{ \begin{array}{ll} (1-a)^{\frac{a+b}{a}} & \text{for} \quad a \neq 0 \\ \\ e^{-b} & \text{for} \quad a = 0 \end{array} \right.$$

$$p_n = \frac{1}{n-1}\left\{\sum_{y=1}^{n-1}[(2a+b)y-a]\,p_y p_{n-y}\right\} \quad for \;\; n = 2,\, 3,\, \ldots, \infty$$

Proof : The results follow from the previous corollary putting $i = 1$. □

Examples:

(i) Borel distribution:

(ii) Consul distribution:

The results could be verified for the basic LPD shown in table 1.

3.4.2 PGF for the general LPD $(a; b)$ family

Let $g(t)$ be a pgf of a member of $(a; b; 0)$ family and $f(t)$ be a pgf of a member of $(a; b)$ family. Then the pgf of general LPD generated by $g(t)$ and $f(t)$ is given by

$$\phi_*(u) = \sum_{n=0}^{\infty} p_n^* u^n \tag{3.21}$$

$$= f(\phi(u))$$

Where $\phi(u)$ is the pgf of basic LPD. Note that p_n^* could be calculated recursively in a two step recursion, namely

Step I : $\phi(u) = \sum_{n=0}^{\infty} p_n u^n$ where p_n is calculated by the recursion (3.21).

Step II : p_n^* could be calculated using Panjer's recursion, since $f(t)$ belongs to $(a; b)$ family.

Thus we have two independent methods of calculating the pf for the Lagrange $(a; b)$ family. The first method is the direct calculation by using the Lagrange expansion as explained in section 3.2 earlier in this article. The second method is the two step recursion as explained above. Both of the methods have their merits and demerits. The main advantage of the second method is that it helps to derive a recursion for the compound $(\boldsymbol{a};\boldsymbol{b})$ family. In the following section, the idea of two step recursion is exploited to find a recursive way of calculating compound Lagrange PDF for Lagrange $(\boldsymbol{a};\boldsymbol{b})$ family. In Step I, we will find recursion for compound basic LPD with the help of an auxiliary sequence and in step II, we will use Panjer's recursion to finally calculate pf for the compound Lagrange $(\boldsymbol{a};\boldsymbol{b})$ family.

4 Compound Lagrange $(a;b)$ family

Let X be a non-negative integer valued random variable, namely claim severity, with $m_x = P(X = x)$ where x is non-negative integer. If the claim frequency follows a distribution of Lagrange $(\boldsymbol{a};\boldsymbol{b})$ family then the total claim distribution follows a compound Lagrange $(\boldsymbol{a};\boldsymbol{b})$ family. The pgf of a compound basic LPD is given by

$$\psi(u) = \phi(M_X(u))$$

where $M_X(u) = \sum m_i u^i$ is the pgf of the random variable X. By definition the pgf of the Lagrange distribution, $\psi(u)$ should satisfy a differential equation and hence we have the following theorem.

Theorem 4.1 *Let the pgf $g(t)$ belong to the $(\boldsymbol{a};\boldsymbol{b};0)$ family. If $\psi(u)$ is the pgf of a compound basic LPD generated by $g(t)$ and $M_X(u)$, where $M_X(u)$ is a pgf of a non-negative integer valued random variable X, then $\psi(u)$ satisfies the following differential equation.*

$$\psi'(u)\left\{1-\sum_{i=1}^{m}[(i+1)a_i+b_i]\psi(u)^i\right\} = \psi(u)\left\{1-\sum_{i=1}^{m}a_i\psi(u)^i\right\}\frac{M_X'(u)}{M_X(u)} \quad (4.22)$$

Proof : By definition

$$\psi(u) = M_X(u)g(\psi(u))$$

Taking log and differentiating both sides with respect to u we have

$$\frac{\psi'(u)}{\psi(u)} - \frac{g'(\psi(u))}{g(\psi(u))}\psi'(u) = \frac{M_X'(u)}{M_X(u)}$$

Substituting the value of g'/g we have

$$\frac{\psi'(u)}{\psi(u)} - \frac{\sum_{i=1}^{m}(ia_i + b_i)\psi(u)^{i-1}}{1 - \sum_{i=1}^{m} a_i\psi(u)^i}\psi'(u) = \frac{M_X'(u)}{M_X(u)}$$

Now by cross multiplying and rearranging, we have the required results. □

Note that for $M_X(u) = u$, (that is X is degenerate at one) $\psi(u)$ is identical with $\phi(u)$ and this theorem reproduces the previous theorem. Now by equating the coefficients, we have a recursive relation to calculate compound basic LPD. Hence we have the following theorem.

Theorem 4.2 *Let the pgf $g(t)$ belong to the $(\boldsymbol{a};\boldsymbol{b};0)$ family and $\psi(u) = \sum_{i=0}^{\infty}\alpha_i u^i$ be the pgf of a compound basic LPD generated by $g(t)$ and $M_X(u)$, where $M_X(u)$ is a pgf of a non-negative integer valued random variable X. Also let*

$$\frac{uM_X'(u)}{M_X(u)} = \sum_{i=0}^{\infty}\xi_i u^i = \pi(u)$$

then the compound pf α_j can be calculated recursively using

$$\alpha_j = \left\{\frac{1}{j - h - \sum_{i=1}^{m}[(i+1)a_i + b_i]j\alpha_0^i}\right\} \times\left\{\sum_{i=1}^{m}[(i+1)a_i + b_i]\sum_{y=1}^{j-1} y\alpha_y\alpha_{j-y}^{*i} + \sum_{y=1}^{j}\xi_y\alpha_{j-y} - \sum_{i=1}^{m}a_i[\sum_{y=0}^{j}\xi_y\alpha_{j-y}^{*(i+1)}]\right\} \tag{4.23}$$

for $j = h+1, h+2, \ldots, \infty$, *and*

the initial value is given by

$$\alpha_i = 0 \quad for\ i < h$$

$$\alpha_h = m_h g(0) \quad for\ h > 0$$

and for $h = 0$, *the initial value is given implicitly by*

$$\alpha_0 = m_0 g(\alpha_0)$$

where ξ_i *can be calculated recursively first using*

$$\begin{aligned} \xi_i &= \frac{1}{m_h}\left((i+h)m_{i+h} - \sum_{j=0}^{i-1} \xi_i m_{i+h-j}\right), \\ & i = 0, 1, 2, \ldots \\ \xi_0 &= h \\ h &= min\{x \ : m_x = Pr(X = x) > 0\} \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} \xi_0 \alpha_j^{*(i+1)} &= h\alpha_j^{*(i+1)} \\ &= \begin{cases} 0 & for \quad h = 0 \\ 0 & for \quad h > 0, \quad and\ j < (i+1)h \\ h\sum_{y=h}^{j-ih} \alpha_y \alpha_{j-y}^{*i} & for \quad h > 0, \quad and \quad j \geq (i+1)h \end{cases} \end{aligned}$$

which is independent of α_j.

Proof : The results follows from previous theorem just by equating coefficients and rearranging. □

Note that for a family of Poisson type with $a = 0$, the above recursion to calculate α_j simplifies to

$$\alpha_j = \left\{\frac{1}{j - h - j\sum_{i=1}^{m} b_i\alpha_0^i}\right\}$$

$$\times\left\{\sum_{i=1}^{m} b_i \sum_{y=1}^{j-1} y\alpha_y\alpha_{j-y}^{*i} + \sum_{y=1}^{j} \xi_y\alpha_{j-y}\right\}$$

for $j = h+1, h+2, \ldots, \infty$.

Corollary 4.1 *For $g(t)$ belonging to $(a, b, 0)$ family, α_j could be calculated recursively by*

$$\alpha_j = \frac{\{(2a+b)\sum_{y=1}^{j-1} y\alpha_y\alpha_{j-y} + \sum_{y=1}^{j} \xi_y\alpha_{j-y} - a\sum_{y=0}^{j} \xi_y\alpha_{j-y}^{*2}\}}{j - h - (2a+b)j\alpha_0} \tag{4.25}$$

for $j = h+1, h+2, \ldots, \infty$, and

where initial values and ξ_i is calculated as in the previous theorem and

$$\xi_0\alpha_j^{*2} = h\alpha_j^{*2}$$

$$= \begin{cases} 0 & \text{for} \quad h = 0 \\ 0 & \text{for} \quad h > 0, \quad \text{and } j < 2h \\ h\sum_{y=h}^{j-h} \alpha_y\alpha_{j-y} & \text{for} \quad h > 0, \quad \text{and} \quad j \geq 2h \end{cases}$$

which is independent of α_j.

Proof : The results follows from previous theorem just by taking $i = 1$. Note that $\alpha_{j-y}^{*1} = \alpha_{j-y}$. □

Also note that for a Poisson family, $a = 0$, and the above recursion to calculate α_j simplifies to

$$\alpha_j = \frac{\{b\sum_{y=1}^{j-1} y\alpha_y\alpha_{j-y} + \sum_{y=1}^{j} \xi_y\alpha_{j-y}\}}{j - h - bj\alpha_0} \tag{4.26}$$

for $j = h+1, h+2, \ldots, \infty$.

By choosing suitable values for a and b from table 4 we can have recursion for the compound Borel distribution, the compound Consul distribution or the compound negative-delta distribution. Finally we are in a position to get two step recursion for compound general LPD. Let $\psi_*(u)$ be the pgf of a compound general LPD generated by the pgf's $f(t)$, $g(t)$, and $M_X(u)$, such that

$$\psi_*(u) = f(\psi(u))$$

where $\psi(u)$ is as in theorem 3.4.1, then the pf of compound general LPD is given by the following theorem.

Theorem 4.3 *Let the pgf $g(t)$ belong to $(\boldsymbol{a};\boldsymbol{b};0)$ family, the pgf $f(t)$ belong to $(\boldsymbol{a};\boldsymbol{b})$ family and $\psi_*(u) = \sum_{i=0}^{\infty} v_i u^i$ be the pgf of a compound general LPD generated by $f(t)$, $g(t)$ and $M_X(u)$, where $M_X(u)$ is a pgf of a non-negative integer valued random variable X. Also let*

$$\frac{uM_X'(u)}{M_X(u)} = \sum_{i=0}^{\infty} \xi u^i.$$

Then the compound pf v_j can be calculated recursively by the following two step recursion.

Step I : $\psi(u) = \sum_{n=0}^{\infty} \alpha_n u^n$ *where α_n is calculated by the theorem 3.4.1.*

Step II : *Compound general LPD given by $\psi_*(u) = \sum_{i=0}^{\infty} v_i u^i$ could be calculated by Panjer's recursion since $\psi_*(u) = f(\sum_{n=0}^{\infty} \alpha_n u^n)$.*

Examples

Generalized negative binomial (GNB) distributions : Let N, a claim count variable, be a GNB random variable and Z_i be the i-th claim amount discrete random variable. Then the total claims variable

$$S = Z_1 + Z_2 + \cdots + Z_N$$

is a compound GNB random variable. Our aim is to calculate the pf of S using the fact that a GNB random variable is a compound binomial sum of Consul distributions. Under the usual assumptions of $Z_1, Z_2, \ldots, Z_N$ are i.i.d. for given N and the distribution of N is free of $Z_i's$, the pgf of a compound GNB distribution is given by

$$G_S(u) = \Big(1 - \theta + \theta G_C(u)\Big)^k \tag{4.27}$$

were $G_C(u)$, the pgf of a compound Consul distribution, is given implicitly by

$$G_C(u) = \Big(1 - \theta + \theta G_C(u)\Big)^\beta G_Z(u) \tag{4.28}$$

and $G_Z(u)$ is the pgf of claim amount distributions. Our aim is to use (3.36) to find a recursion for compound Consul distribution and then use (3.35) to evaluate a compound GNB distribution using Panjer's (1981) results for compound binomial.

Let us take logarithm of both sides of (3.36) and differentiate with respect to u. We have

$$\frac{G_C'(u)}{G_C(u)} = \frac{\beta\theta G_C'(u)}{1 - \theta + \theta G_C(u)} + \frac{G_Z'(u)}{G_Z(u)} \tag{4.29}$$

Rearranging we have

$$uG_C'(u)\{1 - \theta + \theta(1-\beta)G_C(u)\} = G_C(u)\{1 - \theta + \theta G_C(u)\}\frac{uG_Z'(u)}{G_Z(u)} \tag{4.30}$$

Let us define the sequences $\{\alpha_n\}$ and $\{r_n\}$ given by

$$G_C(u) = \sum_{n=0}^{\infty} \alpha_n u^n \tag{4.31}$$

and $$\frac{uG_Z'(u)}{G_Z(u)} = \sum_{n=0}^{\infty} r_n u^n \tag{4.32}$$

The coefficients r_n depend solely on the known probability function (pf) of Z. Let $m_n = P[Z_i = n]$, $n = 0, 1, 2, \ldots$; and

$$m_0 = m_1 = \cdots = m_{h-1} = 0, \quad m_h > 0$$

So we have

$$G_Z(u) = \sum_{n=h}^{\infty} m_n u^n = u^h \sum_{n=0}^{\infty} m_{n+h} u^n \tag{4.33}$$

Hence, from (3.40), cancelling u^h from both numerator and denominator on the left hand side and transposing the denominator to the right hand side we have

$$\sum_{n=0}^{\infty}(n+h)m_{n+h}u^n = \left\{\sum_{n=0}^{\infty} m_{n+h}u^n\right\}\left\{\sum_{n=0}^{\infty} r_n u^n\right\}$$

Now comparing coefficients of u^n we have

$$(n+h)m_{n+h} = \sum_{j=0}^{n} r_j m_{n+h-j} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Then the sequence $\{r_n\}$ can be evaluated recursively as follows:

$$r_n = \frac{1}{m_h}\left\{(n+h)m_{n+h} - \sum_{j=0}^{n-1} r_j m_{n+h-j}\right\} \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{4.34}$$

with $h = min\{j : m_j > 0\}$. Note that $r_0 = h$.

Having evaluated the sequence $\{r_n\}$, we are in a final stage of evaluating the desired sequence $\{\alpha_n\}$, the compound Consul probability distribution. Comparing the coefficients of u^n in (3.38) we have

$$(1-\theta)n\alpha_n + \theta(1-\beta)\sum_{j=0}^{n} j\alpha_j\alpha_{n-j} = (1-\theta)\sum_{j=0}^{n}\alpha_j r_{n-j} + \theta \sum_{j=0}^{n}\alpha_j^{*2} r_{n-j} \tag{4.35}$$

Hence rearranging we have the recursion in its most general form :

$$\alpha_n = \frac{\sum_{j=0}^{n-1}[(1-\theta)\alpha_j + \theta\alpha_j^{*2}]r_{n-j} + \sum_{j=1}^{n-1}\theta[h + (\beta-1)j]\alpha_j\alpha_{n-j}}{(n-h)(1-\theta) - n\theta(\beta-1)\alpha_0}$$
$$\text{for} \quad n = h+1, h+2, \ldots \tag{4.36}$$

where α_n^{*2} is the second order convolution of α_n.

Now the initial values $\alpha_0, \alpha_1, \ldots, \alpha_h$ will be determined from the definition of $G_C(u)$. Since by definition,

$$G_C(u) = \left(1 - \theta + \theta G_C(u)\right)^{\beta} G_Z(u)$$

we have

$$\left\{\sum_{n=0}^{\infty} \alpha_n u^n\right\} = \left(1 - \theta + \theta \sum_{n=0}^{\infty} \alpha_n u^n\right)^{\beta} \left\{\sum_{n=h}^{\infty} m_n u^n\right\}$$

Now comparing the coefficients of u^n we have for $h > 0$

$$\alpha_0 = \alpha_1 = \cdots = \alpha_{h-1} = 0 \quad \text{and} \quad \alpha_h = (1-\theta)^{\beta} m_h$$

For $h = 0$, the value of α_0 is given by the implicit relation

$$\alpha_0 = (1 - \theta + \theta\alpha_0)^{\beta} m_0$$

Even though α_0 does not have an explicit expression in m_0, θ and β, it has a unique value given by the above relation where uniqueness is guaranteed by the GNB distribution parametric restriction namely $\theta\beta < 1$.

Finally, the compound GNB distribution given by $G_S(u) = \sum v_j u^j = (1 - \theta + \theta \sum_{n=0}^{\infty} \alpha_n u^n)^k$, is evaluated in the second step by application of Panjer's recursion for compound binomial distributions. In our case it is given by

$$v_j = \frac{\theta}{j(1 - \theta + \theta\alpha_0)} \sum_{i=1}^{j} \{(k+1)i - j\} \alpha_i v_{j-i}$$

with the starting value $P(S = 0) = G_S(0) = v_0 = (1 - \theta + \theta\alpha_0)^k$. ■

Note that a similar recursive scheme for compound generalized Poisson distributions were derived by Sharif and Panjer (1995).

5 Comments and further research

Our two step recursive scheme for compound Lagrange distribution is simple. Computationally it is more efficient as compared to some other schemes found in the literature for some specific members of the family, such as, Goovaerts and Kaas (1991), and Ambagaspitiya and Balakrishnan (1994). Also our recursion is a generalized result in the sense that h could be any non-negative (including zero). Our approach could be easily used to write down exact recursive relations for all the members of Lagrange $(\boldsymbol{a}; \boldsymbol{b})$-family, in particular for all the members shown in the table 1, table 2 and table 3, to evaluate their compound distributions.

The general Lagrange probability model was originally developed in the field of reliability specially in queueing theory. It had its application in finding the distribution of a busy period in a queueing model. The generalized Poisson model, a popular member of the family, has been used in several other statistical research areas most specifically, biostatistics. Consul (1990) used it to model the distribution of injuries in auto-accidents. Consul (1989) described several chance mechanisms generating this distribution. An actuarial application of the GPD can be found in Gerber (1990) where it is linked to the ruin model.

Because of the peculiarity of being embedded in a Lagrange expansion, the Lagrange distributions have not been very popular. We believe that our through treatment of the distribution and our efficient algorithm to calculate its pf and its compound pf might entice more actuaries to use the Lagrange distribution in actuarial research.

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