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A solution of defective renewal equations with applications to ruin theory

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1. Introduction

In this note we present an approach in which a defective renewal equation is solved in terms of a related compound geometric distribution. The advantage of this approach is discussed. We then apply this approach to a defective renewal equation which involves the time of ruin, the surplus before ruin and the total deficit at the time of ruin. We show how to compute the moments of the time of ruin as well as the joint distribution of the surplus before ruin and the deficit at the time of ruin. Two important cases in terms of claim size distributions are considered in detail. First, we consider exponential claim size distributions. We then consider a combination of two exponentials. The moments of the time of ruin are given in both cases.

2. A general defective renewal equation

Defective renewal equations have been widely used in the analysis of the surplus process in the classical risk model. In this section we present an approach which expresses the solution of a defective renewal equation in terms of the tail probability of a compound geometric distribution.

Consider the following equation

$$\phi(u) = \frac{1}{1+\beta}\int_0^u \phi(u-x)dG(x) + \frac{1}{1+\beta}H(u), u \geq 0, \tag{1}$$

where $\beta > 0$, $G(x)$ is a distribution function which usually represents some claim size distribution in risk theory, and $H(u)$ is a differentiable function for $u > 0$.

To solve (1), we introduce an associated compound geometric distribution. Let

$$\overline{K}(u) = \sum_{n=1}^{\infty} \frac{\beta}{1+\beta}(\frac{1}{1+\beta})^n \overline{G}^{*n}(u). \tag{2}$$

Then $\overline{K}(u)$ is the solution of the integral equation

$$\overline{K}(u) = \frac{1}{1+\beta}\int_0^u \overline{K}(u-x)dG(x) + \frac{1}{1+\beta}\overline{G}(u). \tag{3}$$

It can be shown (Lin and Willmot, 1997) that the solution $\phi(u)$ in (1) can be expressed as

$$\phi(u) = -\frac{1}{\beta}\int_0^u \overline{K}(u-x)H'(x)dx + \frac{1}{\beta}H(u) - \frac{H(0)}{\beta}\overline{K}(u), \tag{4}$$

which may be verified by the Laplace transform. This formula has several advantages. A closed-form solution is available for three important classes of distributions: exponentials, combinations of exponentials and mixtures of Erlangs. We will discuss the first two classes in later sections and the class of mixtures of Erlangs is discussed in Lin and Willmot (1997). The tail of a compound distribution can be evaluated or approximated fairly easily. The asymptotic formula is available when the moment generating function of the claim size distribution exists (Gerber, 1979). More accurate approximations such as one given by Tijms are also available (Tijm, 1994). Simple upper and lower bounds can be obtained even when the claim size distribution is subexponential (Willmot, 1994; Lin, 1996).

3. Applications to ruin theory

In this section we apply our result in the previous section to ruin theory. Consider the classical continuous time risk model, where the number of claims from an insurance portfolio is assumed to follow a Poisson process N_t with mean λ. The individual claim sizes $X_1, X_2, ...$, independent of N_t, are positive, independent and identical random variables with common distribution function (df) $P(x) = 1 - \overline{P}(x) = Pr(X \leq x)$. The aggregate claims process is $\{S_t;\ t \geq 0\}$ where $S_t = X_1 + X_2 + \cdots + X_{N_t}$ (with $S_t = 0$ if $N_t = 0$). The insurer's surplus process is $\{U_t;\ t \geq 0\}$ with $U_t = u + ct - S_t$ where $u \geq 0$ is the initial surplus, $c = \lambda p_1(1+\theta)$ the premium rate per unit time, and $\theta > 0$ the relative security loading, with moments $p_j = \int_0^\infty x^j dP(x)$ for $j = 0, 1, 2, ...$

Define $T = \inf\{t;\ U(t) < 0\}$ to be the first time that the surplus becomes negative and is called the time of ruin. The probability $\psi(u) = Pr\{T < \infty\}$ is called the probability of (ultimate) ruin. Two nonnegative random variables in connection with the time of ruin are the surplus immediately before ruin $U(T-)$, where $T-$ is the left limit of T and the ruin $|U(T)|$. See Bowers et al (1986, Chapter 12) for details.

Gerber and Shiu (1997a) define

$$\phi(u) = E\{e^{-\delta T} w(U(T-), |U(T)|)\ I(T < \infty)\} \tag{5}$$

where $w(x_1, x_2)$, $0 \leq x_1, x_2 < \infty$, is a nonnegative function, $I(T < \infty) = 1$, $T < \infty$, and $I(T < \infty) = 0$ otherwise. This function is useful in the sense that many functions in ruin theory can be viewed as a special case. For instance, if $\delta = 0$ and $w(x_1, x_2) = 1$, $\phi(u)$ becomes the probability of ruin and if $\delta = 0$ and $w(x_1, x_2) = I(x_2 \leq y)$, for fixed y, $\phi(u)$ becomes the distribution function of the deficit at ruin.

Gerber and Shiu (1997a) show that the function $\phi(u)$ satisfies the defective renewal equation

$$\phi(u) = \frac{\lambda}{c}\int_0^u \phi(u-x)\int_x^\infty e^{-\rho(y-x)}dP(y)dx + \frac{\lambda}{c}e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x, y-x)dP(y)dx, \tag{6}$$

where $\rho = \rho(\delta)$ is the unique nonnegative solution of the equation

$$c\rho - \delta = \lambda - \lambda\tilde{p}(\rho). \tag{7}$$

The equation (6) is in the form of (1). In Lin and Willmot (1997), we identify the parameter and the functions β, $G(x)$, and $H(u)$ in (1). In fact,

$$\overline{G}(x) = \frac{e^{\rho x}\int_x^\infty e^{-\rho y}\overline{P}(y)dy}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy}, \qquad x \geq 0, \tag{8}$$

$$\beta = \frac{1+\theta}{\int_0^\infty e^{-\rho y}dP_1(y)} - 1, \tag{9}$$

where $P_1'(y) = \frac{1}{p_1}\overline{P}(y)$, and

$$H(u) = \frac{e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x, y-x)dP(y)dx}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy}. \tag{10}$$

We remark that only the function $H(u)$ depends on the function $w(x_1, x_2)$.

With different choices of $w(x_1, x_2)$, we are able to obtain the distribution of the deficit at ruin, the distribution of the surplus immediately before ruin and the Laplace transform of the time of ruin. The Laplace transform of the time of ruin can be used to calculate the higher moments of the time of ruin. In previous research, the calculation of the moments of the time of ruin has been limited to the first moment.

4. The joint distribution of surplus before ruin and deficit at ruin

In this section, we give the joint distribution of the surplus before and the deficit at ruin using the formula (1). For any fixed x and y, choose

$$w(x_1, x_2) = \begin{cases} 1, & x_1 \leq x, x_2 \leq y \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

Noting that $\delta = 0$ implies $\rho = 0$, the function $H(u)$ defined in (10) can be written as follows: if $u < x$,

$$\begin{aligned} H(u) &= \frac{\lambda(1+\beta)}{c} e^{\rho u} \int_u^x e^{-\rho x_1} \int_{x_1}^{x_1+y} dP(x_2) dx_1 \\ &= [\overline{P}_1(u) - \overline{P}_1(u+y)] - [\overline{P}_1(x) - \overline{P}_1(x+y)]; \end{aligned} \tag{12}$$

if $u \geq x$,

$$H(u) = 0.$$

With a little algebra, we have the joint distribution function of $U(T-)$ and $|U(T)|$, $F(x, y|u)$.

$$F(x,y|u) = \begin{cases} \frac{1}{\theta p_1}[\int_0^x \psi(u-x_1)\overline{P}(x_1)dx_1 - \int_0^x \psi(u-x_1)\overline{P}(x_1+y)dx_1] \\ \quad + \frac{\psi(u)}{\theta}[P_1(x+y) - P_1(x) - P_1(y)], & x \leq u \\ \frac{1+\theta}{\theta}[\psi(u) - \psi(u+y)] + \frac{1}{\theta}\int_0^y \psi(u+y-x_1)dP_1(x_1) \\ \quad + \frac{\psi(u)}{\theta}[P_1(x+y) - P_1(x) - P_1(y)] - \frac{1}{\theta}[P_1(x+y) - P_1(x)], & x > u. \end{cases} \tag{13}$$

The formula for the moments of the time of ruin involves the solutions of a sequence of defective renewal equations. We present our results without derivation. Interested readers can refer to Lin and Willmot (1997) for details.

Let $\psi_k(u) = E\{T^k I(T < \infty)\}$. Thus, the k-th moment of the time of ruin is

$$\frac{\psi_k(u)}{\psi(u)},$$

where $\psi(u)$ is the probability of ruin, or $\psi_k(u)$ is the unconditional k-th moment of the time of ruin. It is worth to point out that when $\delta = 0$, $\psi(u) = \overline{K}(u)$.

The unconditional mean time of ruin is given by

$$\psi_1(u) = \frac{1}{\lambda p_1 \theta}\left\{\int_0^u \psi(u-x)\psi(x)dx + \int_u^\infty \psi(x)dx - \frac{p_2}{2p_1\theta}\psi(u)\right\}. \tag{14}$$

The unconditional k-th moment of the time of ruin is given recursively by

$$\psi_k(u) = \frac{k}{\lambda p_1 \theta}\left\{\int_0^u \psi(u-x)\psi_{k-1}(x)dx + \int_u^\infty \psi_{k-1}(x)dx - \psi(u)\int_0^\infty \psi_{k-1}(x)dx\right\}. \tag{15}$$

5. Exponential distribution

The exponential distribution is one of the simplest distibutions but it has been used widely due to its nice properties and simplicity. In this case, the claim size distribution $P(x)$ is defined as $\overline{P}(x) = e^{-\mu x}$. Then, $\overline{G}(x) = e^{-\mu x}$. Thus,

$$\overline{K}(u) = Ce^{-Ru}, \tag{16}$$

where

$$R = \frac{\beta\mu}{1+\beta} = \frac{\theta\mu}{1+\theta} + \frac{\rho}{\rho+\mu}\frac{\mu}{1+\theta},$$

and

$$C = \frac{1}{1+\beta} = \frac{\mu}{\rho+\mu}\frac{1}{1+\theta}.$$

It can be shown by induction that

$$\psi_k(u) = e^{-Ru}\sum_{j=0}^{k}\overline{C}_{j,k}\frac{(Ru)^j}{j!}. \tag{17}$$

The coefficients $\overline{C}_{j,k}$ satisfy the following recursion relations:

$$\overline{C}_{0,k} = \frac{k(1+\theta)}{c\mu\theta}\sum_{i=0}^{k-1}\overline{C}_{i,k-1} \tag{18}$$

and for $j = 1, 2, \cdots, k$,

$$\overline{C}_{j,k} = \frac{k(1+\theta)^2}{c\mu\theta^2}\Big[\frac{1}{1+\theta}\overline{C}_{j-1,k-1} + \sum_{i=j}^{k-1}\overline{C}_{i,k-1}\Big], \tag{19}$$

with $\sum_{i=k}^{k-1} = 0$ and $\overline{C}_{0,0} = C$. The first two moments can be easily derived using (18) and (19).

$$\begin{aligned}\overline{C}_{0,1} &= \frac{1+\theta}{c\mu\theta}\overline{C}_{0,0} = \frac{1}{c\mu\theta},\\ \overline{C}_{1,1} &= \frac{(1+\theta)^2}{c\mu\theta^2}\frac{1}{1+\theta}\overline{C}_{0,0} = \frac{1}{c\mu\theta^2}.\end{aligned}$$

Thus,

$$E\{TI(T<\infty)\} = \Big[\frac{1}{c\mu\theta} + \frac{1}{c\mu\theta^2}(Ru)\Big]e^{-Ru},$$

which is in agreement with Gerber (1979, p.138).

Also,

$$\begin{aligned}\overline{C}_{0,2} &= \frac{2(1+\theta)}{c\mu\theta}\Big[\overline{C}_{0,1} + \overline{C}_{1,1}\Big] = \frac{2(1+\theta)^2}{c^2\mu^2\theta^3},\\ \overline{C}_{1,2} &= \frac{2(1+\theta)^2}{c\mu\theta^2}\Big[\frac{1}{1+\theta}\overline{C}_{0,1} + \overline{C}_{1,1}\Big] = \frac{2(1+2\theta)}{c^2\mu^3\theta^3(1+\theta)},\\ \overline{C}_{2,2} &= \frac{2(1+\theta)^2}{c\mu\theta^2}\frac{1}{1+\theta}\overline{C}_{1,1} = \frac{2(1+\theta)}{c^2\mu^2\theta^4}.\end{aligned}$$

$$E\{T^2I(T<\infty)\} = \frac{2}{c^2\mu^2\theta^3}\Big[(1+\theta)^2 + \frac{1+2\theta}{\mu(1+\theta)}(Ru) + \frac{1+\theta}{\theta}\frac{(Ru)^2}{2!}\Big]e^{-Ru}.$$

6. Mixture or combination of two exponential distributions

The combination of two exponential distributions is useful in approximating the tail of a compound distribution as pointed out by Tijms (1994). In this case we may assume that the claim size df is a mixture of two exponentials, i.e.

$$\overline{P}(x) = qe^{-\mu_1 x} + (1-q)e^{-\mu_2 x},$$

let

$$q^* = \frac{q(\rho+\mu_1)^{-1}}{q(\rho+\mu_1)^{-1}+(1-q)(\rho+\mu_2)^{-1}}, \ \beta = (1+\theta)\frac{q\mu_1^{-1}+(1-q)\mu_2^{-1}}{q(\rho+\mu_1)^{-1}+(1-q)(\rho+\mu_2)^{-1}} - 1,$$

and

$$\Psi = (1-q^*)\mu_1 + q^*\mu_2.$$

Then,

$$\overline{K}(u) = C_1 e^{-R_1 u} + C_2 e^{-R_2 u}, \tag{20}$$

where $R_1 < R_2$ are two positive roots of the quadratic equation

$$(1+\beta)R^2 - [\Psi + \beta(\mu_1+\mu_2)]R + \beta\mu_1\mu_2 = 0,$$

and

$$C_1 = \frac{\Psi - R_1}{(1+\beta)(R_2-R_1)}, \quad C_2 = \frac{R_2 - \Psi}{(1+\beta)(R_2-R_1)}.$$

The form (20) for $\overline{K}(u)$ actually holds more generally than for combinations of exponentials, as derived above. See Tijms (1994), for example. Furthermore, in cases where explicit evaluation of $\overline{K}(u)$ is complicated or not possible, the Tijms approximation may be used, and it is also of the form (20). This approximation is quite accurate as long as beta is not too large, and is convenient for analytic purposes. In particular, substitution of (20) into (4) leads to a relatively tractable approximation. The results in the remainder of this section require only the form (20) for $\overline{K}(u)$, and do not require the claim size df to be a combination of two exponentials.

The unconditional k-th moment can be expressed as

$$\psi_k(u) = \sum_{j=0}^{k}\left[A_{j,k}e^{-R_1 u} + B_{j,k}e^{-R_2 u}\right]\frac{u^j}{j!}. \tag{21}$$

The coefficients $A_{j,k}$, $B_{j,k}$ are obtainable recursively, similar to the method used in the exponential case above. For example, the mean of the time of ruin is

$$E\{TI(T<\infty)\}$$

$$= \frac{1+\theta}{c\theta}\{C_1^2 u e^{-R_1 u} + C_2^2 u e^{-R_2 u}$$
$$+ \ [\frac{C_1}{(1+\theta)R_1} + \frac{C_1 C_2 (R_1^2 + R_2^2)}{R_1 R_2 (R_2 - R_1)}] e^{-R_1 u} + [\frac{C_2}{(1+\theta)R_2} - \frac{C_1 C_2 (R_1^2 + R_2^2)}{R_1 R_2 (R_2 - R_1)}] e^{-R_2 u}\}.$$

7. Concluding remarks

In this paper, we present a solution of a defective renewal equation in terms of the tail of a compound geometric distribution. With this approach, we are able to obtain the joint distribution of the surplus before ruin and the deficit at ruin for the classical risk model. We are also able to obtain the moments of the time of ruin when the claim size distribution is an exponential or a combination of two exponentials. The derivation of these results are given in Lin and Willmot (1997). Extensions and generalizations are possible. For instance, an explicit solution can be obtained when the claim size distribution is a mixture of Erlang distributions. Mixtures of Erlangs are an important distribution class in the sense that every claim size distribution can be approximated by a mixture of Erlangs with arbitrary accuracy. In Lin and Willmot (1997), we give a detail analysis of our approach. Results on the moments of the surplus before ruin, the moments of the deficit at ruin are given there.

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