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A Geometric Approach to Exact Solutions in Finance and Actuarial Science

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Closed Form Models or “Exact Solutions”

What ? Models in which calculus produces ‘simple’ formulae

Why? Exact Solution of an Approximate Problem

$\approx$ Approximate Solution of an Exact Problem

Insight can be gained by studying formulae (Education)

Can lead to efficient computational tools if model is accepted as practical (e.g. Black Scholes Option formula)

Can serve as starting point for more realistic computer based models

Closed Form Models or "Exact Solutions" (Contd.)

Limitations: "simple" formula is usually the result of a symmetry in the model $\Rightarrow$ model may be too special

Small change to problem may destroy symmetry

Example from classical compound interest theory:

$$PV = \frac{1}{1+i_1} + \frac{1}{(1+i_1)(1+i_2)} + \ldots + \frac{1}{(1+i_1)\ldots(1+i_n)}$$

If $i_1 = i_2 = \ldots = i_n = i$ (symmetry) then

$$PV = \frac{1 - v^n}{i}$$ "Exact Solution"

An important practical result in a low tech environment.

Mathematical Formulation of Standard Problems in Finance

a) State variables $u^1,\ldots,u^n$ evolve according to

$$du^i = \lambda^i(t,u^j)\,dt + \sum_\alpha \sigma^i_\alpha \, dz^\alpha$$

b) Risk free instantaneous rate of return $r = r(t,u^j)$.

c) Vector of risk premia $\pi^i = \pi^i(t,u^j)$.

Then a security $B(t,u^i)$ which pays cash flow $CF(t,u^i)$ satisfies

$$\frac{\partial B}{\partial t} + \sum_i \lambda^i \frac{\partial B}{\partial u^i} + \frac{1}{2}\sum_{i,j} \sigma^{ij} \frac{\partial^2 B}{\partial u^i \partial u^j} = rB + \sum_i \pi^i \frac{\partial B}{\partial u^i} - CF(t,u^i).$$

with boundary condition $B(T,u^i) = F(u^i)$. (Note $\sigma^{ij} = \sum_\alpha \sigma^i_\alpha \sigma^j_\alpha$).

Examples of Related Problems in Actuarial Science

1) Traditional Life Insurance Policy with stochastic surrenders

$$\frac{\partial V}{\partial t} + \sum_i (\lambda^i - \pi^i)\frac{\partial V}{\partial u^i} + \frac{1}{2}\sum_{i,j} \sigma^{ij}\frac{\partial^2 V}{\partial u^i \partial u^j} + \sum_\alpha \mu^\alpha(t,u^j)[B_\alpha(t,u^i) - V] = rV + g - e$$

Where the $\mu^\alpha(t,u^i)$ are forces of mortality, surrender etc.
The B_α are the benefits paid on death, surrender etc.

2) Policy has an account balance R which evolves according to

$$dR = R(\mu - \Delta)dt + \sigma R dz(t)$$

Liability is assumed to be a function $V = V(t,u^i,R)$.

$$\frac{\partial V}{\partial t} + (r - \Delta)\frac{\partial V}{\partial R} + \frac{\sigma^2 R^2}{2}\frac{\partial^2 V}{\partial R^2} + \mu^w(t,R)[R - SC - V] + \mu^d(t)[\max(K,R) - V] = rV - e$$

The Role of Symmetry in Exact Solutions:

Stochastic Process $$du^i = \sum_j A_j^i(\theta^j - u^j)\,dt + dz^i$$

Backward Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \sum_{i,j} A_j^i(\theta^j - u^j)\frac{\partial f}{\partial u^i} + \frac{1}{2}\sum_i \frac{\partial^2 f}{(\partial u^i)^2} = 0 \qquad (*)$$

Symmetry Group $$(t,\boldsymbol{u}) * (\tau,\boldsymbol{g}) = (t+\tau,\boldsymbol{u} + e^{tA}\boldsymbol{g})$$

Theorem: If $f(t,\boldsymbol{u})$ is a solution of (*) then so is

$$\tilde{f}(t,\boldsymbol{u}) = f[(t,\boldsymbol{u})*(\tau,\boldsymbol{g})] = f(t+\tau,\boldsymbol{u}+e^{tA}\boldsymbol{g}).$$

Corollary: The constant function(is a solution of (*) which is mapped into itself under the group action (a similarity solution).

Value Rescaling

If $f(t,\boldsymbol{u})$ satisfies $\dfrac{\partial f}{\partial t} + \sum_i \lambda^i \dfrac{\partial f}{\partial u^i} + \sum_{i,j} \dfrac{\sigma^{ij}}{2} \dfrac{\partial^2 f}{\partial u^i \partial u^j} = 0$

Then $V(t,\boldsymbol{u}) = e^{\psi(t,\boldsymbol{u})} f(t,\boldsymbol{u})$ satisfies

$$\frac{\partial V}{\partial t} + \sum_i \lambda^i \frac{\partial V}{\partial u^i} + \frac{1}{2}\sum_{i,j} \sigma^{ij} \frac{\partial^2 V}{\partial u^i \partial u^j} = rV + \sum_i \pi^i \frac{\partial V}{\partial u^i}$$

where a) $$r = \frac{\partial \psi}{\partial t} + \sum_i \lambda^i \frac{\partial \psi}{\partial u^i} + \sum_{ij} \frac{\sigma^{ij}}{2}\left(\frac{\partial^2 \psi}{\partial u^i \partial u^j} - \frac{\partial \psi}{\partial u^i}\frac{\partial \psi}{\partial u^j}\right),$$

b) $$\pi^i = \sum_j \sigma^{ij} \frac{\partial \psi}{\partial u^j} .$$

Value rescaling is a symmetry preserving operation.

Value Rescaling (Contd.)

Symmetry action on solutions of a rescaled equation:

If $V = e^{\Psi} f$ is a solution then so is

$$\tilde{V}(t,\boldsymbol{u}) = e^{\Psi(t,\boldsymbol{u})} f[(t,\boldsymbol{u}) * (\tau,\boldsymbol{g})]$$

$$= e^{\Psi(t,\boldsymbol{u})} e^{-\Psi[(t,\boldsymbol{u}) * (\tau,\boldsymbol{g})]} e^{\Psi[(t,\boldsymbol{u}) * (\tau,\boldsymbol{g})]} f[(t,\boldsymbol{u}) * (\tau,\boldsymbol{g})].$$

$$= e^{\Psi(t,\boldsymbol{u})} e^{-\Psi[(t,\boldsymbol{u}) * (\tau,\boldsymbol{g})]} V[(t,\boldsymbol{u}) * (\tau,\boldsymbol{g})]$$

Green's function of rescaled equation is given by

$$G(t,\boldsymbol{u};s,\boldsymbol{v}) = e^{\Psi(t,\boldsymbol{u})} \Theta(t,\boldsymbol{u};s,\boldsymbol{v}) e^{-\Psi(s,\boldsymbol{y})}.$$

Application: Gaussian Interest Rate Models

$$du^i = \sum_j A_j^i(\theta^j - u^j)\,dt + dz^i.$$

Zero Coupon Bond $$B(t,u^i,T) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{\psi(t,u^i)}\Theta(t,u^i;T,v^j)\,e^{-\psi(T,v^j)}\,dv^1\ldots dv^n.$$

Where

$$\Theta(t,u^i;T,v^j) = \frac{\exp(-\frac{1}{2}[\mathbf{v} - \boldsymbol{\mu}(t,u^j,t)]^{\mathrm{T}} W^{-1}[\mathbf{v} - \boldsymbol{\mu}(t,u^i,T)])}{\sqrt{(2\pi)^n |W|}},$$

$$\boldsymbol{\mu}(t,\mathbf{u},T) = \mathrm{E}\mathbf{v}(T) = \theta + e^{-(T-t)A}[\mathbf{u} - \theta],$$

And

$$W(T-t) = \mathrm{E}[(\mathbf{v}(T)-\boldsymbol{\mu})(\mathbf{v}(T)-\boldsymbol{\mu})^{\mathrm{T}}] = e^{-(T-t)A}\left(\int_0^{T-t} e^{sA} e^{sA^{\mathrm{T}}}\,ds\right) e^{-(T-t)A^{\mathrm{T}}}.$$

Choose the function ψ so that the integral defining the bond values can be done.

A convenient choice is

$$e^{\psi(t,u^i)} = \frac{\exp[\sum_{ij}\Gamma_{ij}(t)u^iu^j+\sum_{i}\gamma_i(t)u^i+g(t)]}{p(t)+\sum_{a}p_a(t)u^a+\sum_{ab}p_{ab}(t)u^au^b+\dots}$$

The resulting expression for zero coupon bonds is then

$$B(t,u^i,T) = \exp[g(t)-g(T)+\sum_{i}\gamma_i(t)u^i+\sum_{ij}\Gamma_{ij}(t)u^iu^j+\frac{1}{2}(\Sigma_{ij}^{-1}\tilde{\mu}^i\tilde{\mu}^j-W_{ij}^{-1}\mu^i\mu^j)]$$

$$\times\sqrt{\frac{|\Sigma|}{|W|}}\;\frac{p(T)+\sum_{a}p_a(T)\tilde{\mu}^a+\sum_{ab}p_{ab}(T)(\tilde{\mu}^a\tilde{\mu}^b+\Sigma^{ab})+\dots}{p(t)+\sum_{a}p_a(t)u^a+\sum_{ab}p_{ab}(t)u^au^b+\dots}.$$

WHERE $\Sigma^{-1}(t,T) = W^{-1}(T-t) + 2\Gamma(T)$,

AND $\tilde{\boldsymbol{\mu}}(t,u^{i},T) = \Sigma(t,T)(W^{-1}(T-t)\ \boldsymbol{\mu}(t,u,T)-\gamma(T))$.

Can always choose *g(t)* so that the model fits an arbitrary initial yield curve.

Special Cases

3) If ψ is a linear function of the state variables the family includes the models of Vasicek , and Hull & White.

4) If ψ is a quadratic function of the state variables the family includes the quadratic model of Beaglehole and Tenney.

5) If ψ depends on u^{i} only through $\sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}$ and $A_i^j = A\delta_i^j$, $\theta^j = 0$, we get a class of single factor models related to the Cox, Ingersoll & Ross family of models.

Other Facts

$B^*(t,\boldsymbol{u}) = e^{\psi(t,\boldsymbol{u})}$ is a solution which is mapped into itself under the symmetry (a similarity solution).

Since the risk premium $\pi^i = \partial\psi/\partial u^i$ is built from B^*, this security maximizes expected return for a given level of risk.

Time Rescaling - The model of Cox, Ingersoll & Ross

Alternative Symmetry Group $\qquad (\tau, \boldsymbol{u}) * (s, \boldsymbol{y}) = (\tau e^{\boldsymbol{a}\cdot\boldsymbol{y}} + s, \boldsymbol{u} + \boldsymbol{y})$

Base Stochastic Process $\qquad du^i = e^{-\boldsymbol{a}\cdot\boldsymbol{u}} b^i d\tau + e^{-\boldsymbol{a}\cdot\boldsymbol{u}/2} dz^i(\tau)$

Rescale the time line $\qquad d\tau = e^{\gamma t} dt, \quad dz^i(\tau) = e^{\gamma t/2} dz^i(t),$

so that $\qquad du^i = e^{-\boldsymbol{a}\cdot\boldsymbol{u} + \gamma t} dt + e^{-(\boldsymbol{a}\cdot\boldsymbol{u} - \gamma t)/2} dz^i(t).$

Choose $\qquad \psi(t, \boldsymbol{u}) = [(\boldsymbol{a}\cdot\boldsymbol{b}) + \frac{\boldsymbol{a}\cdot\boldsymbol{a}}{2}]t - e^{\boldsymbol{a}\cdot\boldsymbol{u} - \gamma t},$

Then $$r = \frac{\partial \psi}{\partial t} + \sum_i \lambda^i \frac{\partial \psi}{\partial u^i} + \sum_{ij} \frac{\sigma^{ij}}{2} \left(\frac{\partial^2 \psi}{\partial u^i \partial u^j} - \frac{\partial \psi}{\partial u^i} \frac{\partial \psi}{\partial u^j} \right),$$

$$\Rightarrow \quad r = (\gamma - \frac{a^2}{2}) e^{\boldsymbol{a}\cdot\boldsymbol{u} - \gamma t}, \quad \text{and } \pi^i = \sum_j \sigma^{ij} \frac{\partial \psi}{\partial u^j} = a^i.$$

Assume $\gamma \geq a^2/2$ so that interest rates are positive.

Then the risk neutral stochastic process for r is

$$dr = [(\gamma - \frac{a^2}{2})(\mathbf{a}\cdot\mathbf{b} + \frac{a^2}{2}) - (\gamma - a^2)r]dt + \sqrt{\gamma - \frac{a^2}{2}}\,|\mathbf{a}|\sqrt{r}\,dw(t)$$

Zero coupon bond price $B(t,u^i,T)$ depends on u^i only through r.

$$\frac{\partial B}{\partial t} + [(\gamma - \frac{a^2}{2})(\mathbf{a}\cdot\mathbf{b} + \frac{a^2}{2}) - (\gamma - a^2)r]\frac{\partial B}{\partial r} + \frac{1}{2}(\gamma - \frac{a^2}{2})a^2 r\frac{\partial^2 B}{\partial r^2} = rB.$$

Solution has the form $B(t,r,T) = \alpha(t,T)e^{-\beta(t,T)r}$ where

$$\alpha(t,T) = \left[\frac{\gamma e^{(\gamma - a^2/2)(T-t)}}{(\gamma - a^2/2)(e^{\gamma(T-t)}-1)+\gamma}\right]^{(2\frac{\mathbf{b}\cdot\mathbf{a})}{a^2}-1)},$$

$$\beta(t,T) = \frac{e^{\gamma(T-t)}-1}{(\gamma - a^2/2)(e^{\gamma(T-t)}-1)+\gamma}.$$

Formal Geometric Results

A homogeneous exact solution is determined by

a) its symmetry group

b) its similarity solution e^{Ψ}

c) its time scale

The operations of value rescaling, time rescaling and coordinate transformation are the only operations which preserve the local point symmetry of the problem.

Not every (n+1) dimensional Lie Group can serve as the symmetry group of a financial model. Allowed structures are semi-direct products $G_n \times) R^1$, $R^1 \times) G_n$ for some n dimensional group G_n since the model must respect the existence of a global time function.

Formal Geometric Results (Contd.)

If $|\sigma| = \det \sigma^{ij} \neq 0$ define operators L, T, ∇^2 by

$$L(V) := \frac{\partial V}{\partial t} + \sum_i \lambda^i \frac{\partial V}{\partial u^i} + \frac{1}{2}\sum_{i,j} \sigma^{ij} \frac{\partial^2 V}{\partial u^i \partial u^j} - rV$$

$$= \left(\frac{\partial}{\partial t} + \sum_i [\lambda^i - \frac{1}{2}\sum_j \sqrt{|\sigma|} \frac{\partial}{\partial u^j} \frac{\sigma^{ij}}{\sqrt{|\sigma|}}] \frac{\partial}{\partial u^i} \right) V + \frac{1}{2}\left(\sum_{i,j} \sqrt{|\sigma|} \frac{\partial}{\partial u^i} \frac{\sigma^{ij}}{\sqrt{|\sigma|}} \frac{\partial}{\partial u^i} \right) V - rV$$

$$= T(V) + \frac{1}{2}\nabla^2 V - rV$$

Given a vector field $\xi = \xi^0(t)\frac{\partial}{\partial t} + \sum_i \xi^i(t,u^j)\frac{\partial}{\partial u^j}$ and scalar $p = p(t,u^j)$

define operators K and ∇ by

$$K(V) := \xi(V) - pV, \quad \nabla p := \sum_{ij} \sigma^{ij} \frac{\partial p}{\partial u^i} \frac{\partial}{\partial u^j}.$$

Formal Geometric Results (Contd.)

Theorem: If there exists a scalar $\lambda = \lambda(t)$ such that

$$L_\xi \sigma = \lambda \sigma$$

$$L_\xi T = \lambda T - \nabla p$$

$$L_\xi r = \lambda r - [T(p) + \frac{1}{2}\nabla^2 p]$$

Then $[K,L](V) = KL(V) - LK(V) = \lambda L(V)$.

Corollary I: If V is a solution of $L(V) = 0$ then so is $K(V)$.

Corollary II: If V is a solution of $L(V) = 0$ the so is $e^{sK}(V)$ when it is defined.

Corollary III: If K and K` are symmetry operators so is [K,K`].

Formal Geometric Result (Contd.)

Model is homogeneous if there exists a set of (n+1) linearly independent operators $K_A = \xi_A - p_A$ such that $K_A K_B - K_B K_A = \sum_C C_{AB}^{C} K_C$.

Homogeneity implies we can choose scalars ψ, ϕ so that $p_A = \xi_A(\psi)$, $\lambda_A = \xi_A(\phi)$, $[\xi_A, \xi_B] = \sum_C C_{AB}^{C} \xi_C$ and we can write the symmetry conditions as

$$L_{\xi_A} e^{-\phi} \sigma = 0$$

$$L_{\xi_A} e^{-\phi} (T + \nabla\psi) = 0$$

$$L_{\xi_A} e^{-\phi} [r - T\psi - \frac{1}{2}(\nabla^2\psi + \nabla\psi \cdot \nabla\psi)] = 0.$$

This allows us to solve for the quantities σ, T, r in terms of the scalars ϕ, ψ and a set of right invariant vector fields $(R_A) = (R, R_\alpha)$ which satisfy $L_{\xi_A} R_B = [\xi_A, R_B] = 0$, $[R_A, R_B] = -\sum_C C_{AB}^{C} R_C$.

Formal Geometric Result (Conclusion)

$$\sigma^{ij} = e^{\phi} \sum_{\alpha} R_{\alpha}^{i} R_{\alpha}^{j},$$

$$T = e^{\phi} R - \nabla\psi,$$

$$r = \rho e^{\phi} + T(\psi) + \frac{1}{2}[\nabla^2\psi + \nabla\psi\cdot\nabla\psi].$$

The function ψ can be chosen arbitrarily (value rescale). The existence of a universal time function t places restrictions the function ϕ and the group structure. Can show that $R_{\alpha}(t) = 0, \quad T(t) = 1$ imply:

$$[R_{\alpha}, R_{\beta}] = \sum_{\gamma} C_{\alpha\beta}^{\gamma} R_{\gamma},$$

$$[R, R_{\alpha}] = A_{\alpha} R + \sum_{\beta} A_{\alpha}^{\beta} R_{\beta},$$

$$R_{\alpha}(\phi) = A_{\alpha}.$$

This defines the function ϕ up to an arbitrary function $\phi_0(t)$ (time rescale). Additional restrictions apply to A_{α}, A_{α}^{β}, $C_{\alpha\beta}^{\gamma}$ (Jacobi Identity).

Strategy for Solving Problems with American Type Options

Basic Idea: Look for a combination of

1) Value Rescaling

2) Time Rescaling

3) Coordinate Transformations

4) Group Parameter Changes

Which leaves the risk neutral stochastic process unchanged but maps the interest rate r into a new function $r + \mu(t,u^j)$.

Start with

$$du^i = \sum_j [A_j^i(\theta^i - u^j) - \delta^{ij}\frac{\partial\Psi}{\partial u^j})dt + dz^i(t)$$

$$r = \frac{\partial\Psi}{\partial t} + \sum_{ij}[A_j^i(\theta^j - u^j)\frac{\partial\Psi}{\partial u^i} + \frac{1}{2}\delta^{ij}(\frac{\partial^2\Psi}{\partial u^i\partial u^j} - \frac{\partial\Psi}{\partial u^i}\frac{\partial\Psi}{\partial u^j})$$

Example: Value Rescale combined with a change in group

$$\psi \rightarrow \psi + \Delta\psi, \quad A_j^{\,i} \rightarrow A_j^{\,i} + \Delta A_j^{\,i}$$

Risk neutral stochastic process is preserved if

$$\sum_j [(A_j^{\,i} + \Delta A_j^{\,i})(\theta^j - u^j) - \delta^{ij}\frac{\partial(\psi + \Delta\psi)}{\partial u^j}] = \sum_j [A_j^{\,i}(\theta^j - u^j) - \delta^{ij}\frac{\partial\psi}{\partial u^j}].$$

This equation can be solved, **if the matrix** $\Delta A_j^{\,i}$ **is symmetric**, to get

$$\Delta\psi = \Delta\psi_0(t) + \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\Delta A(2\theta - \boldsymbol{u}).$$

The resulting decrement is calculated from

$$r + \mu = \frac{\partial\psi + \Delta\psi}{\partial t} + \sum_{ij}\left((A_j^{\,i} + \Delta A_j^{\,i})(\theta^j - u^j)\frac{\partial\psi + \Delta\psi}{\partial u^i} + \frac{\delta^{ij}}{2}[\frac{\partial^2\psi + \Delta\psi}{\partial u^i \partial u^j} - \frac{\partial\psi + \Delta\psi}{\partial u^i}\frac{\partial\psi + \Delta\psi}{\partial u^j}]\right).$$

and is a quadratic function of the state variables

$$\mu(t, u^j) = \frac{\partial\Delta\psi_0(t)}{\partial t} + \frac{1}{2}[(\theta - \boldsymbol{u})^{\mathrm{T}}\Delta A^{\mathrm{T}}(2A + \Delta A)(\theta - \boldsymbol{u}) - Tr\Delta A].$$

Conclusion: For Gaussian interest rate models we can solve the problem of an American option provided the option holder behaviour can be described by a force of option exercise which is linear or quadratic in the state variables.

For the linear or quadratic interest rate models this means we can solve the problem of option holder behaviour which is linear in an interest rate.