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The Mollification Analysis of Stochastic Volatility

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Abstract

One of the most important problems in Finance is the valuation of financial securities written on underlying assets whose prices are subject to uncertainty. Such uncertainty typically is described by a stochastic process:

$$dS = \mu(S,t)dt + \sigma(S,t)dW$$

where W is a standard Brownian Motion and σ is the instantaneous standard deviation of S which specifies its volatility.

This paper presented a technique to estimate the unknown stochastic volatilities by solving the inverse problem associated with the parabolic partial differential equation governing risk neutral derivative security prices. This technique can be applied in a very general multi-factor setting and numerical examples are provided.

Keywords Diffusion processes; Implied volatility; Ill-posed problems; Inverse Heat Conduction Problems; Mollification.

1. Introduction

The modern valuation theory of financial derivative securities begins with the modeling of the underline asset prices, indexes, or rates. Such modeling process use a stochastic process to describe the price movement of the underline asset , and typically incorporate one or more parameters that may be constants or deterministic functions. Assuming the chosen process is a realistic model for the underline dynamics, the successful application in hedging and other trading activities will depend critically on how the parameters are quantified.

Traditional approach is to assume constant volatility and use the Black-Scholes (1976) option formula and the quoted price to solve the "implied volatility'. In practice, however, the volatility varies with respect to time and correlated to the price movement of the underline assets.

Hull and White (1987) assumed the volatility follows a stochastic process itself.

Lagnado and Osher (1996) provided a calibrating derivative security pricing models w.r.t. observed market prices of all the options in the same class.

This work attempts to identify the unknown volatility from the risk-neutral model itself.

2. The Mathematical Model

Assuming the underlying asset S (stock or stock index) follows the general stochastic process of the form:

$$dS = \mu(S,t)Sdt + \sigma(S,t)SdW \tag{2.1}$$

where μ is the drift of S, W is a standard Brownian motion (Wiener process), and the local volatility σ is a deterministic function that may depend on both the stock or stock index and the time t.

We consider an European call written on assets S that pays continuous proportional dividends at a rate of δ, for example futures contracts or foreign currencies, with exercise price of K and the expiration date T.

We assume that markets are perfect, there are no transactions costs or differential taxes, and trading takes place continuously. We also assume that the asset price S of the underline assets on which the call is written follows a geometric Brownian motion with volatility σ, continuous sample path, and the drift equal to μ where $\mu > 0$ is the expected rate of return on the assets. Hence S is described by the stochastic differential equation. Using standard risk neutral approach (see, for example, Black and Scholes, 1973), one can show that the price of the European call could be evaluated by solving an equivalent partial differential equation (PDE). The associated PDE is discussed in the following section.

3. The PDE Problem

Assume that the underlying asset follows the general stochastic process specified in equation (2.1).

Let $v(S,t)$ be the value of the call option at time t, when the asset value is S and for which the strike price is K and the expiration date is T. Risk-neutral theory of asset pricing shows that v must satisfy the following heat equation:

$$v_t + (r-q)Sv_S + \frac{1}{2}\sigma^2(S,t)S^2 v_{SS} = rv, \qquad (3.1)$$

where r is the riskfree continuously compounded interest rate and q is the continuously dividend yield on the asset(stock or stock index).

In order to solve the above PDE, one also need to specifies boundary conditions. Due to the nature of the problem, the PDE subject to the following boundary conditions:

$$v(S,T) = max(S-K,0), \qquad S > 0, \qquad (3.2)$$

$$v(0,t) = 0, \qquad 0 \le t \le T, \qquad (3.3)$$

$$v_S(S,t) \to e^{-q(T-t)},\ as\, S \to \infty,\ 0 \le t \le T. \qquad (3.4)$$

This a classical heat problem if all the parameters in the PDE are given and can be solved using standard algorithm. Software packages are also available. In practice, however, the volatility $\sigma = \sigma(t, S)$ is usually unknown. In the case of unknown volatility, the problem (3.1)-(3.4) becomes an inverse heat conduction problem.

4. The Inverse Problem

To show that the problem (3.1)-(3.4) is a standard inverse heat conduction problem, let τ to be the time to maturity ($\tau = T - t$), substitute τ in Equation (3.1):

$$v_\tau = (r-q)Sv_S + \frac{1}{2}\sigma^2(S,\tau)S^2 v_{SS} - rv, \tag{4.1}$$

and

$$v(S, 0; \sigma) = max(S - K, 0), \qquad S > 0, \tag{4.2}$$

$$v(0, \tau; \sigma) = 0, \qquad 0 \leq \tau \leq T, \tag{4.3}$$

$$v_S(S, \tau; \sigma) \to e^{-q\tau},\ as\ S \to \infty,\ 0 \leq \tau \leq T. \tag{4.4}$$

Now, let

$$x = \ln S + (r - q)\,\tau, \tag{4.5}$$

and

$$v(S,\tau) = e^{-r\tau}\, u(x,\tau) \tag{4.6}$$

The equation (4.1) is then equivalent to solving $u(x,\tau)$ and $\sigma(x,\tau)$ satisfying

$$u_\tau = \frac{1}{2}\sigma^2(x,\tau)u_{xx}, \qquad (4.7)$$

with the boundary conditions corresponding to (4.2) - (4.4).

This is a standard heat problem of identifying diffusion coefficient. It is well known that the problem is an ill-posed problem because small errors in the data might induce large errors in the computed solution. For this reason, special techniques are needed in order to restore stability with respect to the data.

The one-dimensional IHCP has been discussed by many authors, and several different methods have been proposed for its solution. See Murio, 1993 and the references therein, for a complete description of the algorithms and their historical account. We now use the mollification technique to stabilized the problem.

5. Mollification

In this section we use the mollification method to stabilize system (4.7).

We introduce the function

$$\rho_{\delta,p}(t) = A_p \exp\left(-\frac{t^2}{\delta^2}\right), \qquad (5.1)$$

where $p > 0, \delta > 0$, and

$$A_p = (\int_{-p}^{p} \exp(-s^2)ds)^{-1}.$$

The kernel $\rho_\delta(t)$ falls to nearly zero outside the interval $(-p\delta,\, p\delta)$.

Let $I = [0,T], I_\delta = [p\delta, T - p\delta]$,

If f is integrable on I, we define its δ-mollification by the convolution

$$J_\delta f(t) = \int_0^1 \rho_\delta(t-s) f(s)\, ds, \qquad t \in I_\delta. \qquad (5.2)$$

The Fourier transform of a function $f(t)$ is defined by

$$\hat{f}(\omega) = \frac{1}{(2\pi)} \int_R f(t)\, e^{-i\omega t} dt,$$

$$-\infty < \omega < +\infty, i = \sqrt{-1}.$$

The convolution theorem allow us to evaluate the Fourier transform of $J_\delta f$ from

$$\widehat{J_\delta f}(\omega) = \hat{\rho}_\delta(\omega)\, \hat{f}(\omega) = \exp\left(-\frac{\delta^2}{4}\left(\omega^2\right)\right) \hat{f}(\omega).$$

The basic idea of the mollification method is that instead of attempting to find the point values of the function $f(y,t)$, we attempt to reconstruct the δ-mollification of the functions f at the point (y,t), given by

$$J_\delta f(y,t) = (\rho_\delta * f)(y,t).$$

For the purpose of illustration, we use a model problem in the following section and apply mollification analysis to find the solution.

6. The Model Problem

For the purpose of illustration, we use a model problem in the following section and apply mollification analysis to find the solution.

Find $\sigma(x)$ in I and u throughout the domain $[0,X] \times [0,T]$ of the (x,t) plane, from measured approximations of $f(t), g(t), \sigma$ and $h(x)$ satisfying

$$u_t = \frac{1}{2}\sigma^2(x)u_{xx}, \qquad 0 < t < T,\, 0 < x < X, \tag{6.1}$$

and

$$u(0,t) = f(t), \qquad 0 \le t \le T, \tag{6.2}$$

$$u_x(X,t) = g(t), \qquad 0 \le t \le T, \tag{6.3}$$

$$\sigma(0) = \sigma, \tag{6.4}$$

$$u(x,0) = h(x), \qquad 0 \le x \le X. \tag{6.5}$$

Notice that $f(t), g(t), \sigma$ and $h(x)$ are not known exactly. The available data $f^\epsilon, g^\epsilon, \sigma^\epsilon$ and $h(x)^\epsilon$ are measured approximation of $g(t), g(t), \sigma$ and $h(x)$, respectively, and satisfy the estimates

$$||f - f^\epsilon||_{\infty,I} \leq \epsilon,$$
$$||g - g^\epsilon||_{\infty,I} \leq \epsilon,$$
$$||\sigma - \sigma^\epsilon||_{\infty,I} \leq \epsilon,$$

and

$$||h - h^\epsilon||_{\infty,I} \leq \epsilon.$$

Mollifying system (6.1), we have the following associated problem: For some $\delta > 0$, find $v(x,t) = J_\delta u\,(x,t)$ and $\sigma\,(x)$ at the (x,t) points of interest, given that the mollified temperature function $J_\delta u\,(x,t)$ satisfies

$$v_t = \frac{1}{2}\sigma^2(x)v_{xx}, \quad 0 < x < X, \quad t > 0, \tag{6.6}$$

$$v(0,t) = J_\delta f^\epsilon(t), \qquad 0 \leq t \leq T, \tag{6.7}$$

$$v_x(X,t) = J_\delta g^\epsilon(t), \qquad 0 \leq t \leq T, \tag{6.8}$$

$$\sigma(0) = \sigma^\epsilon, \tag{6.9}$$

$$v(x,0) = J_\delta h^\epsilon(x), \qquad 0 \leq x \leq X. \tag{6.10}$$

Upon applying mollification method to filter measured data, Problem (6.6) - (6.10) become a stabilized problem. For the stability analysis, see Murio, 1993.

7. The Numerical Example

Since in practice only a discrete set of points is generally available, we shall assume that the date functions f^ϵ, g^ϵ σ^ϵ, and h^ϵ are discrete functions measured at sample points.

Let $K = \{t_1, t_2, \cdots, t_n\}$
$(0 \leq t_1 < t_2 < \cdots, < t_n \leq T)$ and

$\Delta t = max_j|t_{j+1} - t_j|$.

Let $G = \{g_j\}_{j=1}^n$ be a discrete function defined on K.
We define the discrete δ-mollification of G as follows:
for $t \in I_\delta$,

$$g_\delta(t) = \sum_{j=1}^{n} (\int_{s_{j-1}}^{s_j} \rho_\delta (t - s) \, ds) \, g_j, \tag{7.1}$$

where $s_0 = 0, s_n = 1$, and $s_j = \frac{1}{2}(t_j + t_{j+1}), (1 \leq j \leq n-1)$.

Let C represent a generic constant independent of the mollification parameter δ, and the grid size Δt. The stability analysis of discrete mollification technique were established by Mejia and Murio (1996).

Let $h = \Delta x = \frac{1}{M}$ and $k = \Delta t = \frac{1}{N}$ be the parameters of the finite difference discretization.

$\epsilon = 0.01$ and $\Delta x = \Delta t = 128$.
We solve numerically, system (6.1)-(6.5) with

$$f(t) = 2e^t, \quad g(t) = 0, \quad \sigma(0) = 2,$$

and

$$h(x) = 2 + x^2.$$

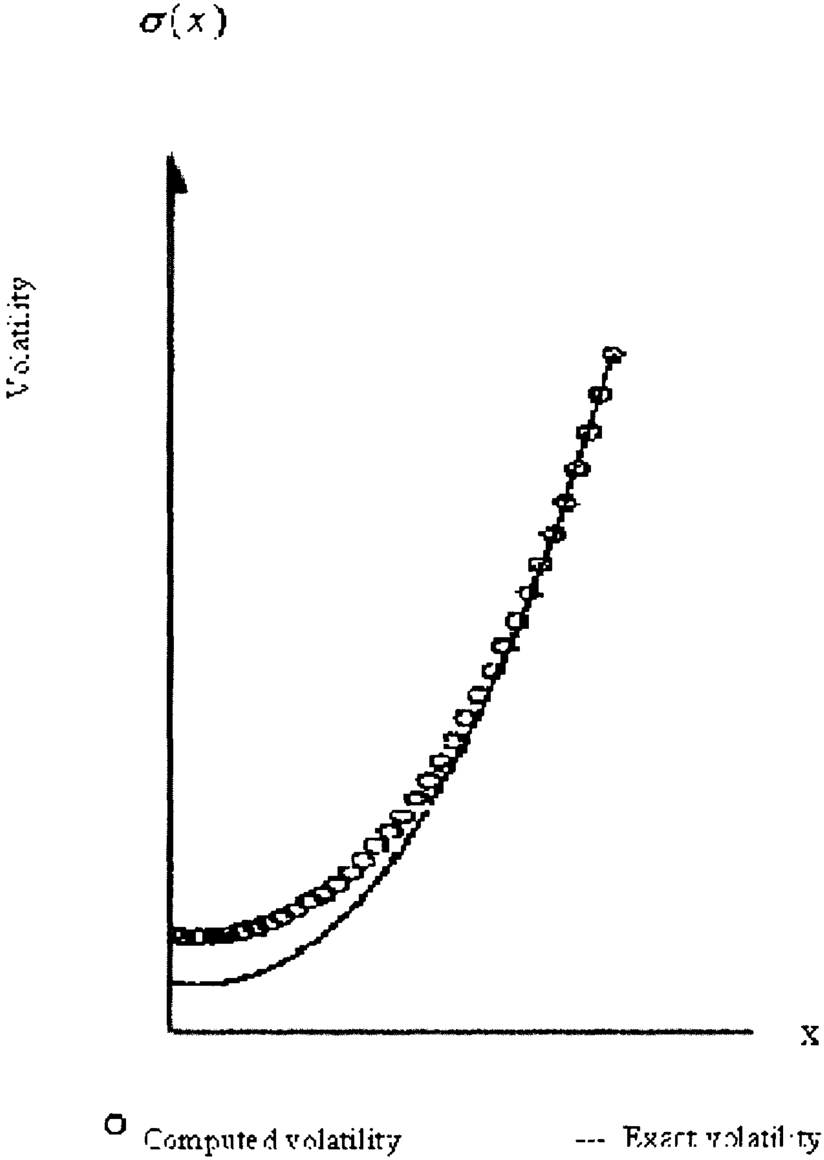
Discretized measured approximations of the initial and bounded data are modeled by adding random errors to the exact data functions. For example, for a boundary function $g(t)$, its discrete noisy version is

$$g_n^\epsilon = g(t_n) + \epsilon_n, n = 0, 1, \cdots, N,$$

where the (ϵ_n)'s are Gaussian random variables with variance ϵ^2. The exact solution for $\sigma(x)$ is

$$\sigma(x) = 2 + x^2.$$

Using mollification and finite difference method and solving σ and v iteratively, the computed (o o o) value of the volatility σ is shown in the following figure together with the exact (__) value for comparison.



σ(x)
Volatility
X
Computed volatility
--- Exact volatility

8. Discussion

This paper presented a preliminary study on solving an inverse option pricing problem.

Using the mollification technique, the stability of the inverse problem was restored and a numerical example on the recovering local volatility was presented. The approach and analysis presented in this paper could also be modified to solve other inverse problems in finance and insurance such as American option pricing and pricing of default risk. Rigorous proof of the theorems, complete error analysis of the algorithm and several numerical experiments involving more general model will be described elsewhere.

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