

# Catastrophe Risk Bonds

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## Abstract

We examine the pricing of catastrophe risk bonds. Catastrophe risk cannot be hedged by traditional securities. Therefore the pricing of catastrophe risk bonds requires an incomplete markets setting and this creates special difficulties in the pricing methodology. We briefly discuss the theory of equilibrium pricing and its relationship to the standard arbitrage-free valuation framework. Equilibrium pricing theory is used to develop a pricing method based on a model of the term structure of interest rates and a probability structure for the catastrophe risk. This pricing methodology can be used to assess the default spread on catastrophe risk bonds relative to traditional defaultable securities.

# Catastrophe Risk Bonds

“It is indeed most wonderful to witness such desolation produced in three minutes of time.” - Charles Darwin commenting on the February 20, 1835 earthquake in Chile.

## 1 Introduction

Catastrophe risk bonds provide a mechanism for direct transfer of catastrophe risk to capital markets, in contrast to transfer through a traditional reinsurance company. The bondholder’s cash flows (coupon or principal) from these bonds are linked to particular catastrophic events such as earthquakes, hurricanes, or floods. Although several deals involving catastrophe risk bonds have been announced recently the concept has been around a while. Goshay and Sandor [8] proposed trading reinsurance futures in 1973. In 1984, Svensk Exportkredit launched a private placement of earthquake bonds that are immediately redeemable if a major earthquake hits Japan [14]. Insurers in Japan bought the bonds agreeing to accept lower than normal coupons in exchange for the right to put the bonds back to the issuer at face value if an earthquake hits Japan. This is the earliest catastrophe risk bond deal we know about.

In the early 1990s the Chicago Board of Trade introduced exchange traded futures (later they were dropped) and options based on industry-wide loss indices. More recently catastrophe risk has been embedded in privately placed bonds which allow the borrower to transfer risk to the lender. In the event of a catastrophe, a catastrophe risk bond behaves much like a defaultable corporate bond. The “default” of a catastrophe risk bond is triggered by a catastrophe as defined by the bond indenture. Unlike corporate bonds, the default risk of a catastrophe risk bond is uncorrelated with the underlying financial market variables such as interest rate levels or aggregate consumption [7]. Consequently, the payments from a catastrophe risk bond cannot be hedged<sup>1</sup> by a portfolio of traditional bonds or common stocks. The pricing of catastrophe risk bonds requires an incomplete markets framework since no portfolio of primitive securities replicates the catastrophe risk bond. Fortunately, the fact that catastrophe risk is uncorrelated with movements in underlying economic variables renders the incomplete markets theory somewhat simpler than the case of significant correlation. We use this to develop a simple approach to pricing catastrophe risk bonds.

The model we present for pricing catastrophe risk bonds is based on equilibrium pricing. The model is practical in that the valuation may be done in a two stage

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<sup>1</sup>Financial economists would say that the payments from catastrophe risk bonds cannot be *spanned* by primitive assets (ordinary stocks and bonds).

procedure. First, we select or estimate the interest rate dynamics<sup>2</sup> in the states of the world which do not involve the catastrophe. Constructing a term structure model which is a relatively well understood and practiced procedure. Second, we estimate the probability<sup>3</sup> of the catastrophe occurring. Valuation for the full model is then accomplished by combining the probability of the catastrophe occurring and the interest rate dynamics from the term structure model.

One may implement the valuation using the standard tool of a risk-neutral valuation measure. The full model is arbitrage-free.

The paper begins in section 2 by describing how catastrophe risk bonds arise from the securitization of liabilities. We also describe some recent catastrophe bond deals. Section 3 provides a quick overview and motivation of how pricing may be carried out for catastrophe risk bonds. We work out a numerical example which illustrates the principles underlying catastrophe bond deals. Section 4 details the inherent pricing problems one faces with catastrophe risk bonds because of the incomplete markets setting. Section 5 describes our formal model and provides a numerical example and section 6 concludes the paper.

## 2 Catastrophe Reinsurance as a High-Yield Bond

Most investment banks, some insurance brokers and most large reinsurers developed over-the-counter insurance derivatives by 1995. This is a form of liability securitization, but instead of exchange traded contracts these securities are handled like private placements or customized forwards or options. Tilley [18] describes securitized catastrophe reinsurance in terms of a high-yield bond. Froot *et al.* [7] describe a similar one-period product. These products illustrate how catastrophe risk can be distributed through capital markets in a new way. The following description is an abstraction and simplification but useful for illustrating the concepts.

Consider a one-period reinsurance contract under which the reinsurer agrees to pay a fixed amount  $L$  at the end of the period if a defined catastrophic event occurs. The reinsurer issues a one-period reinsurance contract that pays  $L$  at the end of the period, if there is a catastrophe. It pays nothing if no catastrophe occurs.  $L$  is known when the policy is issued. If  $q_b$  denotes the probability of a catastrophic event and  $P$  the price of

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<sup>2</sup>Those familiar with state prices recognize that this is equivalent to estimating local state prices for states of the world which are independent of the catastrophe.

<sup>3</sup>More generally, one estimates the probability distribution for the varying degrees of severity of the catastrophe risk.

the reinsurance, then the fair value of the reinsurance is

$$P = \frac{1}{1+r} q_{\text{cat}}$$

where  $r$  is the one period effective default-free interest rate. This defines a one-to-one correspondence between bond prices and probabilities of a catastrophe. Since the reinsurance market will determine the price  $P$ , it is natural to denote the corresponding probability with a subscript "cat". That is,  $q_{\text{cat}}$  is the reinsurance market assessment of the probability of a catastrophe.

From where does the capital to support the reinsurer come? Astute buyers and regulatory authorities will want to be sure that the reinsurer has the capital to pay the catastrophe loss. Usual risk based capital requirements based on diversification over a portfolio do not apply since the reinsurer has a single risk. The appropriate risk based capital requirement is full funding. That is, the reinsurer will have no customers unless it can convince them that it has secure capital at least equal to  $L$ . To obtain capital before it sells the reinsurance, the reinsurer borrows capital by issuing a defaultable bond, i.e., a junk bond. Investors know when they buy a junk bond that it may default but they buy anyway because the bonds do not often default and they have higher returns than more reliable bonds. (Indeed, we will see that the recent deals were popular with investors.) The reinsurer issues enough bonds to raise an amount of cash  $C$  determined so that

$$(P + C)(1 + r) = L.$$

This satisfies the reinsurer's customers. They see that the reinsurer has enough capital to pay for a catastrophe. The bondholders know that the bonds will be worthless if there is a catastrophe. In this case they get nothing. If there is no catastrophe, they get their cash back plus a coupon  $R = LC$ . The bond market will determine the price per unit of face value. In terms of discounted expected cash flow, the price per unit can be written in the form

$$\frac{1}{1+r}(1+c)(1-q_b)$$

where  $c = R/C$  is the coupon rate and  $q_b$  denoted the bondholders assessment of the probability of default on the bonds. We can assume that the investment bank designing the bond contract sets  $c$  so that the bonds sell at face value. Thus,  $c$  is determined so that investors pay 1 in order to receive  $1 + c$  one year later, if there is no catastrophe. This is expressed as

$$1 = \frac{1}{1+r}(1+c)(1-q_b).$$

Of course, default on the bonds and a catastrophe are equivalent events. The probabilities may differ because bond investors and reinsurance customers may have different information about catastrophes. The reinsurance company sells bonds once  $c$  is determined to raise the required capital  $C$ . The corresponding bond market probability is found by solving for  $q_b$ :

$$q_b = \frac{c - r}{1 + c}$$

The implied price for reinsurance is

$$P_b = \frac{1}{1 + r} \frac{c - r}{1 + c} L.$$

Provided the reinsurance market premium  $P$  (the fair price determined by the reinsurance market) is at least as large as  $P_b$ , the reinsurance company will function smoothly. It will collect  $C$  from the bond market and  $P$  from the reinsurance market at the beginning of the policy period. The sum invested for one period at the risk free rate will be at least  $L$ . This is easy to see mathematically using the relation  $R = L - C$ :

$$\begin{aligned} (P + C)(1 + r) &\geq (P_b + C)(1 + r) \\ &= \frac{c - r}{1 + c} L + (1 + r) C \\ &= \frac{R - rC}{C + R} L + (1 + r)C \\ &= \frac{R - rC}{L} L + (1 + r)C = R + C = L \end{aligned}$$

So long as  $P_b$  does not exceed  $P$ , or equivalently, so long as

$$q_{\text{cat}} \geq \frac{c - r}{1 + c},$$

there will be an economically viable market for reinsurance capitalized by borrowing in the bond market. Borrowing (issuing bonds) to finance losses is not new. In the late 1980s, when US liability insurance prices were high and interest rates were moderate, some traditional insurance customers replaced insurance with self-insurance programs financed by bonds. Of course this is not a securitization of insurance risk but is an example of insurance customers turning to capital markets to finance losses. More recently, several state run hurricane and windstorm pools extended their claims paying ability with bank-arranged contingent borrowing agreements in lieu of reinsurance [12].

The catastrophe property market in the 1990s may not satisfy this condition – insurance prices are high enough to attract investors – but is close enough to attract a lot of interest and entice capital market advocates such as Froot *et al.* [7], Lane [10], and Tilley [18] to offer cat risk products. A catastrophe risk bond market is developing.

In our model the fund always has adequate cash to pay the loss if a catastrophic event occurs. If no catastrophe occurs, the fund goes to the bond owners. From the bond owners' perspective, the bond contract is like lending money subject to credit risk, except the risk of “default” is really the risk of a catastrophic event. Tilley describes this as a fully collateralized reinsurance contract since the reinsurer has adequate cash at the beginning of the period to make the loss payment with probability one. This scheme is a simple version of how a traditional reinsurer works with the following differences.

- The traditional reinsurance company owners buy shares of stock instead of bonds.
- Traditional reinsurer losses affect investors (stockholders) on a portfolio basis rather than a single exposure.
- Simplifying and specializing makes it possible to sell single exposures through the capital markets, in contrast to shares of stock of a reinsurer, which are claims on the aggregate of outcomes.

Tilley [18] demonstrates this technique in a more general setting in which the reinsurance and bond are  $N$  period contracts. This one period model illustrates the key ideas. Now we describe three recent catastrophe bonds which have recently appeared on the market. In the last section we describe a hypothetical example which illustrates how catastrophe bonds increase insurer capacity to write catastrophe coverages.

## USAA Hurricane Bonds

USAA is a personal lines insurer based in San Antonio. It provides personal financial management products to current or former US military officers and their dependents. *Business Insurance* [19] in reporting on the USAA deal, described USAA as “over exposed” to hurricane risk in its personal automobile and homeowners business along the US Gulf and Atlantic coasts. In June 1997, USAA arranged for its captive Cayman Islands reinsurer, Residential Re, to issue \$477 million face amount of one-year bonds with coupon and/or principal exposed to property damage risk to USAA policyholders due to Gulf or East coast hurricanes. Residential Re issued reinsurance to USAA based on the capital provided by the bond sale.

The bonds were issued in two series (also called tranches), according to an article in *The Wall Street Journal* [17]. In the first series coupons only are exposed to hurricane

risk – the principal is guaranteed. For the second series both coupons and principal are at risk. The risk is defined as damage to USAA customers on the Gulf or East coast during the year beginning in June. The coupons and/or principal will not be paid to investors if these losses exceed one billion dollars. That is, the risk begins to reduce coupons at \$1 billion and at \$1.5 billion the coupons in the first series are completely gone and in the second series the coupons and principal are lost. The coupon-only tranche has a coupon rate of LIBOR plus 2.73%, The principal and coupon tranche has a coupon rate of LIBOR + 5.76%. The press reported that the issue was “over-subscribed,” meaning there were more buyers than anticipated. The press reports indicated that the buyers were life insurance companies, pension funds, mutual funds, money managers, and, to a very small extent, reinsurers. As a point of reference for the risk involved, we note that industry losses due to hurricane Andrew in 1992 amounted to \$16.5 billion and USAA’s Andrew losses amounted to \$555 million. Niedzielski reported in the *National Underwriter* that the cost of the coverage was about 6% rate on line plus expenses.<sup>4</sup> According to Niedzielski’s (unspecified) sources the comparable reinsurance coverage is available for about 7% rate on line. The difference is probably more than made up by the fees related to establishing Residential Re and the fees to the investment bank for issuing the bonds. The rate on line refers only to the cost of the reinsurance. The reports did not give the sale price of the bonds, but the investment bank probably set the coupon so that they sold at face value.

As successful as this issue has turned out so far (two months after issue everyone is happy), it was a long time coming. Despite advice of highly regarded advocates such as Morton Lane and Aaron Stern [7, 10, 13], catastrophe bonds have developed more slowly than many experts expected. According to press reports, USAA has obtained 80% of the coverage of its losses in the \$1.0 to \$1.5 billion layer with this deal. On the other hand, we have to wonder why it is a one year deal. Perhaps it is a matter of getting the technology in place. The off-shore reinsurer is re-usable. And the next time USAA goes to the capital market investors will be familiar with these exposures. If the traditional catastrophe reinsurance market gets tight, USAA will have a capital market alternative. The cost of this issue is offset somewhat by the gain in access to alternative sources of reinsurance.

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<sup>4</sup>Rate on line is the ratio of premium to coverage layer. The reinsurance agreement provides USAA with 80 percent of \$500 million in excess of \$1 billion. The denominator of the rate on line is  $(0.80)(500) = 400$  million, so this implies USAA paid Residential Re a premium of about  $(0.06)(400) = 24$  million.

## Winterthur Windstorm Bonds

Winterthur is a large insurance company based in Winterthur, Switzerland. In February 1997, Winterthur issued three year annual coupon bonds with a face amount of CHF 4700. The coupon rate is 2.25%, subject to risk of windstorm (most likely hail) damage during a specified exposure period each year to Winterthur automobile insurance customers. The deal was described in the trade press and Schmock has written an article in which he values the coupon cash flow [16]. The deal has been mentioned in US publications (for example, *Investment Dealers Digest* [11]), but we had to go to *Euroweek* for a published report on the contract details [2]. If the number of automobile windstorm claims during the annual observation period exceeds 6000, the coupon for the corresponding year is not paid. The bond has an additional financial wrinkle. It is convertible at maturity; each face amount of CHF 4700 is convertible to five shares of Winterthur common stock at maturity.

## Swiss Re California Earthquake Bonds

The Swiss Re deal is similar to the USAA deal in that the bonds were issued by a Cayman Islands reinsurer, evidently created for issuing catastrophe risk bonds, according to an article in *Business Insurance* [20]. However, unlike USAA's deal, the underlying California earthquake risk is measured by an industry-wide index rather than Swiss Re's own portfolio of risks. The index is developed by Property Claims Services. Evidently the bond contract is written on the same (or similar) California index underlying the Chicago Board of Trade Catastrophe Options. The CBOT options have been the subject of numerous scholarly and trade press articles [3, 4, 5, 6].

Zolkos reported in *Business Insurance* details on the Swiss Re bonds. There were earlier reports that Swiss Re was looking for a ten year deal. This is not it, so perhaps they are still looking. According to Zolkos, SR Earthquake Fund (a company Swiss Re set up evidently for this purpose) issued Swiss Re \$122.2 million in California reinsurance coverage based on funds provided by the bond sale. In the next section we will provide a numerical example which illustrates the principles underlying these three deals.

## 3 Modeling Catastrophe Risk Bonds

In the previous section we discussed the securitization underlying catastrophe risk bonds. In this section we adopt a standardized definition of a catastrophe risk bond for the purposes of analyzing this security using financial economics. We are informal in this section, leaving the definition of some technical terms until section 5.



A *catastrophe risk bond* with face amount of 1 is an instrument that is scheduled to make a coupon payment of  $c$  at the end of each period and a final principal repayment of 1 at the end of the last period [labeled time  $T$ ] so long as a specified catastrophic event (or events) does not occur. The investment banker designing the bond knows the market well enough to know what coupon is required for the bond to sell at face value. However, we will take the view that the coupon is set in the contract and we will determine the market price. This is an equivalent approach.

We will focus most of our attention on bonds which have coupons and principal exposed to catastrophe risk. These are defined as follows. The bond coupons are made with only one possible cause of default -- a specified catastrophe. The bond begins paying at the rate  $c$  per period and continues paying to  $T$  with a final payment of  $1 + c$ , if no catastrophe occurs. If a catastrophe should occur during a coupon period, the bond makes a fractional coupon payment and a fractional principal repayment that period and is then wound up. The fractional payment is assumed to be of the fraction  $f$  so that if a catastrophe occurs, the payment made at the end of the period in which the catastrophe occurs is equal to  $f(1 + c)$ .

At present we are not allowing for varying severity in the claims associated with the catastrophe. Varying severity would occur in practice and we mention this modeling issue later. Financial economics theory tells us that when an investment market is *arbitrage-free*, there exists a probability measure, which we denote by  $Q$ , referred to as the risk-neutral measure, such that the price at time 0 of each uncertain cash flow stream  $\{c(k) : k = 1, 2, \dots, T\}$  is given by the following expectation under the probability measure  $Q$ ,

$$E^Q \left[ \sum_{k=1}^T \frac{1}{[1 + r(0)][1 + r(1)] \cdots [1 + r(k-1)]} c(k) \right]. \quad (1)$$

The process  $\{r(k) : k = 1, \dots, T-1\}$  is the stochastic process of one-period interest rates. The interest rate for the first period  $r(0)$  is known at time 0; the factor  $1 + r(0)$  could be moved out of the expectation. We denote the price at time 0 of a non-defaultable zero coupon bond with a face amount of 1 maturing at time  $n$  by  $P(n)$ . Therefore we have, for  $n = 1, 2, \dots, T$ ,

$$P(n) = E^Q \left[ \frac{1}{[1 + r(0)][1 + r(1)] \cdots [1 + r(n-1)]} \right]. \quad (2)$$

We shall let  $\tau$  denote the time of the first occurrence of a catastrophe. A catastrophe may or may not occur prior to the scheduled maturity of the catastrophe risk bond at time  $T$ . If a catastrophe occurs then evidently  $\tau \in \{1, 2, \dots, T\}$ . For a catastrophe

bond with coupons and principal at risk (like the second tranche of the USAA bond issue or the Swiss Re bonds), the cash flow stream to the bondholder may be described (using indicator functions<sup>5</sup>) as follows:

$$c(k) = \begin{cases} c 1_{\{\tau > k\}} + f(c+1)1_{\{\tau = k\}} & k = 1, 2, \dots, T-1 \\ c 1_{\{\tau > T\}} + f(c+1)1_{\{\tau = T\}} & k = T \end{cases} \quad (3)$$

For a catastrophe bond with coupons only at risk (like the first tranche of the USAA bonds), we replace the factor  $f(1+c)$  in (3) by  $fc$  and adjust the payment in the event  $\tau = T$  to reflect the return of principal guarantee:

$$c(k) = \begin{cases} c 1_{\{\tau > k\}} + fc 1_{\{\tau = k\}} & k = 1, 2, \dots, T-1 \\ 1 + c 1_{\{\tau > T\}} + fc 1_{\{\tau = T\}} & k = T \end{cases} \quad (4)$$

We will consider a bond with principal and coupon at risk, but the analysis is identical, involving only re-specification of the contingent cash flows, for coupon only at risk bonds.

Let us assume that we are trading catastrophe risk bonds in an investment market which is arbitrage-free with risk-neutral valuation measure  $Q$  and that the time of the catastrophe is independent of the term structure under the probability measure  $Q$ . We shall formalize these notions<sup>6</sup> in section 5. We may apply relation (1) to the cash flow stream in (3) and find that the price at time 0 of the cash flow stream provided by the catastrophe risk bond is given by the expression

$$c \sum_{k=1}^T P(k) Q(\tau > k) + P(T) Q(\tau > T) + f(1+c) \sum_{k=1}^T P(k) Q(\tau = k). \quad (5)$$

The term  $Q(\tau > k)$  is the probability under the risk neutral valuation measure that the catastrophe does not occur within the first  $k$  periods. The other probabilistic terms may be verbalized similarly. No assumption has been made about the distribution of  $\tau$  but the assumption that only one degree of severity can occur is clearly being used here. Of course, the distribution of  $\tau$  will depend on the structure of the catastrophe risk exposure.

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<sup>5</sup>For an event  $A$ , the indicator function is the random variable which is 1 if  $A$  occurs and zero otherwise. It is denoted  $1_A$ .

<sup>6</sup>These are the assumptions made by Tilley [18] although they are not stated in quite this terminology.

Formula (5) expresses the price of the catastrophe risk bond in terms of known parameters, including the coupon rate  $c$ . As we described at the beginning of this section, the principal amount of the catastrophe risk bond is fixed at the time of issue and the coupon rate is varied to ensure that the price of the cash flows provided by the bond are equal to the principal amount. One may apply the valuation formula (5) to obtain a formula for the coupon rate as

$$c = \frac{1 - P(T)Q(\tau > T) - f \sum_{k=1}^T P(k)Q(\tau = k)}{\sum_{k=1}^T P(k)Q(\tau > k) + f \sum_{k=1}^T P(k)Q(\tau = k)}. \quad (6)$$

Let  $F(x)$  denote the conditional severity distribution the bondholders' cash flow  $X$ , given a catastrophe occurs. Formula (5) becomes

$$c \sum_{k=1}^T P(k) Q(\tau > k) + P(T) Q(\tau > T) + \sum_{k=1}^T P(k) Q(\tau = k) \int_0^\infty x dF(x). \quad (7)$$

On comparing formula (5) and (7) we see that there is little difference between the two formulas. Generally, the conditional severity distribution is embedded as part of the risk-neutral measure  $Q$ .

Let us suppose that the catastrophe risk structure is such that the conditional probability under the risk neutral measure of no catastrophe for a period is equal to a constant  $\theta_0$ . Furthermore, suppose that should a catastrophe occur there is a single severity level resulting in a payment equal to  $f(1+c)$  at the end of the period in which the catastrophe occurs. Let  $\theta_1 = 1 - \theta_0$ . In this case, formula (5) simplifies to the expression given by Tilley [18] for the price at time 0 of the catastrophe risk bond, namely

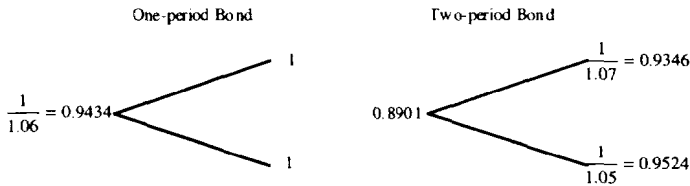
$$c \sum_{k=1}^T P(k)(1 - \theta_1)^k + P(T)(1 - \theta_1)^T + f(1 + c) \sum_{k=1}^T P(k)\theta_1(1 - \theta_1)^{k-1}. \quad (8)$$

In order to apply Tilley's formula (8), one must know what the conditional risk-neutral probability [or equivalently  $\theta_0$ ] is. At this point,  $\theta_1$  has not been related to the empirical conditional probability of a catastrophe occurring. Therefore, the formula (8) is not quite "closed". In order to close the model we need to link the valuation formula (8) with observable quantities that can be used to estimate the parameters needed to apply the valuation model. Although we began the discussion of the pricing model with an assumption about the existence of a valuation measure  $Q$ , it is possible to justify an interpretation of  $\theta_1$  as the empirical conditional probability of a catastrophe occurring. We shall address and clarify this point in section 5.

## 4 Incompleteness in the Presence of Catastrophe Risk

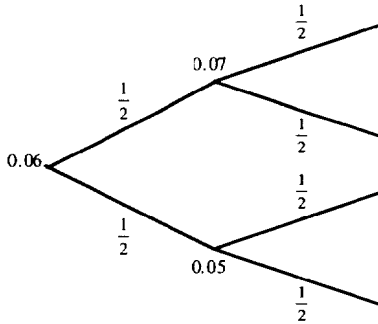
The introduction of catastrophe risk into a securities market model implies that the resulting model is incomplete. The pricing of uncertain cash flow streams in an incomplete model is substantially weaker in the interpretation of the pricing results that can be obtained than is the case for pricing in complete securities markets. In this section we discuss market completeness and explain the nature of the incompleteness problem for models with catastrophe risk exposures. For simplicity, we work with a one-period model although similar notions may be developed for multi-period models. Let us consider a single-period model in which two bonds are available for trading, one of which is a one-period bond and the other a two period bond. For convenience we shall assume that both bonds are zero coupon bonds. We further assume that the financial markets will evolve to one of two states at the end of the period, “interest rates go up” or “interest rates go down” and that the price of each bond will assume to behave according to the binomial model depicted in the following figures.

Figure 1



The bond prices for this model could be derived from the equivalent information in the following tree diagram for which the one-period model is embedded. We specified the bond prices directly to avoid bringing a two-period model into our discussion of the one-period case. The prices reported in figure 1 have been rounded from what one would compute from figure 2. For example, we rounded  $\frac{1}{1.06}(\frac{1}{2})(\frac{1}{1.07} + \frac{1}{1.05})$  to 0.8901.

**Figure 2**  
One-Period Rates and Risk-Neutral Probabilities



Suppose that we select a portfolio of the one-period and two-period bonds. Let us denote the number of one-period bonds held in this portfolio by  $n_1$  and the number of two-period bonds held in this portfolio by  $n_2$ . This portfolio will have a value in each of the two states at time 1. Let us represent the state dependent price of each bond at time 1 using a column vector. Then we may represent the value of our portfolio at time 1 by the following matrix equation.

$$\begin{bmatrix} 1 & \frac{1}{1.07} \\ 1 & \frac{1}{1.05} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \quad (9)$$

The cost of this portfolio is given by

$$\frac{1}{1.06}n_1 + 0.8901n_2. \quad (10)$$

The  $2 \times 2$  matrix of bond prices at time 1 appearing in equation (9) is nonsingular. Therefore, any vector of cash flows at time 1 may be generated by forming the appropriate portfolio of these two bonds. For instance, if we want the vector of cash flows at time 1 given by the column vector,

$$\begin{bmatrix} c^u \\ c^d \end{bmatrix} \quad (11)$$

then we form the portfolio

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{1.07} \\ 1 & \frac{1}{1.05} \end{bmatrix}^{-1} \begin{bmatrix} c^u \\ c^d \end{bmatrix}$$

at a cost of  $\frac{1}{1.06}n_1 + 0.8901n_2$ . Carrying out the arithmetic, one finds that the price of each cash flow of the form (11) is given by the expression

$$\left(\frac{1}{2}\right) \frac{1}{1.06}c^u + \left(\frac{1}{2}\right) \frac{1}{1.06}c^d = 0.4717c^u + 0.4717c^d. \quad (12)$$

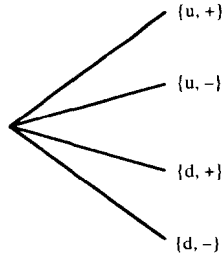
Since every such set of cash flows at time 1 can be obtained and priced in the model we say that the one-period model is complete. The notion of pricing in this complete model is justified by the fact that the price we assign to each uncertain cash flow stream is exactly equal to the price of the portfolio of one-period and two-period bonds that generates the value of the cash flow stream at time 1.

Let us see how the model is changed when catastrophe risk exposure is incorporated as part of the information structure. Suppose that we have the framework of the previous model with the addition of catastrophe risk. Furthermore, let us suppose that the catastrophic event occurs independently of the underlying financial market variables. Therefore, there will be four states in the model which we may identify as follows.

$$\begin{aligned} \{\text{interest rate goes up, catastrophe occurs}\} &\equiv \{u, +\} \\ \{\text{interest rate goes up, no catastrophe occurs}\} &\equiv \{u, -\} \\ \{\text{interest rate goes down, catastrophe occurs}\} &\equiv \{d, +\} \\ \{\text{interest rate goes down, no catastrophe occurs}\} &\equiv \{d, -\} \end{aligned} \quad (13)$$

The reader will note that the symbol  $\{u, +\}$  is shorthand for “interest rates go up” and “catastrophe occurs” and so forth. This information structure is represented on a single-period tree with four branches such as is shown in figure 3.

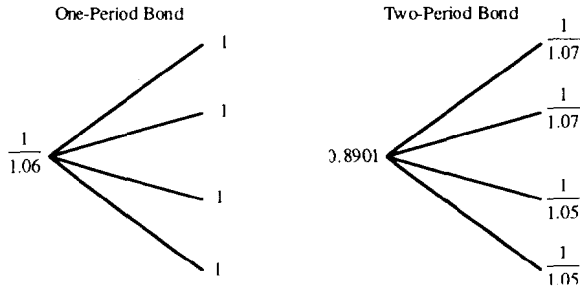
**Figure 3**



The values at time 1 of the one-period bond and the two-period bond are not linked to

the occurrence or nonoccurrence of the catastrophic event and therefore do not depend on the catastrophic risk variable. We may represent the prices of the one-period and two-period bond in the extended model as shown in figure 4.

**Figure 4**



In contrast to equation (9), the value at time 1 of a portfolio of the one-period and two-period bonds is now given by the following matrix equation.

$$\begin{bmatrix} 1 & \frac{1}{1.07} \\ 1 & \frac{1}{1.07} \\ 1 & \frac{1}{1.05} \\ 1 & \frac{1}{1.05} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \tag{14}$$

The cost of this portfolio is still given by  $\frac{1}{1.06}n_1 + 0.8901n_2$ . The most general vector of cash flows at time 1 in this model is of the following form:

$$\begin{bmatrix} c^{u,+} \\ c^{u,-} \\ c^{d,+} \\ c^{d,-} \end{bmatrix} \tag{15}$$

On reviewing equation (14) we see that the span of the assets available for trading in the model [*i.e.* the one-period and two-period bonds] are not sufficient to span all cash flows of the form (15). Consequently, we cannot derive a pricing relation such as (12) that is valid for all cash flow vectors of the form (15). The best we can do is to obtain bounds on the price of a general cash flow vector so that its price is consistent with the absence of arbitrage. This one-period securities market model is arbitrage-free if and only if there exists a vector (see Pliska [15, chapter 1])

$$\Psi \equiv [\Psi^{u,+}, \Psi^{u,-}, \Psi^{d,+}, \Psi^{d,-}], \tag{16}$$

each component of which is positive, such that

$$[\Psi^{u,+}, \Psi^{u,-}, \Psi^{d,+}, \Psi^{d,+}] \begin{bmatrix} 1 & \frac{1}{1.07} \\ 1 & \frac{1}{1.07} \\ 1 & \frac{1}{1.05} \\ 1 & \frac{1}{1.05} \end{bmatrix} = \begin{bmatrix} 0.9434 \\ 0.8901 \end{bmatrix}. \quad (17)$$

Such a vector is called a state price vector<sup>7</sup>. One may solve (17) for all such vectors to find that the class of all state price vectors for this model is of the form

$$\Psi = [0.4717 - s, s, 0.4717 - t, t] \quad (18)$$

for  $0 < s < 0.4717$  and  $0 < t < 0.4717$ . For each cash flow of the form (15), there is a range of prices that are consistent with the absence of arbitrage. This is given by the expression

$$0.4717c^{u,+} + 0.4717c^{d,+} + s(c^{u,-} - c^{u,+}) + t(c^{d,-} - c^{d,+}), \quad (19)$$

where  $s$  and  $t$  range through all feasible values  $0 < s < 0.4717$  and  $0 < t < 0.4717$ . Note that a security with cash flows which do not depend on the catastrophe are uniquely priced. This is not true of catastrophe risk bonds. For instance, the price of the cash flow stream which pays 1 if no catastrophe occurs and 0.5 if a catastrophe occurs has the price range given by the expression

$$0.4717(0.5) + 0.4717(0.5) + s(1 - 0.5) + t(1 - 0.5) = 0.4717 + (s + t)(0.5).$$

The range of prices for this cash flow stream is the open interval (0.4717, 0.9434). These price bounds are not very tight. However, they are all that can be said working solely from the absence of arbitrage.

Let us consider the case of a one-period catastrophe risk bond with  $f = 0.3$ . In return for a principal deposit of \$1 at time 0, the investor will receive an uncertain cash flow stream at time 1 of the form:

$$(1 + c) \begin{bmatrix} 0.3 \\ 1.0 \\ 0.3 \\ 1.0 \end{bmatrix} \quad (20)$$

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<sup>7</sup>The reader may check that the components of the state price vector are precisely the risk neutral probabilities of each state discounted by the short rate.



We may apply the relation (19) to find that the range of values on the coupon that must be paid to the investor have the range in the open interval (0.06, 2.5333). The coupon rate of the catastrophe risk bond is not uniquely defined. There is but a range of values for the coupon that are consistent with the absence of arbitrage. Although this is a very wide range of coupon rates, this is the strongest statement about how the coupon values can be set subject only to the criterion that the resulting securities market is arbitrage-free. Evidently, we need to bring in some additional theory if we are to obtain useful, if benchmark, pricing formulas for catastrophe risk bonds. In fact, we shall see that we can tighten these bounds, even to the point of generating an explicit price, by embedding in the model the probabilities of the catastrophe occurring. For this example, let us assume that investors agree on the probability  $q$  of a catastrophe and they agree that the catastrophe bond price should be its discounted expected value. The expected cash flow to the bondholder is

$$(1 + c) \begin{bmatrix} 0.3q + 1.0(1 - q) \\ 0.3q + 1.0(1 - q) \end{bmatrix} = (1 + c)(0.3q + 1.0(1 - q)) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where we have only uncertainty with regard to interest rates remaining. This bond has the same (expected) value in each interest rate state, so its price  $V$  is that value times the price of the one-year default free bond:

$$V = (1 + c)(0.3q + 1.0(1 - q)) \frac{1}{1.06}$$

Now we could determine the coupon  $c$  so that the bond sells at par ( $V = 1$ ) initially, or we could determine the price for a specified coupon. Given the probability distribution of the catastrophe and the assumption that prices are discounted expected values (over both risks), then we can get unique prices.

This illustrates the difficulty with the financial markets approach. The price can no longer be justified by arbitrage considerations alone [*i.e.* the cost of a portfolio of existing assets that gives the appropriate payoffs - since there is no such portfolio]. We lose the uniqueness of prices - and it is recovered only at the expense of introducing the probability distribution of the catastrophe risk. Such is the nature of incomplete markets. In the following section we shall describe a method of obtaining explicit prices for catastrophe risk bonds and describe some examples.

## 5 A Formal Model

In section 3 we gave a preliminary presentation of the basic formulation of a valuation model for catastrophe risk bonds and discussed the type of valuation formulas described

in Tilley [18]. The discussion offered in section 3 should be considered as motivation for the formal model that we now develop. The formal model we describe is designed to combine primary financial market variables with catastrophe risk variables to yield a theoretical valuation model for catastrophe risk bonds. Of course, the mathematics of the model may be used in other contexts regardless of the interpretation we give to the components of the model.

The financial market variables are assumed to be modeled on the filtered probability space  $(\Omega^{(1)}, \mathbf{P}^{(1)}, P_1)$ . We briefly review the concepts and notation based on Pliska's full account ([15, chapter 3]). The sample space  $\Omega^{(1)}$  is finite and it represents all the paths the financial variables may take over the times  $k = 0, 1, \dots, T$ . The filtration  $\mathbf{P}^{(1)}$  represents how information evolves in the financial market and may be thought of as an information tree. More precisely, the filtration is an increasing sequence

$$\mathbf{P}^{(1)} = \{\mathbf{P}_0^{(1)} \subseteq \mathbf{P}_1^{(1)} \subseteq \dots \subseteq \mathbf{P}_T^{(1)}\}$$

of sets of events indexed by time  $k = 0, 1, \dots, T$ . The events in  $\mathbf{P}_k^{(1)}$  represent the investment information available to the market at time  $k$  - essentially past security prices. The increasing feature formalizes the idea that no information is lost from one time to the next. The probability measure  $P_1$  is defined on  $\mathbf{P}_T^{(1)}$  and so  $P_1(A)$  is defined for all events  $A \in \mathbf{P}_k^{(1)}$  for  $k \leq T$ .

The catastrophe risk variables are assumed to be modeled on the filtered probability space  $(\Omega^{(2)}, \mathbf{P}^{(2)}, P_2)$ .  $P_2$  is the probability measure governing the catastrophe structure. The filtration  $\mathbf{P}^{(2)}$  is indexed over the same times  $k = 0, 1, \dots, T$ . The probability space for our full model is taken to be the product space  $\Omega := \Omega^{(1)} \times \Omega^{(2)}$ .  $\Omega$  is also referred to as the sample space for the full model. Therefore, a typical element of the probability space for the full model is of the form  $\omega = (\omega^{(1)}, \omega^{(2)})$  with  $\omega^{(1)} \in \Omega^{(1)}$  and  $\omega^{(2)} \in \Omega^{(2)}$ . Such an element (or state of the world for the full model) describes the state of the financial market variables and the catastrophe risk variables.

It should again be emphasized that under this construction, the embedded sample space  $\Omega^{(1)}$  represents the primary financial market variables, which for the purposes of valuing catastrophe risk bonds is essentially the term structure of interest rates, while the embedded sample space  $\Omega^{(2)}$  represents information related to the catastrophe exposure. The probability measure on the sample space  $\Omega$  is given by the product measure structure. The probability of a state of the world  $\omega = (\omega^{(1)}, \omega^{(2)})$  is  $P(\omega) = P_1(\omega^{(1)}) P_2(\omega^{(2)})$ . This assumption ensures the independence of the economic and catastrophe risk variables. It is easily checked that the events in  $\mathbf{P}_T^{(1)}$  and  $\mathbf{P}_T^{(2)}$  are independent under the probability measure  $P$ .

The benchmark financial economics technique used to price uncertain cash flow streams in an incomplete markets setting is the representative agent. We now describe

this technique in the context of the probability structure we have just defined. The representative agent technique consists of an assumed representative utility function and an aggregate consumption process. The agent makes choices about future consumption, represented by the stochastic process  $\{c(k) : k = 0, 1, \dots, T\}$ . The aggregate consumption process may be thought of as the total consumption available in the economy at each point in time and in each state. We shall denote the aggregate consumption stochastic process by

$$\{C^*(k) | k = 0, 1, \dots, T\}.$$

Only the first choice is known with certainty at time  $k = 0$ . The other choices are random,  $C^*(\omega, k)$ , depending on the random state  $\omega$ . We shall assume that the representative agent's utility is time-additive and separable as well as differentiable. Time-additive and separable means that there are utility functions  $u_0, u_1, \dots, u_T$  such that the agent's expected utility for a generic consumption process  $\{c(k) | k = 0, 1, \dots, T\}$  is given by

$$E^P \left[ \sum_{k=0}^T u_k(c(k)) \right]. \tag{21}$$

It follows from the theory of the representative agent<sup>8</sup> that the price  $V(c)$  of a generic future cash flow process  $c = \{c(k) | k = 1, \dots, T\}$  at time 0 is given by the expectation

$$V(c) = E^P \left[ \sum_{k=1}^T \frac{u'_k(C^*(k))}{u'_0(C^*(0))} c(k) \right]. \tag{22}$$

Note that the aggregate consumption process plays a role in the pricing relation. In many implementations of this pricing relation the aggregate consumption process is assumed to evolve according to an exogenous process. This will not be an issue for us. Both the form of the utility function and the aggregate endowment process will be removed from the pricing analysis by relating the pricing relation to the valuation measure approach of arbitrage-free pricing.

In order to proceed further from relation (22), we assume that aggregate consumption [or equivalently, the aggregate endowment since we are in equilibrium] does not depend on the catastrophe risk variables. This assumption is the condition that, for all

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<sup>8</sup>See Karatzas [9] for a rigorous discussion of the theory of the representative agent. Embrechts and Meister [6] apply a related method from an alternative viewpoint.

$\omega = (\omega^{(1)}, \omega^{(2)}) \in \Omega$  and all  $k$ , the aggregate consumption  $C^*(\omega, k) = C^*(\omega^{(1)}, \omega^{(2)}, k)$  depends only on  $\omega^{(1)}$  and  $k$ .

The hypothesis that aggregate consumption does not depend on catastrophe risk variables is a reasonable approximation since the overall economy is only marginally influenced by localized catastrophes such as earthquakes or hurricanes.

In order to relate the representative agent valuation formula to the usual valuation measure approach in arbitrage-free pricing we need to define the one-period interest rates implicit in the representative agent pricing model. We define the one-period interest rates

$$\{r(k) | k = 0, 1, 2, \dots, T - 1\}$$

through the conditional expectations

$$\frac{1}{1 + r(k)} := \frac{1}{u'_k(C^*(k))} E^P [u'_{k+1}(C^*(k + 1)) | \mathbf{P}_k] \quad (23)$$

for  $k = 0, 1, 2, \dots, T - 1$ . The prices and interest rates in the model are known at time  $k = 0$ . The reader may check that the one-period interest rate process is independent of the catastrophe risk exposure. Indeed, the random variable under the expectation operator depends only on the financial market information available at time  $k$ . Full information is represented by  $\mathbf{P}_k$ , but the aggregate consumption process depends only on the financial information.

We define a new probability measure  $Q$  in terms of  $P$  and the positive random variable, called the *Radon-Nikodym derivative* of  $Q$  with respect to  $P$ . We change for convenience: Under the new measure prices are discounted (with respect to the term structure  $\{r(k)\}$ ) expected values. The Radon-Nikodym derivative is

$$\frac{dQ}{dP}(\omega) := [1 + r(0)][1 + r(\omega, 1)] \cdots [1 + r(\omega, T - 1)] \frac{u'_T(C^*(\omega, T))}{u'_0(C^*(\omega, 0))}. \quad (24)$$

We use it as follows. For any random variable  $X$ , the  $P$  and  $Q$  expectations are related by

$$E^Q[X] = E^P[X \frac{dQ}{dP}].$$

To see the convenience consider the term of equation (22) corresponding to  $k = T$ :

$$\begin{aligned} E^P \left[ \frac{u'_T(C^*(T))}{u'_0(C^*(0))} c(T) \right] &= E^P \left[ \frac{1}{[1 + r(0)][1 + r(1)] \cdots [1 + r(T - 1)]} \frac{dQ}{dP} c(T) \right] \\ &= E^Q \left[ \frac{1}{[1 + r(0)][1 + r(1)] \cdots [1 + r(T - 1)]} c(T) \right] \end{aligned}$$

The right side is the more convenient expression. Other terms in the sum are transformed similarly, using conditional expectations. For example, the term corresponding to  $k = T - 1$  is transformed as follows:

$$\begin{aligned}
 & \mathbb{E}^P \left[ \frac{u'_{T-1}(C^*(T-1))}{u'_0(C^*(0))} c(T-1) \right] \\
 &= \mathbb{E}^P \left[ \frac{u'_T(C^*(T))}{u'_0(C^*(0))} c(T-1) \frac{u'_{T-1}(C^*(T-1))}{u'_T(C^*(T))} \right] \\
 &= \mathbb{E}^P \left[ \frac{c(T-1)}{[1+r(0)][1+r(1)] \cdots [1+r(T-2)]} \frac{dQ}{dP} Y \right] \\
 &= \mathbb{E}^P \left[ \frac{c(T-1)}{[1+r(0)][1+r(1)] \cdots [1+r(T-2)]} \frac{dQ}{dP} \mathbb{E}^P [Y | \mathbf{P}_{T-1}] \right] \\
 &= \mathbb{E}^Q \left[ \frac{c(T-1)}{[1+r(0)][1+r(1)] \cdots [1+r(T-2)]} \right]
 \end{aligned}$$

where  $Y = \frac{u'_{T-1}(C^*(T-1))}{(1+r(T-1))u'_T(C^*(T))}$  and we used the definition of  $1+r(T-1)$  to see that

$$\mathbb{E}^P [Y | \mathbf{P}_{T-1}] = \mathbb{E}^P \left[ \frac{u'_{T-1}(C^*(T-1))}{(1+r(T-1))u'_T(C^*(T))} \mid \mathbf{P}_{T-1} \right] = 1.$$

Now we can rewrite the valuation formula (22) as

$$V(c) = \mathbb{E}^Q \left[ \sum_{k=1}^T \frac{1}{[1+r(0)][1+r(1)] \cdots [1+r(k-1)]} c(k) \right]. \quad (25)$$

Equation (25) recasts the equilibrium valuation formula as a standard risk-neutral expectation. The cash flow  $c(\omega, k) = c(w^{(1)}, \omega^{(2)}, k)$  depends, in general, on both interest rate states and catastrophe states. However, the discount factors depend only on the interest rate states. To make this more explicit, we re-write the formula by breaking the expectation into two expectations, the first conditional on the interest rates. The catastrophe bond (and in general catastrophe derivatives) can be evaluated by first calculating the conditional random variables

$$\bar{c}(k) = \mathbb{E}^Q [c(k) | \mathbf{P}^{(1)}]$$

which are expectations over the loss distribution. The value of  $\bar{c}(k)$  reflects the random interest rate events represented by  $\mathbf{P}^{(1)}$ . The  $\bar{c}(k)$  depend only on the financial variables.

Then we obtain this valuation formula:

$$\begin{aligned}
 V(c) &= E^Q \left[ \sum_{k=1}^T \frac{1}{[1+r(0)][1+r(1)] \cdots [1+r(k-1)]} c^{(k)} \right] \\
 &= E^Q \left[ \sum_{k=1}^T \frac{1}{[1+r(0)][1+r(1)] \cdots [1+r(k-1)]} E^Q [c^{(k)} | \mathbf{P}^{(1)}] \right] \\
 &= E^Q \left[ \sum_{k=1}^T \frac{1}{[1+r(0)][1+r(1)] \cdots [1+r(k-1)]} \bar{c}^{(k)} \right] \\
 &= E^{Q_1} \left[ \sum_{k=1}^T \frac{1}{[1+r(0)][1+r(1)] \cdots [1+r(k-1)]} \bar{c}^{(k)} \right]
 \end{aligned} \tag{26}$$

This shows we can calculate the catastrophe bond price in stages:

1. Calculate the equivalent risk neutral probabilities and interest rates using the aggregate consumption process.
2. For each interest rate state of the world, calculate the bond's expected cash flows conditionally on the interest rate path.
3. Calculate the price of the expected cash flows using the equilibrium valuation formula.

We note also that the valuation measure,  $Q(\omega) = Q(\omega^{(1)}, \omega^{(2)})$ , can be written as a product

$$Q(\omega^{(1)}, \omega^{(2)}) = Q_1(\omega^{(1)}) P_2(\omega^{(2)})$$

where

$$\begin{aligned}
 Q_1(\omega^{(1)}) &= P_1(\omega^{(1)}) [1+r(0)][1+r(\omega^{(1)}, 1)] \cdots \\
 &\quad \cdots [1+r(\omega^{(1)}, T-1)] \frac{u'_T(C^*(\omega, T-1))}{u'_0(C^*(\omega, 0))}.
 \end{aligned} \tag{27}$$

This is well-defined because the aggregate consumption process does not depend on the catastrophe risk. We know that  $\mathbf{P}^{(1)}$  and  $\mathbf{P}^{(2)}$  are independent under the probability measure  $P$ . It is also true that  $\mathbf{P}^{(1)}$  and  $\mathbf{P}^{(2)}$  are independent under the probability

measure  $Q$ . Indeed, suppose that  $A_1 \in \mathbf{P}^{(1)}$  and  $A_2 \in \mathbf{P}^{(2)}$ . Then we find by direct calculation that,

$$\begin{aligned} Q(A_1 \cap A_2) &= \sum_{\omega \in \Omega} 1_{A_1}(\omega) 1_{A_2}(\omega) Q(\omega) \\ &= \sum_{\omega^{(1)} \in \Omega^{(1)}} \sum_{\omega^{(2)} \in \Omega^{(2)}} 1_{A_1}(\omega^{(1)}) 1_{A_2}(\omega^{(2)}) Q_1(\omega^{(1)}) P_2(\omega^{(2)}) \\ &= Q_1(A_1) P_2(A_2) = Q(A_1) Q(A_2) \end{aligned} \quad (28)$$

The independence of  $\mathbf{P}^{(1)}$  and  $\mathbf{P}^{(2)}$  under the probability measure  $Q$  simplifies the valuation problem for catastrophe risk bonds as we now illustrate.

For simplicity, we shall suppose that the catastrophe risk variables have a stationary and finite tree structure of the following nature. The stochastic process  $\{X(k) : k = 1, 2, \dots, T\}$  denotes the catastrophe losses allocated to each period<sup>9</sup>. Stationary means that the distribution of  $X(k)$  does not change with  $k$ . Since there are only finitely many states, we can denote the range of  $X(k)$  by  $0, x_1, x_2, \dots, x_n$ . The event  $X(k) = 0$  indicates no catastrophe; its probability is  $\theta_0$ . For the positive loss amounts  $x_i$ , we use the notation  $\Pr[X(k) = x_i] = \theta_i$ . Recall that the losses depend only on  $\Omega^{(2)}$ . In other words, for a state  $\omega = (\omega^{(1)}, \omega^{(2)}) \in \Omega$ , the value of  $X(\omega, k)$  is independent of  $\omega^{(1)}$ .

## The Morgan Stanley Bond

This bond pays coupons at a rate of  $c$  per period until a catastrophe occurs. If no catastrophe occurs, the bond matures with a final payment at time  $T$  of  $1 + c$ . We define the time  $\tau$  of the first catastrophe as

$$\tau \equiv \min\{k | X(k) > 0\} \quad (29)$$

where  $\tau = \infty$  if  $X(k) = 0$  for all  $k$ . If a catastrophe occurs, the bondholders get a final payment of  $g(X(\tau))$  where  $g(x)$  is a function specified in the bond contract. The cash flows to the bondholder are

$$c(k) = \begin{cases} c 1_{\tau > k} + g(X(k)) 1_{\tau = k} & k = 1, 2, \dots, T-1 \\ (1+c) 1_{\tau > T} + g(X(T)) 1_{\tau = T} & k = T \end{cases} \quad (30)$$

where the fixed rate coupon  $c$  and the face amount 1 are paid until a catastrophe occurs.

<sup>9</sup>The bond indenture may talk of "incurred" or "paid" as ways of allocating losses to periods. We assume some well-defined unambiguous allocation is defined.

Direct calculation shows that, because the coupons are independent of  $\mathbf{P}^{(1)}$ ,

$$\begin{aligned}\bar{c}(k) &= E^Q[c(k) | \mathbf{P}^{(1)}] \\ &= \begin{cases} c E^Q[1_{\tau > k} + g(X(k)) 1_{\tau = k}] & k = 1, 2, \dots, T-1 \\ (1+c) E^Q[1_{\tau > T} + g(X(T)) 1_{\tau = T}] & k = T \end{cases}\end{aligned}$$

Now use  $E^Q[1_{\tau > k}] = \theta_0^k$ ,

$$E^Q[g(X(k)) 1_{\tau = k}] = \theta_0^{k-1}(1 - \theta_0)E^Q[g(X(k)|X(k) > 0)]$$

and let

$$\begin{aligned}\mu &= E^Q[g(X(k))|X(k) > 0] \\ &= g(x_1)\frac{\theta_1}{1 - \theta_0} + g(x_2)\frac{\theta_2}{1 - \theta_0} + \dots + g(x_n)\frac{\theta_n}{1 - \theta_0}.\end{aligned}$$

Then, we have the expected cash flow (averaged over the loss distribution) conditional on the financial states:

$$\bar{c}(k) = \begin{cases} c\theta_0^k + \theta_0^{k-1}(1 - \theta_0)\mu & k = 1, 2, \dots, T-1 \\ (1+c)\theta_0^T + \theta_0^{T-1}(1 - \theta_0)\mu & k = T \end{cases}$$

It turned out for this bond that the expected coupons are constant. Then, relying on the independence relation to simplify the expectation (25) for the cash flow stream (30) shows the price of the catastrophe risk bond is given by the expression

$$\begin{aligned}V &= E^{Q_1} \left[ \sum_{k=1}^T \frac{1}{[1+r(0)][1+r(1)] \dots [1+r(k-1)]} \bar{c}(k) \right] \\ &= \sum_{k=1}^T E^{Q_1} \left[ \frac{1}{[1+r(0)][1+r(1)] \dots [1+r(k-1)]} \bar{c}(k) \right] \\ &= \sum_{k=1}^T E^{Q_1} \left[ \frac{1}{[1+r(0)][1+r(1)] \dots [1+r(k-1)]} \right] \bar{c}(k) \\ &= \sum_{k=1}^T P(k) \bar{c}(k) \\ &= c \sum_{k=1}^T P(k) \theta_0^k + P(T) \theta_0^T + \mu \sum_{k=1}^T P(k) \theta_0^{k-1} (1 - \theta_0)\end{aligned}\tag{31}$$



Examining relation (31) allows us to draw the following conclusion. We have established that valuation by a representative agent is equivalent to selecting a term structure model which is independent of the catastrophe risk structure and combining this term structure model with the probabilities of a catastrophe occurring to price the catastrophe risk bond. The evolving catastrophe risk bond prices [i.e. prices at times other than time 0] may be obtained from computing conditional expectations.

The general intertemporal valuation formula for the price of this type of catastrophe risk bond at time  $n$ , given the market information  $\mathbf{P}_k^{(1)}$  and assuming no catastrophe has occurred as of time  $n$ , is given by

$$V_n = c \sum_{k=n+1}^T P(n, k)Q(\tau > k | \tau > n) + P(n, T)Q(\tau > T | \tau > n) + \mu \sum_{k=n+1}^T P(n, k)Q(\tau = k | \tau > n) \quad (32)$$

where  $P(n, k)$  denotes the price at time  $n$  of a zero coupon bond maturing for 1 at time  $k$ . For our stationary model, the conditional probabilities are easy to compute.

This formula (31) has already made an appearance in section 3 [equation (8)] with  $\theta_0$  replaced by  $1 - \theta_1$ . The model developed in Tilley [18] may be thought of as the selection of a short-rate process  $\{r(k)\}$  on the filtered space  $(\Omega^{(1)}, \mathbf{P}^{(1)})$  and a risk-neutral probability measure  $Q^{(1)}$  on the probability space  $\Omega^{(1)}$  [i.e. a term structure model defined by  $\{r(k)\}$  and  $\Omega^{(1)}$ ] crossed with a conditional binomial catastrophe structure. The independence of the financial market risk from the catastrophe risk has permitted us to easily fit together these two probability structures to obtain a practical and economically meaningful model. The binomial formula is easy to apply as all that is needed for pricing the catastrophe risk bond is an estimate of the probability of a catastrophe occurring within one-period and a knowledge of the current yield curve. The expression (31) is theoretically equivalent to Tilley's formula except that we are able to interpret the parameter  $\theta_0$  in a traditional actuarial fashion because we closed our model using the theory of a representative agent which naturally involves the empirical probabilities of the various risks in the model. The fact that a catastrophe risk model is necessarily incomplete means that there is no unique interpretation of the prices that we assign to the catastrophe risk bonds. This problem is inherent in any model that is used to attach a price to catastrophe risk bonds. The utility function of the representative agent, which we could loosely refer to as the risk aversion of the market or the market's attitude towards risk, is part of the assumed structure of the pricing rule. In our equivalent formulation of the pricing problem in terms of risk-neutral valuation,

the incompleteness is embedded in the selection of the term structure model rather than as part of the catastrophe probabilities. In other words, for varying economic and catastrophe variables, the effect on the price dynamics of the catastrophe risk bond [equation (32)] appears through the implicit selection of the embedded term structure model. Although the bond pricing formula (31) seems to not depend on the embedded risk aversion, the dynamics of the catastrophe risk bond prices as shown in formula (32) depend on the full term structure model and thus on the embedded risk aversion. The fact that it is natural to select a term structure model for actuarial valuation problems hides the inherent difficulty associated with the fact that the catastrophe risk market is incomplete.

The Morgan Stanley bond was proposed as a means of financing a layer of risk in the California Earthquake Authority program to provide earthquake coverage. At the last minute it was under-bid by Berkshire Hathway with an offer of traditional reinsurance to cover a \$1.5 billion layer for four years for a premium of \$161 million, a rate on line of  $161/1500 = 10.73\%$ . According to a report in *Institutional Investor* [1], the corresponding Morgan Stanley deal would have had a rate on line of 11-14%.

Now we look at a few more examples.

## Winterthur's Bonds

For this example we need the set of financial assets to include the default free bonds maturing at each coupon date as usual, and also an equity security, Winterthur's common stock. We are relying on Schmock's paper [16] and the trade press (including [2]) for this description. We let the stochastic process  $\{S(k)\}$  denote the price of Winterthur's stock. For simplicity we assume no dividend payments to stock holders are expected during the term of the bond. This bond's cash flow depends on the number of claims rather than the severity or occurrence of a catastrophe. Therefore we let  $\{N(k)\}$  denote the number of windstorm claims per year to the 750,000 autos Winterthur insures in Switzerland. We will write the coupon in terms of this claim number process. As we did earlier, we will assume that the loss variables are stationary. The "added flexibility," as the trade press describes it, is a conversion option at maturity  $T = 3$ . The conversion option allows the bondholder to take five shares of stock in lieu of the payment that is otherwise due. The face amount is 4700 Swiss francs and the coupon rate is 2.25%. Thus the bondholder's cash flow can be described as follows:

$$c(k) = \begin{cases} 4700(0.0225)1_{N(k) \leq 6000} & k = 1, 2 \\ \max\{5S(3), 4700 + 4700(0.0225)1_{N(3) \leq 6000}\} & k = 3 \end{cases}$$

Let  $\Pr(N(k) > 6000) = q$  and compute the expected coupons, conditionally on the financial variables, to obtain

$$\bar{c}(1) = \bar{c}(2) = 4700(0.0225)(1 - q)$$

and

$$\bar{c}(3) = \max\{5S(3), 4700(1.0225)\}(1 - q) + \max\{5S(3), 4700\}q.$$

These expected payments to bondholders of course still have financial risk since  $S(3)$  is random, but in principle we could proceed now with the financial measure and compute the market value of the bond.

## USAA's bonds

The first tranche of the USAA deal has a face amount of \$163.8 million. Only the coupon is at risk and the coupon rate is LIBOR plus 2.73%. If there is a catastrophe (as described earlier), the coupon is not paid to the bondholders. Their principal is safe, but according to the bond indenture as described by Zolkos [19], the principal will not be repaid for ten years, during which time no coupons will be paid. In effect, some of the bondholder's principal is lost because each dollar of principal due at maturity is replaced by a dollar due 10 years later. Let  $X$  be the losses as described in the bond indenture. Assume that the term structure is based on LIBOR. The press articles do not specify exactly how the coupon depends on  $X$ , but we will assume an all or nothing payoff. Then the coupon per 1000 of face value can be written as follows:

$$c(k) = \begin{cases} 1000(1 + r(0) + 0.0273) 1_{\{X \leq 10^9\}} & k = 1 \\ 0 & 1 < k < 10 \\ 1000 1_{\{X > 10^9\}} & k = 10 \end{cases}$$

Let  $\Pr(X > 10^9) = q$ . Then we have

$$\bar{c}(k) = \begin{cases} 1000(1 + r(0) + 0.0273)(1 - q) & k = 1 \\ 0 & 1 < k < 10 \\ 1000 q & k = 10 \end{cases}$$

We let  $P(1, 11)$  denote the price at time 1 (when the cat bond matures) of a default free zero coupon bond providing a payment at time 11. At the time the cat bond is issued,  $P(1, 11)$  is random so this contract, like Winterthur's, has financial risk blended with the cat risk. The expected coupon is equivalent to a single payment at  $k = 1$ :

$$1000(1 + r(0) + 0.0273)(1 - q) + 1000 P(1, 11) q$$

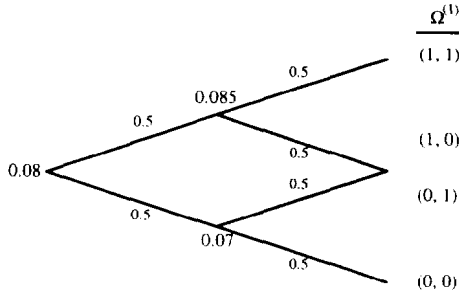
Now we will work out an example completely.

## A Final Example

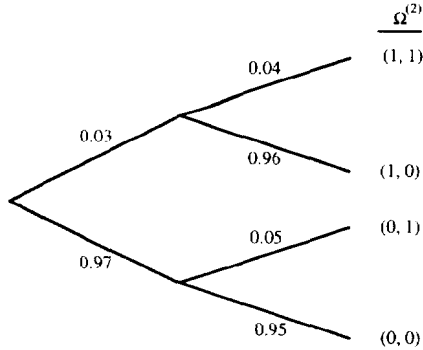
We illustrate the pricing model for a two-period case combining a binomial term structure model and a binomial catastrophe risk structure. The bond is Winterthur-style, but without the conversion option. The face amount is 100. Coupons only are at risk so the 100 is paid to the bondholder at  $k = 2$  with probability 1. A coupon of 12 is paid at  $k = 1, 2$  provided no catastrophe occurs during the period  $[k - 1, k]$ . The term structure is shown in Figure 5. The catastrophe states and probabilities are shown in Figure 6.

**Embedded Term Structure Model - Figure 5**

[one-period rates and risk neutral probabilities]



**Catastrophe Risk Structure - Figure 6**



The expected bondholder payments, averaged over the catastrophe distribution, are  $\bar{c}(1) = 12(0.97) = 11.64$  and  $\bar{c}(2) = 100 + 12(0.97)(0.95) + 12(0.03)(0.96) = 111.4036$ .

The discounted expected value, using the term structure, is the price of the cat bond:

$$\frac{1}{1.08} \left[ 11.64 + 111.4036 \left( \frac{1}{1.085} + \frac{1}{1.07} \right) \frac{1}{2} \right] = 106.51$$

Consider a bond that has the same prospective cash flow, but no possibility of default. This is called a *straight bond*. The price of the straight bond at the time the cat bond is issued is found by using the term structure:

$$\frac{1}{1.08} \left[ 12 + 112 \left( \frac{1}{1.085} + \frac{1}{1.07} \right) \frac{1}{2} \right] = 107.36$$

Suppose an insurer (like Winterthur, Swiss Re, or USAA) issues the cat bond and simultaneously buys the straight bond. The straight bond is more expensive. The trades cost the insurer 0.85 per 100 of face value. What does the insurer get in return? In each of the two future periods, if there is no catastrophe, the insurer's net cash flow is zero because it receives the straight bond coupon and pays the cat bond coupon. However, if there is a catastrophe in either period, it still receives the straight bond coupon (12), but does not pay the cat bond coupon. In effect, the insurer has purchased a two year catastrophe reinsurance contract which pays 12 in case a catastrophe occurs during either period. This increases the insurers capacity to sell insurance for the next two years by 12 at cost of 0.85. The actual deals we have described all increase the bond issuer's capacity. The cost may be high, but the technology is being developed so the cost will probably come down. Moreover, investors are becoming more familiar with the product so future deals might be relatively less costly. And, as others have pointed out, the insurance industry would be strained by a \$30 billion hurricane loss, but the capital markets could withstand it with relative calm. Catastrophe bonds may become a routine method of transferring catastrophe risk.

It is worth mentioning again that the line of insurance is immaterial to the capital market -- it does not have to be catastrophe risk. At the 1997 Swiss Actuarial Summer School held at the University of Lausanne we heard from Winterthur actuaries of a proposal to issue bonds which would transfer mortality risk to bondholders. Winterthur has issued long term pension policies and face the risk of unexpected improvement in pension beneficiary mortality. A security with bondholder cash flow tied to a mortality index would provide Winterthur with very long term coverage that is not available in the traditional reinsurance market.

## 6 Concluding Remarks

We have discussed the financial economics involved in the pricing of catastrophe risk bonds. Furthermore, we have demonstrated how this theory may be utilized to construct a practical valuation model which can be justified within the framework of a representative agent equilibrium. A full implementation of the representative agent model could have been made but there is little point since in practice one is more likely to choose to work with the non-defaultable term structure model backing the valuation procedure. It is quite natural that the inputs to a valuation procedure for catastrophe risk bonds should be assumptions about the term structure dynamics and the probability structure governing the occurrence of a catastrophe. As a first approximation to the pricing of catastrophe risk bonds, such a valuation framework seems to hold reasonable intuition and is theoretically sound. A catastrophe risk bond cannot be fully hedged because of the lack of traditional securities that can be used to closely approximate the payoffs from the catastrophe risk bond [i.e. inherent market incompleteness]. Consequently, implicit in the coupon rate [or equivalently the price] of a catastrophe risk bond is the investor's attitude towards risk. Although we have provided a framework in which to attach a specific price to a catastrophe risk bond, the fact that the catastrophe risk bond cannot be perfectly hedged necessarily implies that there is a range of prices at which the catastrophe risk bond could sell without the existence of arbitrage in the market. The inability of investor's to efficiently hedge the risk in catastrophe risk bonds also suggests that were Charles Darwin to observe a catastrophe bond market during a major catastrophe he might comment "[i]t is indeed most wonderful to witness such financial desolation produced in three minutes of time." At such a time, catastrophe risk bondholders would generally find that the "high yields" they were receiving were insufficient to protect them from the bare risk that is inherent in such an unhedgeable security. There is substantial literature dealing with the problem of incomplete markets. In the end however, no matter how one chooses to look at the valuation problem in incomplete markets there is simply no way to assign exact prices to securities. Readers interested in these pricing issues may consult Chan and van der Hoek (1996) as an introduction to several techniques for pricing cash flows in incomplete markets. In the end, one is hard pressed to come up with completely convincing pricing theories for catastrophe risk bonds.

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