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Asymptotics in the Subexponential case

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This is a summary of the presentation given during the ARC Conference. Its purpose was to give a brief introduction to subexponential behavior and to show that the family of subexponential distributions provides ideal characteristics to model insurance risks. Although the literature related to this topic accumulated quickly during the last years, a textbook explaining the applications of extreme value theory with intuition as well as insight was still to be written. Embrechts et al. (1997) satisfies this need and most of the results presented here can be found in this new classic in the actuarial literature.

The framework

The most common model of insurance risk theory is, by far, the random sum model

$$S(t) = X_1 + X_2 + \dots + X_{N(t)},$$

where the random variables $X_1, X_2, \dots$ are independent, identically distributed[1] and independent of $N(t)$. The distribution of $S(t)$ is obtained by conditioning on the number of terms in the sum:

$$F_{S(t)}(s) = \sum_{n=0}^{\infty} P[N = n] F_X^{*n}(s),$$

where $F_X^{*n}(s)$ denotes de n[th] convolution of the common distribution of the X_i's. These types of functions are usually called compound distributions and, in the particular case of Poisson claims, take the form:

$$F_{S(t)}(s) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} F_X^{*n}(s).$$

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[1] Usually concentrated in $(0,\infty)$

We are interested in the right tail of compound distributions. The main asymptotic results at hand are:

- The "light-tailed" (Lundberg type) asymptotics
- The "heavy-tailed" subexponential asymptotics

In this work, we focus on the following questions:

- How do we characterize heavy-tailed distributions?
- What is an appropriate class of heavy-tailed distributions with suitable asymptotic properties?
- How do these asymptotic properties compare to Lundberg asymptotics?

Tails and failure rates

The tail of a claim size distribution can be analyzed via the failure-rate μ (force of mortality) in the following way:

Notice that directly from its definition we get

$$\frac{1}{\mu(y)} = \frac{1-F(y)}{f(y)} = \int_y^\infty \frac{f(x)}{f(y)}dx = \int_0^\infty \frac{f(x+y)}{f(y)}dx.$$

If the function $\frac{f(x+y)}{f(y)}$ is increasing[2] in y for fixed x, the integral $\int_0^\infty \frac{f(x+y)}{f(y)}dx$ will also be increasing and therefore $\mu(y)$ will be decreasing. In the same way, if that quotient is decreasing, the failure-rate will be increasing. Now, we just have to relate the behavior of μ with the thickness of the right tail of a distribution.

Intuitively speaking, if the function $\frac{f(x+y)}{f(y)}$ is increasing in y, the values of $f(y)$ decrease faster than $f(x+y)$. The distribution is "pushing" most of its probability to the tail and therefore we get a thick right tail. In summary, a decreasing failure rate is associated to heavy tails. In the same way, if the function $\frac{f(x+y)}{f(y)}$ is decreasing in y, the distribution goes to one faster, accumulating most of its probability in its center, giving a thin right tail: Increasing failure rates are therefore associated to light tails.

Figure1 illustrates this non-rigorous explanation.

[2] Non-decreasing, strictly speaking

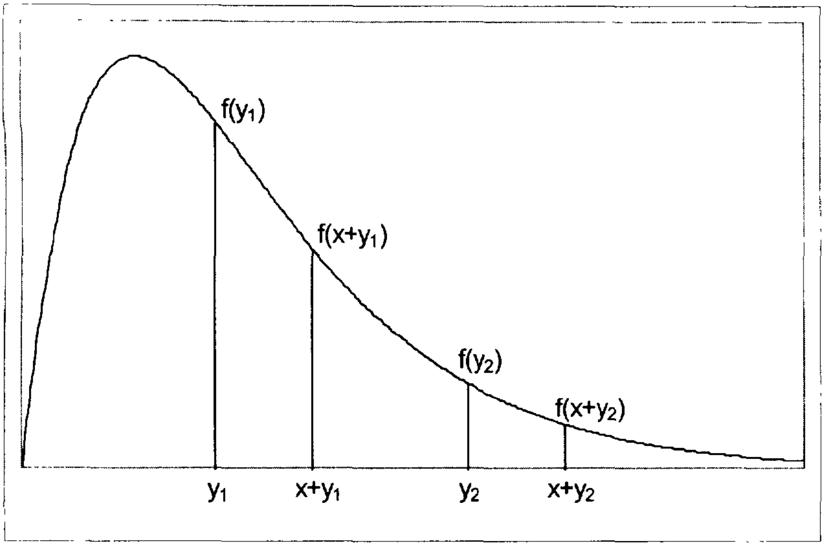

Fig. 1

Notice that by Cauchy-Schwarz inequality, decreasing failure rate behavior is preserved under mixing, whereas convolutions preserve increasing failure rate behavior. Therefore, mixing is a procedure that gives us heavy tailed distributions, while the sum of random variables generally generates light tailed distributions.

An example

We have learned that typical examples of heavy tailed distributions appear in modeling catastrophes. In order to have a sequence of losses that most of us would consider as catastrophes, I decided to plot in figure 2 the 40 most costly insurance losses during the period 1970-1998 as registered in the bulletin Sigma(1999)[3]. The amounts of the losses[4] are plotted against time and the line denotes the record loss up to that time. This graph includes, for example, hurricane Celia (1970), the "Autumn storm" (1987), hurricane Daria (1990), the earthquake in Southern California (1994) and the most costly insurance loss of all times: hurricane Andrew (1992). Most actuaries would agree that in order to model these losses - and obtain a good fit - we must choose a heavy tailed distribution.

[3] Which is actually an update of a similar table found in Embrechts et al (1997)
[4] in million $US at 1998 prices.

The 40 most costly insurance losses (1970-1998)

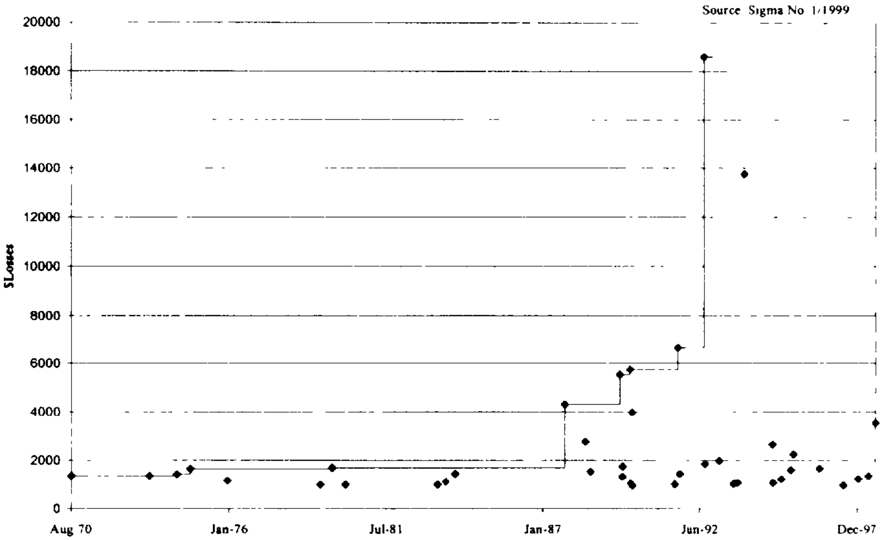

Fig. 2

But while analyzing these data, one notices a very peculiar behavior: the record losses seem to have the wrong concavity! If we were observing a realization of a sequence of independent, identically distributed random variables, the record loss should look like a logarithmic curve, not like an exponential one As Embrechts (1997) on p. 4 notes:

"Intuition tells us that successive records for iid data should become more and more rare as time goes by: it becomes more and more difficult to exceed all past observations"

In order to verify this intuition, at least at an exploratory level, we can simulate losses occurring exactly at the same dates as our original data, preserving the mean and variance of the observed losses. Figures 3 and 4 simulate several realizations assuming normal and gamma losses respectively. The drastic change in the record processes is due to the tail of each distribution: heavy tails generate successive record losses that exceed the past observations quite "easily".

Now that we have confirmed the presence of heavy tailed behavior, we are interested in finding an appropriate class of distributions that fits our data and has desirable properties from the actuarial point of view.

Simulated process of records, assuming normal distribution

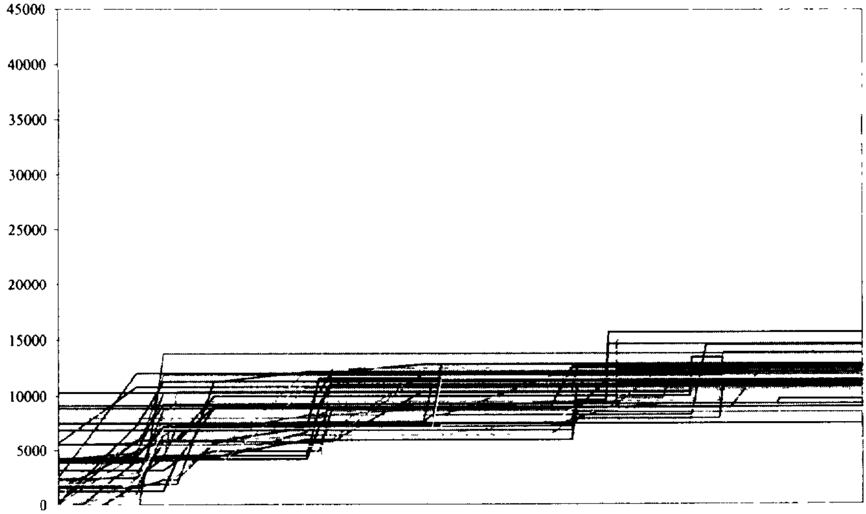

Fig. 3

Simulated process of records, assuming gamma distribution

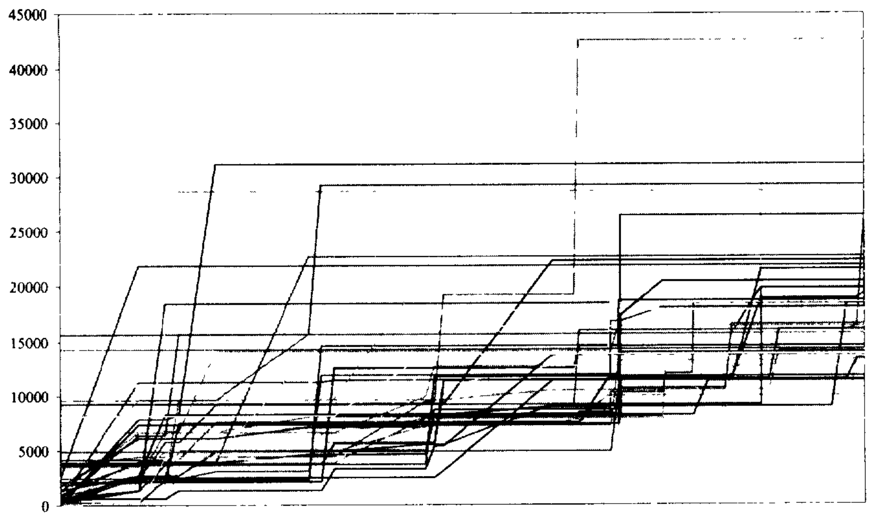

Fig. 4

Coefficient of adjustment and tails

The coefficient of adjustment v can be defined in several ways, a useful one is:

$v>0$ such that $\int_0^\infty e^{vx} dF_I(x) = 1+\theta$, where θ is the safety loading and F_I the equilibrium distribution of the claim amount distribution F.

One notices that the existence of v implies that F_I and F are exponentially bounded, i.e. $1-F(x) \le \frac{E(e^{vX})}{e^{vx}}$. In other words, large claims are very unlikely (with exponentially small probability) to occur.

This is the case for distributions like: exponential, gamma, Weibull (light tail), truncated normal and those distributions with bounded support. But it is not the case for many other distributions, for example: lognormal, Pareto, Burr, Benktander I and II, Loggamma and Weibull (heavy tail). Therefore we need some additional tools to expand our family of possible distributions to use.

Regular variation theory: just a glance

Regular varying distribution functions are of the form $1-F(x) = x^{-\alpha} L(x)$, where L is a slowly varying function, that is, one for which $L(tx) \sim L(x)$[5] $t>0$.

These functions have the nice property of closeness under convolution operations, in particular, $1-F^{*n}(x) \sim n[1-F(x)]$. But also notice that

$$P[\max(X_1,\ldots,X_n) > x] = 1-[F(x)]^n = [1-F(x)]\sum_{k=0}^{n-1} F^k(x) \sim n[1-F(x)]$$

i.e., the distribution of the maximum and the distribution of the sum of n claims are of the same order of convergence! This gives a very nice expression for the probability of ultimate ruin:

$$\psi(u) = \frac{\theta}{1+\theta}\sum_{n=0}^{\infty}(1+\theta)^{-n}\left(1-F_I^{*n}(u)\right)$$

$$\frac{\psi(u)}{[1-F_I(u)]} = \frac{\theta}{1+\theta}\sum_{n=0}^{\infty}(1+\theta)^{-n}\frac{[1-F_I^{*n}(u)]}{[1-F_I(u)]} \to \frac{\theta}{1+\theta}\sum_{n=0}^{\infty}(1+\theta)^{-n} n = \frac{1}{\theta} \text{ as } u\to\infty$$

[5] In other words, they are of the same order of convergence. Please see the details in Embrechts (1982).

Therefore,

$$\psi(u) \sim \frac{1}{\theta E[X]} \int_u^{\infty} [1 - F(y)] dy \quad \text{as } u \to \infty.$$

This result covers the Pareto, Burr, loggamma and truncated stable distributions. But we want more! Is it possible to expand this class of distributions (regular varying) in such a way that we preserve this asymptotic result for the probability of ultimate ruin?

The answer is yes. The class can be expanded further more, to the class of *Subexponential distributions*.

Subexponential distributions

Although the theory of subexponential distributions is far from trivial, we can benefit from all the research done in this field and use simplified but completely accurate definitions. For a comprehensive review of this topic, see Embrechts(1982).

A distribution F is subexponential if $\lim_{x\to\infty} \frac{[1 - F^{*n}(x)]}{[1 - F(x)]} = n$, or equivalently if

$$P[X_1 + X_2 + ... + X_n > x] \sim P[\max\{X_1, X_2, ..., X_n\}] \text{ as } x \to \infty.$$

Relevant properties are:

- $\lim_{x\to\infty} \frac{1 - F(x - y)}{1 - F(x)} = 1$ uniformly for compact sets in the support of F.
- $e^{\varepsilon x}(1 - F(x)) \to \infty$, for all $\varepsilon > 0$. (Which justifies the name "subexponential")
- For any $\varepsilon > 0$, $\int_y^{\infty} e^{\varepsilon x} dF(x) \geq e^{\varepsilon y}(1 - F(y))$, $y \geq 0$. Therefore this integral - that defines the coefficient of adjustment - goes to infinity, which comprises the heavy-tailed cases we are interested in.

Applications

These results allow us to formulate the Cramér-Lundberg theorem for the subexponential class:

For the classical ruin model, with F_I an element of the subexponential class of distributions:

$$\psi(u) \sim \frac{(1 - F_I(u))}{\theta} \text{ as } u \to \infty.$$

This result generalizes the asymptotic formula for regular varying functions, including now the lognormal, Benktander I and II and the heavy-tailed Weibull.

Even more, the class of subexponential distributions is the *biggest* class such that this asymptotic result holds. Therefore, this is the natural class of distributions whenever the coefficient of adjustment does not exist:

$$F_I \text{ subexponential} \Leftrightarrow 1 - \psi(u) \text{subexponential} \Leftrightarrow \psi(u) \sim \frac{(1 - F_I(u))}{\theta}$$

The formulas for **total claim amounts** are as easy as those for ruin probabilities:

Let G denote the distribution of total claim amounts, then

$G_t(x) = \sum_{n=0}^{\infty} P_t[N = n] F^{*n}(x)$. If F is subexponential and the probability generating function of N exists around one,[6] then G_t will also be subexponential and

$1 - G_t(x) \sim E[N(t)][1 - F(x)]$ as $x \to \infty$. A notably simple formula under such general conditions.

[6] This simply means that the distribution for the number of claims is not heavy-tailed.

References

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