UNDERSTANDING RELATIONSHIPS USING COPULAS

by

Edward W. (Jed) Frees
and
Emiliano A. Valdez

University of Wisconsin - Madison

August 20, 1997

Abstract

This article introduces actuaries to the concept of "copulas," a tool for understanding relationships among multivariate outcomes. A copula is a function that links univariate marginals to their full multivariate distribution. Copulas were introduced in 1959 in the context of probabilistic metric spaces. Recently, there has been a rapidly developing literature on the statistical properties and applications of copulas. This article explores some of these practical applications, including estimation of joint life mortality and multidecrement models. In addition, we describe basic properties of copulas, their relationships to measures of dependence and several families of copulas that have appeared in the literature. An annotated bibliography provides a resource for researchers and practitioners who wish to continue their study of copulas. This article will also be useful to those who wish to use copulas for statistical inference. Statistical inference procedures are illustrated using insurance company data on losses and expenses. For this data, we (1) show how to fit copulas and (2) describe their usefulness by pricing a reinsurance contract and estimating expenses for pre-specified losses.

Keywords and Phrases: Dependence, Multivariate Distributions.

Please address correspondence to:
Edward W. (Jed) Frees
Time Insurance Professor of Actuarial Science
School of Business
975 University Avenue
Madison, Wisconsin 53706
jfrees@bus.wisc.edu

Table of Contents

Abstract

1. Introduction

As emphasized in the recent release of the *General Principles of Actuarial Science* (1997), actuaries strive to understand stochastic outcomes of financial security systems. Because these systems are generally complex, outcomes are measured in several dimensions. Describing relationships among different dimensions of an outcome is a basic technique that actuaries use to help explain the behavior of financial security systems to concerned business and public policy decision-makers. This article introduces the concept of a "copula" function as a tool for relating different dimensions of an outcome.

Understanding relationships among multivariate outcomes is a basic problem in statistical science; it is not specific to actuarial science nor is it new. To illustrate, a fundamental contribution to understanding multivariate relationships was made by Sir Francis Galton in the late nineteenth century with his introduction of regression analysis. In one data set described in his 1885 presidential address to the anthropological section of the British Association of the Advancement of Sciences, Galton linked the distribution of heights of adult children to the distribution of their parents' heights. Galton showed that not only was each distribution approximately normal but also that the joint distribution could be described as a bivariate normal. Thus, the conditional distribution of adult children's height, given the parents' height, could also be described using a normal distribution. As a by-product of his analysis, Galton also observed that "tall parents tend to have tall children although not as tall as the parents" (and vice-versa for short children). From this, he incorrectly inferred that children would "regress to mediocrity" in subsequent generations, hence suggesting the term that has become known as *regression analysis*. See Seal (1967) and Stigler (1986) for additional accounts of the works of Galton and other early contributors to statistical science.

Regression analysis has developed into the most widely applied statistical methodology; see, for example, Frees (1996) for an introduction. It is an important component of multivariate analysis because it allows researchers to focus on the effects of explanatory variables. To illustrate, in the Galton data set of family heights, regression allows the analyst to describe the effect of parents' height on a child's adult height. Regression analysis is also widely applied in actuarial science; as evidence, it is a required component of the two main educational actuarial bodies in the US and Canada, the Society of Actuaries and the Casualty Actuarial Society.

Although widely applicable, regression is limited by the basic set-up that requires the analyst to identify one dimension of the outcome as the primary measure of interest (the dependent variable) and other dimensions as supporting or "explaining" this variable (the independent variables). This article examines problems where this relationship is not of primary interest; hence, we focus on the more basic problem of understanding the distribution of several outcomes, a *multivariate* distribution. For an example in actuarial

science, when two lives are subject to failure as under a joint life insurance or annuity policy, we are concerned with joint distribution of lifetimes. As another example, when we simulate the distribution of a scenario that arises out of a financial security system, we need to understand the distribution of several variables interacting simultaneously, not in isolation of one another.

The normal distribution has long dominated the study of multivariate distributions. For example, leading references on multivariate analysis, such as Anderson (1958) and Johnson and Wichern (1988), focus exclusively on the multivariate normal and related distributions that can be derived from normal distributions, including multivariate extensions of Student's t- and Fisher's F-distributions. Multivariate normal distributions are appealing because the marginal distributions are also normal. For example, in the Galton data set, the distributions of adult children's height and the distribution of parents' height are each approximately normal, in isolation of the other. Multivariate normal distributions are also appealing because the association between any two random outcomes can be fully described knowing only (i) the marginal distributions and (ii) one additional parameter, the correlation coefficient.

More recent texts on multivariate analysis, such as Krzanowski (1988), have begun to recognize the need for examining alternatives to the normal distribution set-up. This is certainly true for actuarial science applications such as for lifetime random variables (Bowers et al., 1997, Chapter 3) and long-tailed claims variables (Hogg and Klugman, 1984) where the normal distribution does not provide an adequate approximation to many data sets. There is a wide literature in statistics dealing with nonnormal multivariate distributions, see, for example, Johnson and Kotz (1973) and Johnson, Kotz and Balakrishnan (1997). However, historically many multivariate distributions have been developed as immediate extensions of univariate distributions, examples being the bivariate Pareto, bivariate Gamma and so on. The drawbacks of these types of distributions are that (i) one needs a different family for each marginal distribution, (ii) extensions to more than just the bivariate case are not clear and (iii) measures of association often appear in the marginal distributions.

A construction of multivariate distributions that does not suffer from these drawbacks is based on the *copula* function. To define a copula, begin as you might in a simulation study by considering "p" uniform (on the unit interval) random variables, $u_1, u_2, \ldots, u_p$. Here, "p" is the number of outcomes that you wish to understand. Unlike many simulation applications, we do not assume that $u_1, u_2, \ldots, u_p$ are independent, yet that they may be related. This relationship is described through their joint distribution function

$$C(u_1, u_2, \ldots, u_p) = \Pr ob(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_p \leq u_p).$$

8

Here, we call the function C a *copula*. Further, U is a (ex-ante) uniform random variable whereas u is the corresponding (ex-post) realization. To complete the construction, we select arbitrary marginal distribution functions $F_1(x_1), F_2(x_2), ..., F_p(x_p)$. Then, the function

$$C\big(F_1(x_1), F_2(x_2), ..., F_p(x_p)\big) = F\big(x_1, x_2, ..., x_p\big) \tag{1.1}$$

defines a multivariate distribution function, evaluated at $x_1, x_2, ..., x_p$, with marginal distributions $F_1, F_2, ..., F_p$.

With copula construction in equation (1.1), we select different marginals for each outcome. For example, suppose we are considering modeling male and female lifetimes for a joint-life annuity product. Then, with p=2, we might choose the Gompertz distribution to represent mortality at the older ages, yet with different parameters to reflect gender differences in mortality. As another example, in Section 4, we consider a bivariate outcome associated with the loss and the expense associated with administering a property and casualty claim. There, we could elect to use a lognormal distribution for expenses and a longer tail distribution, such as Pareto, for losses associated with the claim. The copula construction does not constrain the choice of marginal distributions.

In Section 2 we will see that the copula method for understanding multivariate distributions has a relatively short history in the statistics literature; most of the statistical applications have arisen in the last ten years. However, copulas have been studied in the probability literature for about forty years (Schweizer, 1991) and thus several desirable properties of copulas are now widely known. To begin, it is easy to check from the construction in equation (1.1) that F is a multivariate distribution function. Sklar (1959) established the converse. He showed that any multivariate distribution function F can be written in the form of equation (1.1), that is, using a copula representation. Sklar also showed that, if the marginal distributions are continuous, then there is a unique copula representation. In this sense copulas provide a unifying theme for our study of multivariate distributions. Sections 3 and 5 describe other desirable properties of copulas.

Given that copulas are fundamental building blocks for studying multivariate distributions, we now turn to the question as to how one builds a copula function for a problem at hand. Despite Sklar's result that a copula function always exists, the following Example 1.1 shows that it is *not* always convenient to identify the copula. Example 1.2 will illustrate a useful way of building a copula, using the method of *compounding*. We describe this method of constructing copulas in detail in Section 3.1.

Example 1.1 **Marshall-Olkin (1967) Exponential Shock Model**

Suppose that we wish to model p=2 lifetimes that we suspect are subject to some common disaster, or "shock," that may induce a dependency between the lives. For simplicity, let us assume that Y_1 and Y_2 are two independent (underlying) lifetimes with distribution functions H_1 and H_2. We further assume there exists an independent exponential random variable Z with parameter λ that represents the time until common disaster. Both lives are subject to the same disaster so that actual ages-at-death are represented by $X_1 = \min(Y_1, Z)$ and $X_2 = \min(Y_2, Z)$. Thus, the marginal distributions are

$$\text{Prob}(X_j \le x_j) = F_j(x_j) = 1 - \exp(-\lambda x_j)(1 - H_j(x_j)), \text{ for } j = 1,2.$$

Basic calculations show that the joint distribution is[1]

$$F(x_1, x_2) = F_1(x_1) + F_2(x_2) - 1 + \exp(\lambda \max(x_1, x_2))(1 - F_1(x_1))(1 - F_2(x_2)).$$

This expression, although intuitively appealing, is not in the form of the copula construction (1.1) because the joint distribution function F is not a function of the marginals $F_1(x_1)$ and $F_2(x_2)$. See Frees et al. (1996) and Bowers et al. (1997, Section 9.6) for further discussions in the actuarial literature of this bivariate distribution.

Example 1.2 **Bivariate Pareto Model**

Consider a claims random variable X that, given a risk classification parameter γ, can be modeled as an exponential distribution. That is, $\text{Prob}(X \le x|\gamma) = 1 - e^{-\gamma x}$. As is well known in credibility theory (see, for example, Klugman, et al., 1997), if γ has a Gamma distribution, then the marginal distribution (over all risk classes) of X is Pareto. That is, if γ is $\text{Gamma}(\alpha, \lambda)$, then

$$F(x) = \text{Prob}(X \le x) = \int \text{Prob}(X \le x|\gamma) \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma} d\gamma$$

$$= 1 - \int e^{-\gamma x} \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma} d\gamma = 1 - (1 + x/\lambda)^{-\alpha}, \tag{1.2}$$

a Pareto distribution.

---

[1] $F(x_1, x_2) = \text{Prob}(X_1 \le x_1, X_2 \le x_2) = 1 - \text{Prob}(X_1 > x_1) - \text{Prob}(X_2 > x_2) + \text{Prob}(X_1 > x_1, X_2 > x_2)$

$= 1 - \exp(-\lambda x_1)(1 - H_1(x_1)) - \exp(-\lambda x_2)(1 - H_2(x_2)) + \exp(-\lambda \min(x_1, x_2))(1 - H_1(x_1))(1 - H_2(x_2))$

$= 1 - (1 - F_1(x_1)) - (1 - F_2(x_2)) + \exp(-\lambda \min(x_1, x_2))\exp(\lambda(x_1 + x_2))(1 - F_1(x_1))(1 - F_2(x_2)).$

Suppose, conditional on the risk class γ, that X_1 and X_2 are independent and identically distributed. Assuming that they come from the same risk class γ induces a dependency. The joint distribution is[2]

$$F(x_1, x_2) = F_1(x_1) + F_2(x_2) - 1 + \left[\left(1 - F_1(x_1)\right)^{-1/\alpha} + \left(1 - F_2(x_2)\right)^{-1/\alpha} - 1\right]^{-\alpha}.$$

This yields the copula function

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[\left(1 - u_1\right)^{-1/\alpha} + \left(1 - u_2\right)^{-1/\alpha} - 1\right]^{-\alpha}. \tag{1.3}$$

With this function, we can express the bivariate distribution function as $H(x_1, x_2) = C\left(F_1(x_1), F_2(x_2)\right)$.

Alternatively, we can consider the copula

$$C_*(u_1, u_2) = C(1 - u_1, 1 - u_2) = \left(u_1^{-1/\alpha} + u_2^{-1/\alpha} - 1\right)^{-\alpha} - 1$$

and express the joint survival function as $\mathrm{Prob}(X_1 > x_1, X_2 > x_2) = C_*\left(S_1(x_1), S_2(x_2)\right)$, where $S = 1 - F$. Because our motivating examples in Section 2 concern lifetime (positive) random variables, we will often find it intuitively appealing to work with survival in lieu of distribution functions.

There are several methods for constructing multivariate distributions. See Hougaard (1987) and Hutchinson and Lai (1990) for detailed reviews. Example 1.1 illustrates the so-called "variables-in-common" technique where a common element serves to induce dependencies among several random variables. This article focuses on the compounding method illustrated in Example 1.2 for two reasons. First, there is a long history of using compound distributions for risk classification in the actuarial science literature, particularly within the credibility framework. Second, Marshall and Olkin (1988) showed that compounding can be used to generate several important families of copulas. Additional discussion of this point appears in Sections 2 and 3.

Examples 1.1 and 1.2 each describe bivariate distributions through probabilistic interpretations of random quantities. We will also find it useful to explore (in Section 3) a class of functions called "Archimedean copulas" that arise from the mathematical theory of associativity. An important special case of this class, due to Frank (1979), is

---

[2] $F(x_1, x_2) = 1 - \mathrm{Prob}(X_1 > x_1) - \mathrm{Prob}(X_2 > x_2) + \mathrm{Prob}(X_1 > x_1, X_2 > x_2)$

$$= 1 - \left(1 + x_1 / \lambda\right)^{-\alpha} - \left(1 + x_2 / \lambda\right)^{-\alpha} + \int \mathrm{Prob}(X_1 > x_1 | \gamma) \mathrm{Prob}(X_2 > x_2 | \gamma) \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma} d\gamma$$

$$= 1 - \left(1 + x_1 / \lambda\right)^{-\alpha} - \left(1 + x_2 / \lambda\right)^{-\alpha} + \int e^{-\gamma x_1} e^{-\gamma x_2} \frac{\alpha^\lambda}{\Gamma(\lambda)} \gamma^{\alpha-1} e^{-\lambda\gamma} d\gamma$$

$$= 1 - \left(1 + x_1 / \lambda\right)^{-\alpha} - \left(1 + x_2 / \lambda\right)^{-\alpha} + \left[1 + (x_1 + x_2) / \lambda\right]^{-\alpha}.$$

$$C(u,v) = \frac{1}{\alpha} \ln\left(1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1}\right). \tag{1.4}$$

Although Frank's copula does not appear to have a natural probabilistic interpretation, its other desirable properties make it well suited for empirical applications (Nelson, 1986 and Genest, 1987).

The purpose of this paper is to introduce the reader to copulas, their characteristics and properties, and their applicability to specific situations. Section 2 reviews empirical applications of copulas in analyzing survival of multiple lives and competing risks. Both are familiar topics to actuaries. Section 3 discusses properties and characteristics of copulas. In particular, we show (i) how to specify a copula, (ii) how the association structure of copulas can be summarized in terms of familiar measures of dependence, and (iii) how simulation of multivariate outcomes can be easily accomplished when the distribution is expressed as a copula. Section 4 provides an illustration of fitting a copula to insurance company losses and expenses. Section 5 reviews additional applications of copulas. We conclude in Section 6.

2. Empirical Applications

As seen in the introduction, copulas are useful for examining the dependence structure of multivariate random vectors. In this section, we describe two biological science subject areas that are related to actuarial science and that have used copulas to understand empirical relationships among multivariate observations.

2.1 Survival of Multiple Lives

In epidemiological and actuarial studies, it is often of interest to study the joint mortality pattern of groups of more than a single individual. This group could be, for example, a husband and wife, a family with children, or twins (identical or non-identical). There is strong empirical evidence that supports the dependence of mortality of pairs of individuals. For example, statistical analyses of mortality patterns of married couples are frequently made to test the "broken heart" syndrome. Using a data set consisting of 4,486 55-year old widowers, Parkes et al. (1969) showed that there is a 40% increase in mortality among the widowers during the first few months after the death of their wives, see also Ward (1976). Intuitively, pairs of individuals exhibit dependence in mortality because they share common risk factors. These factors may be purely genetic, as in the case of twins, or environmental, as in the case of a married couple.

The first application of copulas in joint-life models arose indirectly through the work of Clayton (1978) in his study of bivariate life tables of fathers and sons. Clayton developed the bivariate distribution function given in equation (1.3) as the solution of a second-order partial differential equation. Clayton also

pointed out the random effects interpretation of the model that was subsequently developed by Oakes (1982). See also Cook and Johnson (1981).

Random effects models are important in biological and epidemiological studies because they provide a method of modeling heterogeneity. A random effects model that is particularly suited for multivariate survival analysis is the *frailty* model, due to Vaupel, Manton and Stallard (1979) and Hougaard (1984). To describe frailty models, we first introduce some notation. In survival analysis, it is customary to consider the complement of the distribution function, the survival function, and the negative derivative of its logarithmic transform, the hazard function. Thus, for a continuous random survival time T, we define

$$S(t) = \text{Prob}(T > t) = 1 - F(t) \text{ and } h(t) = -\partial \ln S(t) / \partial t = f(t) / S(t).$$

Actuaries know the hazard function $h(t)$ as the force of mortality (see, for example, Bowers et al., 1997, Chapter 3).

To understand explanatory variables Z in survival analysis, one way is to use the Cox (1972) proportional hazards model which represents the hazard function as

$$h(t, Z) = e^{\beta Z} b(t),$$

where $b(t)$ is the so-called "baseline" hazard function and β is a vector of regression parameters. It is proportional in the sense that all the information contained in the explanatory variables is in the multiplicative factor $\gamma = e^{\beta Z}$. Integrating and exponentiating the negative hazard, Cox's proportional hazard model can also be expressed as

$$S(t|\gamma) = \exp\left(-\int_0^t h(s, Z) ds\right) = B(t)^\gamma.$$

Here, $B(t) = \exp\left(-\int_0^t b(s) ds\right)$ is the survival function corresponding to the baseline hazard. Frailty models arise when Z, and hence γ, is unobserved. The factor γ is called a *frailty* because larger values of γ imply a smaller survival function, $S(t|\gamma)$, indicating poorer survival. As demonstrated in Example 1.2, the marginal distribution for a single life T is obtained by taking expectations over the potential values of γ, that is, $S(t) = E_\gamma S(t|\gamma)$.

Oakes (1989, 1994) described how frailties can be used to model the dependencies among multiple lives. Other studies, such as Jagger and Sutton (1991), used a Cox regression model with *known* explanatory variables Z to account for the dependencies among multiple lives. Multivariate frailty models are obtained when the investigator does not wish to attribute, or does not have knowledge of, specific characteris-

tics that may induce dependencies. For multivariate frailty models, we assume that "p" lives $T_1, T_2, ..., T_p$ are independent given the frailty γ. That is,

$$\Pr ob\left(T_i > t_1, ..., T_p > t_p | \gamma\right) = \Pr ob\left(T_i > t_i | \gamma\right) \cdots \Pr ob\left(T_p > t_p | \gamma\right)$$

$$= S_1(t_1 | \gamma) \cdots S_p(t_p | \gamma) = B_1(t_1)^\gamma \cdots B_p(t_p)^\gamma .$$

The joint multivariate survival function is defined as

$$\Pr ob\left(T_1 > t_1, ..., T_p > t_p\right) = E_\gamma \left\{B_1(t_1) \cdots B_p(t_p)\right\}^\gamma \tag{2.1}$$

Example 2.1 **Hougaard's Copula Family**

To illustrate, an important frailty model was given by Hougaard (1986) who assumed that the distribution of γ could be modeled as a positive "stable distribution" with Laplace transform $E_\gamma e^{-\pi} = \exp(-s^\alpha)$ and parameter α. Recall that the Laplace transform of a positive random variable γ is defined by $\tau(s) = E_\gamma e^{-\pi} = \int e^{-s t} dG_\gamma(t)$, where G_γ is the distribution function of γ. This is also the moment generating function evaluated at -s; thus, knowledge of $\tau(s)$ determines the distribution.

With a positive stable distribution for γ, using equation (2.1) we have

$$\Pr ob\left(T_1 > t_1, ..., T_p > t_p\right) = E_\gamma \exp\left(\gamma \ln\left\{B_1(t_1) \cdots B_p(t_p)\right\}\right) = \exp\left(-\left\{-\ln B_1(t_1) - \cdots - \ln B_p(t_p)\right\}^\alpha\right).$$

Because $S_i(t_i) = \exp\left(-\left\{-\ln B_i(t_i)\right\}^\alpha\right)$, we can write the joint survival function as

$$\Pr ob\left(T_1 > t_1, ..., T_p > t_p\right) = \exp\left(-\left[\left\{-\ln S_1(t_1)\right\}^{1/\alpha} + \cdots + \left\{-\ln S_p(t_p)\right\}^{1/\alpha}\right]^\alpha\right), \tag{2.2}$$

a copula expression. In particular, for bivariate lifetimes with p=2, Hougaard proposed examining Weibull marginals so that $B_i(t) = \exp\left(-a_i t^{b_i}\right)$ and $S_i(t|\gamma) = \exp\left(-a_i \gamma t^{b_i}\right)$. This yields the bivariate survivor function

$$\Pr ob\left(T_1 > t_1, T_2 > t_2\right) = \exp\left(-\left[a_1 t_1^{b_1} + a_2 t_2^{b_2}\right]^\alpha\right). \tag{2.3}$$

This is desirable in the sense that both the conditional and marginal distributions are Weibull.

Equations (1.2) of Example 1.2 and (2.2) of Example 2.1 show that these frailty models can be written as copulas. Marshall and Olkin (1988) showed that these are special cases of a more general result;

they demonstrated that all frailty models of the form in equation (2.1) can be easily written as copulas. Further, the copula form is a special type called an Archimedean copula that we will introduce in Section 3.

In addition to the Clayton and Oakes studies, other works have investigated the use of copula models in studying behavior of multiple lives. Hougaard et al. (1992) analyzed the joint survival of Danish twins born between 1881 and 1930. They used the frailty model arising from a positive stable distribution as well as Cox's proportional hazard model. Frees et al. (1995) investigated mortality of annuitants in joint and last-survivor annuity contracts using Frank's copula (equation 1.3). They found that accounting for dependency in mortality produced approximately a three to five percent reduction in annuity values when compared to standard models that assume independence.

2.2 Competing Risks - Multiple Decrement Theory

The subject of competing risks deals with the study of the lifetime distribution of a system subject to several competing causes; this subject is called multiple decrement theory in actuarial science (see, for example, Bowers et al., 1997, Chapters 10 and 11). The problem of competing risks arises in survival analysis, systems reliability and medical studies as well as actuarial science. For example, a person dies because of one of several possible causes: cancer, heart disease, accident, and so on. As yet another example, a mechanical device fails because one of its components fails. For mathematical convenience, the general framework begins with an unobserved multivariate lifetime vector $(T_1, T_2, ..., T_p)$; each element in the vector denotes the lifetime due to one of "p" competing causes. The quantities typically observed are $T = \min(T_1, T_2, ..., T_p)$ and the cause of failure J. To illustrate, in life insurance, T usually denotes the lifetime of the insured individual and J denotes the cause of death such as cancer or accidental death. Several texts lay the foundation of the theory of competing risks. For example, see Bowers et al. (1997), Cox and Oakes (1984), David and Moeschberger (1978), and Elandt-Johnson and Johnson (1980).

When formulating the competing risk model, it is often assumed that the component lifetimes T_i are statistically independent. With independence, the model is easily tractable and avoids the problem of identifiability encountered in inference. However, many authors, practitioners, and academicians recognize that this assumption is not practical, realistic nor reasonable. See Carriere (1994), Makeham (1874) and Seal (1977).

To account for dependence in competing risk models, one general approach is to apply copulas. In particular, the frailty model seems well suited for handling competing risks. Assuming that causes of death

are independent given a frailty γ , we have

$$\Pr ob(T > t|\gamma) = \Pr ob\big(\min(T_1,\ldots,T_p) > t\big) = \Pr ob(T_1 > t|\gamma) \cdots \Pr ob(T_p > t|\gamma) = B_1(t)^\gamma \cdots B_p(t)^\gamma .$$

Thus, similar to equation (2.1), the overall survival function is

$$\Pr ob(T > t) = E_\gamma \left\{ B_1(t) \cdots B_p(t) \right\}^\gamma . \tag{2.4}$$

Example 2.1 - Continued

For a positive stable distribution for γ , the survival function is

$$\Pr ob(T > t) = \exp\left(- \left[\left\{- \ln S_1(t)\right\}^{1/\alpha} + \cdots + \left\{- \ln S_p(t)\right\}^{1/\alpha} \right]^\alpha \right),$$

similar to equation (2.2). For bivariate lifetimes with Weibull marginals, we have

$$\Pr ob(T > t) = \exp\left(-\left[a_1 t^{b_1} + a_2 t^{b_2}\right]^\alpha \right).$$

There have been several applications of frailty models for studying competing risk situations. Oakes (1989) discusses the number of cycles of two chemotherapy regimes tolerated by 109 cancer patients. Shih and Louis (1995) analyze HIV-infected patients using Clayton's family, positive stable frailties as well as Frank's copula. Zheng and Klein (1995) consider data from a clinical trial of patients with non-Hodgkin's lymphoma using gamma copula (as in Clayton's family). In a non-biological context, Hougaard (1987) describes how dependent competing risk models using positive stable copulas can be used to assess machine failure.

3. Properties of Copulas

This section discusses several properties and characteristics of copulas, specifically (i) how to generate copulas, (ii) how copulas can summarize association between random variables and (iii) how to simulate copula distributions.

3.1 Specifying Copulas: Archimedean and Compounding Approaches

As we saw in Section 1, copulas provide a general structure for modeling multivariate distributions. The two main methods for specifying a family of copulas are the Archimedean approach and the compounding approach, the latter being first illustrated in Example 1.2.

The Archimedean representation allows us to reduce the study of a multivariate copula to a single univariate function. For simplicity, we first consider bivariate copulas so that $p=2$. Assume that ϕ is a

convex, decreasing function with domain $(0,1]$ and range $[0,\infty)$ such that $\phi(1) = 0$. Use ϕ^{-1} for the inverse function of ϕ. Then the function

$$C_\phi(u,v) = \phi^{-1}\big(\phi(u) + \phi(v)\big) \text{ for } u,v \in (0,1]$$

is said to be an Archimedean copula. We call ϕ a *generator* of the copula C_ϕ. Genest and McKay (1986a,b) give proofs of several basic properties of C_ϕ, including the fact that it is a distribution function. As seen in Table 3.1, different choices of generator yield several important families of copulas.

Table 3.1. Archimedean Copulas and their Generators			
Family	Generator $\phi(t)$	Dependence Parameter (α) Space	Bivariate Copula $C_\phi(u,v)$
Independence	$-\ln t$	Not applicable	uv
Clayton (1978), Cook-Johnson (1981), Oakes(1982)	$t^{-\alpha} - 1$	$\alpha > 1$	$\left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-1/\alpha}$
Gumbel (1960), Hougaard (1986)	$(-\ln t)^\alpha$	$\alpha \geq 1$	$\exp\left\{-\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{1/\alpha}\right\}$
Frank (1979)	$\ln \dfrac{e^{\alpha t} - 1}{e^\alpha - 1}$	$-\infty < \alpha < \infty$	$\dfrac{1}{\alpha}\ln\left(1 + \dfrac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^\alpha - 1}\right)$

A generator uniquely determines (up to a scalar multiple) an Archimedean copula. Thus, this representation helps identify the copula form. This point will be further developed in Section 3.2.

Examples 1.2 and 2.1 showed that compound distributions can be used to generate copulas of interest. These examples are special cases of a general method for constructing copulas due to Marshall and Olkin (1988). To describe this method, suppose that X_i is a random variable whose conditional, given a positive latent variable γ_i, distribution function is specified by $H_i(x|\gamma_i) = H_i(x)^{\gamma_i}$, where $H_i(\cdot)$ is some baseline distribution function, for i=1, ..., p. Marshall and Olkin considered multivariate distribution functions of the form

$$F(x_1, x_2, ..., x_p) = E\, K\big(H_1(x_1)^{\gamma_1}, ..., H_p(x_p)^{\gamma_p}\big).$$

Here, K is a distribution function with uniform marginals and the expectation is over $\gamma_1, \gamma_2, ..., \gamma_p$. As a special case, take all latent variables equal to one another so that $\gamma_1 = \gamma_2 = ... = \gamma_p = \gamma$ and use the distribution function corresponding to independent marginals. Marshall and Olkin (1988) showed that

$$F(x_1, x_2, \ldots, x_p) = E_\gamma \left(H_1(x_1)^\gamma \cdots H_p(x_p)^\gamma \right) = \tau \left(\tau^{-1} \{ F_1(x_1) \} + \cdots + \tau^{-1} \{ F_p(x_p) \} \right) \qquad (3.2)$$

where F_i is the ith marginal distribution of F and $\tau(\cdot)$ is the Laplace transform of γ, defined by $\tau(s) = E_\gamma e^{-s\gamma}$.

It is known that Laplace transforms have well-defined inverses. Thus, from equation (3.2), we see that the inverse function τ^{-1} serves as the generator for an Archimedean copula. In this sense, equation (3.2) provides a probabilistic interpretation of generators. To illustrate, Table 3.2 provides the inverse Laplace transform for the generators listed in Table 3.1. Here, we see how well known distributions can be used to generate compound distributions. Because generators are defined uniquely only up to scalar multiple, any positive constant in the family of Laplace transforms determines the same class of generators. (Indeed, this methodology suggests new copula families.) Thus, the inverse of a Laplace transform represents an important type of generator for Archimedean copulas.

Table 3.2. Archimedean Generators and their Inverses

Family	Generator $\phi(t)$	Inverse Generator (Laplace Transform) $\tau(s) = \phi^{-1}(s)$	Laplace Transform Distribution
Independence	$-\ln t$	$\exp(-s)$	Degenerate
Clayton (1978), Cook-Johnson (1981), Oakes (1982)	$t^{-\alpha} - 1$	$(1+s)^{-1/\alpha}$	Gamma
Gumbel (1960), Hougaard (1986)	$(-\ln t)^\alpha$	$\exp(-s^{1/\alpha})$	Positive Stable
Frank (1979)	$\ln \dfrac{e^{\alpha t} - 1}{e^\alpha - 1}$	$\alpha^{-1} \ln \left[1 + e^s (e^\alpha - 1) \right]$	Logarithmic Series Distribution on the Positive Integers

To summarize, assume that $X_1, X_2, \ldots, X_p$ are conditionally, given γ, independent with distribution functions $H_i(x)^\gamma$. Then, the multivariate distribution is given by the copula form with the generator being the inverse of the Laplace transform of the latent variable γ. Because of the form of the conditional distribution, we follow Joe (1997) and call this a *mixture of powers* distribution. We remark that $\tau(-\ln H_i(x)) = E_\gamma \exp(-(-\ln H_i(x))\gamma) = F_i(x)$ so that $H_i(x) = \exp(-\tau^{-1}(F_i(x)))$. This provides a way of specifying the baseline function given the marginal distribution and the distribution of the latent variable.

For the applications involving lifetimes in Section 2, we found it natural to discuss distributions in terms of survival functions. Marshall and Olkin (1988) pointed out that the construction as in equation (3.2) could also be used for survivor functions. That is, with the frailty model $\Pr ob(T_i > t | \gamma) = B_i(t)^\gamma$ and conditional independence of $T_1, T_2, \ldots, T_p$, from equation (2.1) we have

$$\Pr ob(T_1 > t_1, \ldots, T_p > t_p) = E_\gamma \{B_1(t_1) \cdots B_p(t_p)\}^\gamma = \tau \left(\tau^{-1} \{S_1(x_1)\} + \cdots + \tau^{-1} \{S_p(x_p)\} \right) \tag{3.3}$$

As before, $\tau(-\ln B_i(t)) = S_i(t) = 1 - F_i(t)$ and $B_i(t) = \exp\left(-\tau^{-1}(S_i(t))\right)$. Assuming the mixing distribution remains the same, the Laplace transform and hence the generator remains the same. However, it is interesting to note that the two constructions yield different multivariate distributions because $\Pr ob(T \le t | \gamma) = 1 - B(t)^\gamma \ne \left(1 - F(t)\right)^\gamma$. We follow Marshall and Olkin and first present the copula construction in equation (3.2) using distribution functions because it is useful for all random variables. However, for positive lifetime random variables, the concept of frailty models is intuitively appealing; thus, using survival functions in the construction is preferred for these applications. We finally remark, unlike the gamma and positive stable families, that Frank's copula $C_\phi(u, v)$ is symmetric about the point (1/2, 1/2) (for example, $C_\phi(u, v) = C_\phi(1/2 - u, 1/2 - v)$). Thus, it is invariant to the choice of F or S = 1- F in the construction (Genest, 1987).

3.2 Measures of Association

Recall the copula representation of a distribution function in equation (1.1), $F(x_1, x_2, \ldots, x_p) = C\left(F_1(x_1), F_2(x_2), \ldots, F_p(x_p)\right)$. With this expression, we see that F is a function of its marginals and the copula. As pointed out by Genest and Rivest (1993), this suggests that a natural way of specifying the distribution function is to examine the copula and marginals separately. Moreover, because the case of independence is a special form of the copula $C(u_1, u_2, \ldots, u_p) = u_1 \cdot u_2 \cdots u_p$ (regardless of the marginals), this suggests that we examine the copula function to understand the association among random variables. Because we are concerned with correlation measures, we restrict our consideration to $p = 2$.

Schweizer and Wolff (1981) established that the copula accounts for all the dependence between two random variables X_1 and X_2 in the following sense. Consider g_1 and g_2, strictly increasing (but otherwise arbitrary) functions over the range of X_1 and X_2. Then, Schweizer and Wolff showed that the transformed variables $g_1(X_1)$ and $g_2(X_2)$ have the same copula as X_1 and X_2. Thus, the manner in

which X_1 and X_2 "move together" is captured by the copula, regardless of the scale in which each variable is measured.

Schweizer and Wolff also showed that two standard nonparametric correlation measures could be expressed solely in terms of the copula function. These are Spearman's correlation coefficient, defined by

$$\rho(X_1, X_2) = 12E\{(F_1(x_1) - 1/2)(F_2(x_2) - 1/2)\} = 12\int\int\{C(u, v) - uv\}dudv$$

and Kendall's correlation coefficient, defined by

$$\tau(X_1, X_2) = \text{Prob}\{(X_1 - X_1^*)(X_2 - X_2^*) > 0\} - \text{Prob}\{(X_1 - X_1^*)(X_2 - X_2^*) < 0\} = 4\int\int C(u, v)dC(u, v) - 1.$$

For these expressions, we assume that X_1 and X_2 have a jointly continuous distribution function. Further, the definition of Kendall's tau uses an independent copy of $(X_1, X_2), (X_1^*, X_2^*)$, in order to define the measure of "concordance." See Section 5 for more details. We also remark that the widely used Pearson correlation coefficient, $\text{Cov}(X_1, X_2) / (\text{Var}X_1 \text{Var}X_2)^{1/2}$, depends not only on the copula but also on the marginal distributions. Thus, this measure is affected by (nonlinear) changes of scale.

Table 3.3. Archimedean Copulas and their Measures of Dependence			
Family	Bivariate Copula $C_\phi(u,v)$	Kendall's tau	Spearman's rho
Independence	u v	0	0
Clayton (1978), Cook-Johnson (1981), Oakes (1982)	$\left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-1/\alpha}$	$\alpha / (\alpha + 2)$	Complicated form
Gumbel (1960), Hougaard (1986)	$\exp\left\{-\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{1/\alpha}\right\}$	$1 - \alpha^{-1}$	No closed form
Frank (1979)	$\dfrac{1}{\alpha}\ln\left(1 + \dfrac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^\alpha - 1}\right)$	$1 - \dfrac{4}{\alpha}\{D_1(-\alpha) - 1\}$	$1 - \dfrac{12}{\alpha}\{D_2(-\alpha) - D_1(-\alpha)\}$

Table 3.3 illustrates the calculation of these correlation measures. The correlations from Frank's copula rely on the so-called "Debye" functions, defined as

$$D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t - 1}dt,$$

for k=1,2. To evaluate negative arguments of the Debye function D_k, basic calculus shows that $D_k(-x) = D_k(x) + kx / (k + 1)$. An important point of this table is that there is a one to one relationship between each correlation measure and the association parameter α. Further, Table 3.3 allows us to see a

drawback of the Clayton/Cook-Johnson/Oakes and Gumbel-Hougaard copula families. Because of the limited dependence parameter space as shown in Table 3.1, these families permit only nonnegative correlations, a consequence of the latent variable model. However, Frank's family permits negative as well as positive dependence.

Correlation measures summarize information in the copula concerning the dependence, or association, between random variables. Following a procedure due to Genest and Rivest (1993), we can also use the dependence measure to specify a copula form in empirical applications, as follows.

Genest and Rivest's procedure for identifying a copula begins by assuming that we have available a random sample of bivariate observations, $(X_{11}, X_{21}), ..., (X_{1n}, X_{2n})$. Assume that the distribution function F has associated Archimedean copula C_ϕ; we wish to identify the form of ϕ. We work with an intermediate (unobserved) random variable $Z_i = F(X_{1i}, X_{2i})$ that has distribution function $K(z) = Prob(Z_i \leq z)$. Genest and Rivest showed that this distribution function is related to the generator of an Archimedean copula through the expression $K(z) = z - \phi(z) / \phi'(z)$. To identify ϕ, we:

1. Estimate Kendall's correlation coefficient using the usual (nonparametric or distribution-free) estimate

$$\tau_n = \binom{n}{2}^{-1} \sum_{i<j} \text{sign}\left[(X_{1i} - X_{1j})(X_{2i} - X_{2j})\right]$$

2. Construct a nonparametric estimate of K, as follows:

   a. First, define the pseudo-observations

   $Z_i = \left\{\text{number of } (X_{1j}, X_{2j}) \text{ such that } X_{1j} < X_{1i} \text{ and } X_{2j} < X_{2i}\right\} / (n-1)$ for i=1, ..., n.

   a. Second, construct the estimate of K as $K_n(z) = $ proportion of Z_i's $\leq z$.

3. Now construct a parametric estimate of K using the relationship $K_\phi(z) = z - \phi(z) / \phi'(z)$. For example, refer to Table 3.1 for various choices of ϕ and use the estimate τ_n to calculate an estimate of α, say α_n. Use α_n to estimate $\phi(x)$, say $\phi_n(x)$. Finally, use $\phi_n(x)$ to estimate $K_\phi(z)$, say $K_{\phi_n}(z)$.

Repeat Step 3 for several choices of ϕ. Then, compare each parametric estimate to the nonparametric estimate constructed in Step 2. Select the choice of ϕ so that the parametric estimate $K_{\phi_n}(z)$ most closely resembles the nonparametric estimate $K_n(z)$. Measuring "closeness" can be done by minimizing a distance such as $\int \left[K_{\phi_n}(z) - K_n(z)\right]^2 dK_n(z)$ or graphically. Graphical representations in-

clude (i) plots of $K_n(z)$ and $K_{\phi_n}(z)$ versus z, and (ii) the corresponding quantile plots. See Section 4 for an example.

3.3 Simulation

Actuaries routinely deal with complex nonlinear functions, such as present values, of random variables. Simulation is a widely used tool for summarizing the distribution of stochastic outcomes and for communicating the results of complex models. The copula construction allows us to simulate outcomes from a multivariate distribution easily.

There are two primary simulation strategies, corresponding to the Archimedean and compounding methods for constructing copulas. As we will see, each has relative advantages and disadvantages.

We begin with the Archimedean construction. Thus, our goal is to construct an algorithm to generate $X_1, X_2, ..., X_p$ having known distribution function $F(x_1, x_2, ..., x_p) = C\big(F_1(x_1), F_2(x_2), ..., F_p(x_p)\big)$, where the copula function is

$$C(u_1, u_2, ..., u_p) = \phi^{-1}\big(\phi(u_1) + ... + \phi(u_p)\big).$$

For this construction, Genest and Revest (1986b) and Genest (1987) introduced the idea of simulating the full distribution of $\big(X_1, X_2, ..., X_p\big)$ by recursively simulating the conditional distribution of X_k given $X_1, ..., X_{k-1}$, for k=2, ..., p. This idea was subsequently developed by Lee (1993) and is given as follows. For simplicity, we assume that the joint probability density function of $X_1, X_2, ..., X_p$ exists. Using the copula construction, the joint probability density function of $X_1, ..., X_k$ is

$$f_k(x_1, ..., x_k) = \frac{\partial^k}{\partial x_1 ... \partial x_k} \phi^{-1}\big(\phi(F_1(x_1)) + ... + \phi(F_k(x_k))\big)$$

$$= \phi^{-1(k)}\big(\phi(F_1(x_1)) + ... + \phi(F_k(x_k))\big) \prod_{j=1}^{k} \phi^{(1)}(F_j(x_j)) F_j^{(1)}(x_j).$$

Here, the superscript notation "(j)" means the jth mixed partial derivative. Thus, the conditional density of X_k given $X_1, ..., X_{k-1}$ is

$$f_k(x_k | x_1, ..., x_{k-1}) = \frac{f_k(x_1, ..., x_k)}{f_{k-1}(x_1, ..., x_{k-1})}$$

$$= \phi^{(1)}\big(F_k(x_k)\big) F^{(1)}(x_k) \frac{\phi^{-1(k-1)}\big(\phi(F_1(x_1)) + ... + \phi(F_k(x_k))\big)}{\phi^{-1(k-1)}\big(\phi(F_1(x_1)) + ... + \phi(F_{k-1}(x_{k-1}))\big)}.$$

Further, the conditional distribution function of X_k given $X_1, \ldots, X_{k-1}$ is

$$F_k(x_k | x_1, \ldots, x_{k-1}) = \int_{-\infty}^{x_k} f_k(x | x_1, \ldots, x_{k-1}) \, dx$$

$$= \frac{\phi^{-1(k-1)}(\phi(F_1(x_1)) + \ldots + \phi(F_k(x_k)))}{\phi^{-1(k-1)}(\phi(F_1(x_1)) + \ldots + \phi(F_{k-1}(x_{k-1})))} = \frac{\phi^{-1(k-1)}(c_{k-1} + \phi(F_k(x_k)))}{\phi^{-1(k-1)}(c_{k-1})},$$

where $c_k = \phi(F_1(x_1)) + \ldots + \phi(F_k(x_k))$. With this distribution function, we may now use the usual procedure of solving for the inverse distribution function and evaluating this at a uniform random number (that is, use $F^{-1}(U) = X_k$).

To summarize, the algorithm is (Lee, 1993):

Algorithm 3.1 Generating Multivariate Outcomes from an Archimedean Copula

1. Generate $U_1, U_2, \ldots, U_p$ independent uniform (0,1) random numbers.

2. Set $X_1 = F_1^{-1}(U_1)$ and $c_0 = 0$.

3. For k=2, ..., p, recursively calculate X_k as the solution of

$$U_k = F_k(X_k | x_1, \ldots, x_{k-1}) = \frac{\phi^{-1(k-1)}(c_{k-1} + \phi(F_k(x_k)))}{\phi^{-1(k-1)}(c_{k-1})}. \tag{3.4}$$

This algorithm was initially introduced in the context of Frank's copula for p=2. Here, pleasant calculations show that the algorithm reduces to:

Algorithm 3.2 Generating Bivariate Outcomes from Frank's Copula

1. Generate U_1, U_2 independent uniform (0,1) random numbers.

2. Set $X_1 = F_1^{-1}(U_1)$.

3. Calculate X_2 as the solution of

$$U_2 = e^{-\alpha U_1} \left(\frac{e^{-\alpha} - e^{-\alpha F_2(x_2)}}{e^{-\alpha F_2(x_2)} - 1} + 1 \right)^{-1}.$$

That is, calculate $X_2 = F_2^{-1}(U_{\cdot 2})$ where

$$U_{\cdot 2} = \frac{U_2 e^{-\alpha} - e^{-\alpha U_1}(1 - U_{\cdot 2})}{U_2 + e^{-\alpha U_1}(1 - U_{\cdot 2})}.$$

Genest (1987) gave this algorithm.

The algorithm can also be readily used to simulate distributions from Clayton's family. From Table 3.2, we have $\phi^{-1}(s) = (1+s)^{-1/\alpha}$ so that $\phi^{-1(1)}(s) = -\alpha^{-1}(1+s)^{-(1/\alpha)-1}$. This expression with p=2 and equation (3.4) show that

$$U_2 = \frac{-\alpha^{-1}\left(1 + F_2(X_2)^{-1/\alpha} - 1 + U_1^{-1/\alpha} - 1\right)^{-(1/\alpha)-1}}{-\alpha^{-1}\left(1 + U_1^{-1/\alpha} - 1\right)^{1(1/\alpha)-1}}.$$

That is, calculate $X_2 = F_2^{-1}(U_{*2})$ where

$$U_{*2} = \left(1 + U_1^{-1/\alpha}(U_{*2}^{-\alpha/(\alpha+1)} - 1)\right)^{-\alpha}.$$

For the Gumbel-Hougaard copula, determining X_2 in equation (3.4) requires an iterative solution. Although straightforward, this is computationally expensive in that many applications require large numbers of simulated values. This drawback leads us to introducing an alternative algorithm suggested by Marshall and Olkin (1988) for compound constructions of copulas.

To generate $X_1, X_2, ..., X_p$ having a mixture of powers distribution specified in equation (3.2), the algorithm is:

Algorithm 3.3 Generating Multivariate Outcomes from a Compound Copula

1. Generate a (latent) random variable γ having Laplace transform τ.

2. Independently of Step 1, generate $U_1, U_2, ..., U_p$ independent uniform (0,1) random numbers.

3. For k=1, ..., p, calculate $X_k = F_k^{-1}(U_{*k})$ where

$$U_{*k} = \tau\left(-\gamma^{-1}\ln U_K\right). \tag{3.5}$$

Recall that the marginal distribution function may be calculated from the baseline distribution function using $F_k(x) = \tau(-\ln H_k(x))$. To illustrate for the Gumbel-Hougaard copula, from equation (3.5), we have

$$U_{*k} = \exp\left(-\left(-\gamma^{-1}\ln U_K\right)^{-1/\alpha}\right).$$

The algorithm is straightforward for most copulas of interest that are generated by the compounding method. Like the conditional distribution approach, it can be easily implemented for more than two dimensions (p>2). It is computationally more straightforward than the conditional distribution approach. A disadvantage is that it requires the generation of an additional variable, γ. For bivariate applications, this means generating 50% more uniform random variates; this can be expensive in applications.

4. Insurance Company Loss and Expense Application

This section illustrates methods of fitting copulas to insurance company indemnity claims. The data consists of 1,500 general liability claims randomly chosen from late settlement lags and were provided by Insurance Services Office, Inc. Each claim consists of an indemnity payment (the loss, X_1) and an allocated loss adjustment expense (ALAE, X_2). Here, ALAE are types of insurance company expenses that are specifically attributable to the settlement of individual claims such as lawyer's fees and claims investigation expenses; see, for example, Hogg and Klugman (1984). Our objective is to describe the joint distribution of losses and expenses.

Estimation of the joint distribution of losses and expenses is complicated by the presence of censoring, a common feature of loss data (Hogg and Klugman, 1984). Specifically, in addition to loss and expense information, for each claim we have a record of the policy limit, the maximal claim amount. With the presence of the policy limit, the loss variable is censored because the amount of claim cannot exceed the stated policy limit. For some claims, the policy limit was unknown and, for these policies, we assumed there was no policy limit.

Table 4.1 summarizes the data. Here, we see that only 34 of 1,500 policies had claims that equaled the policy limit and thus are considered censored. However, the censored losses cannot be ignored; for example, the mean loss of censored claims is much higher than the corresponding mean for uncensored claims. Table 4.1 also shows that our sample consists of only claims with positive losses and expenses. Separate models would be required for claims with positive losses but no expenses, or zero losses but positive expenses.

Table 4.1. Summary Statistics of Losses and Expenses					
	ALAE	Loss	Policy Limit	Loss (Uncensored)	Loss (Censored)
Number	1,500	1,500	1,352	1,466	34
Mean	12,588	41,208	559,098	37,110	217,491
Median	5,471	12,000	500,000	11,048	100,000
Standard Deviation	28,146	102,748	418,649	92,513	258,205
Minimum	15	10	5,000	10	5,000
Maximum	501,863	2,173,595	7,500,000	2,173,595	1,000,000
25th Percentile	2,333	4,000	300,000	3,750	50,000
75th Percentile	12,577	35,000	1,000,000	32,000	300,000

Figure 4.1 is a scatter plot of Loss versus ALAE. The corresponding correlation coefficient is 0.41. This statistic, together with the plot, suggests a strong relationship between losses and expenses.
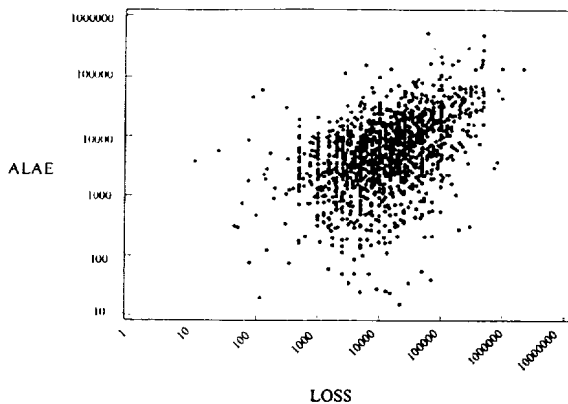
Figure 4.1 Plot of ALAE versus Loss. Both variables are on a logarithmic scale. Thirty-four Loss observations are censored. This plot demonstrates a strong relationship between ALAE and Loss.

## 4.1 Fitting Marginal Distributions

The initial step in our model fitting is to determine the appropriate marginals. Determining appropriate parametric distributions for univariate data is well described in Hogg and Klugman (1984). For this data, this step was examined in detailed earlier by Klugman and Parsa (1995). Thus, for simplicity, we present here only the fit of the univariate marginals using a Pareto distribution. With parameters λ and θ, the distribution function is

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x} \right)^{\theta}.$$

The quality of the fit of the marginal distributions can be examined with a graphical comparison of the fitted distribution function against their empirical versions, as displayed in Figures 4.2a and 4.2b. Because of censoring, we used the Kaplan-Meier empirical distribution function for the loss variable.
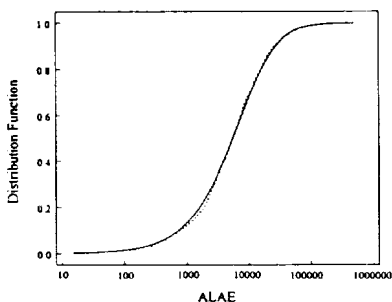
Figure 4.2a. Fitted Distribution Functions of ALAE. The dotted curve is the empirical distribution function. The smooth curve is a fit using the Pareto distribution.
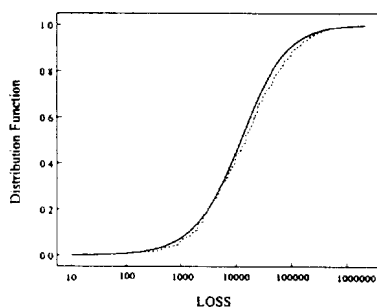
Figure 4.2b. Fitted Distribution Functions of Loss. The dotted curve is a Kaplan-Meier empirical distribution function. The smooth curve is a fit using the Pareto distribution.

4.2 Fitting a Copula to the Bivariate Data

To fit the copula, we first identify the form of the copula in Subsection 4.2.1 and then estimate it using maximum likelihood in Subsection 4.2.2.

4.2.1 Identifying a Copula

We use the procedure developed by Genest and Rivest (1993) for identifying an appropriate copula, outlined in Section 3.2. According to the procedure, we examine the degree of closeness of the parametric and nonparametric versions of the distribution function $K(z)$. This procedure is based on estimates of $K(z)$, the distribution function of pseudo-observations $Z=F(X_1, X_2)$. The idea behind the procedure is to compare a nonparametric estimate of $K(z)$ to those based on a specific form of the copula. To compare the two estimates of $K(z)$, we examine quantiles from each estimated distribution. Scatter plots of the two sets of quantiles are widely known in statistics as quantile-quantile, or q-q, plots. For identification purposes, we ignore the mild censoring in the loss variable although we do accommodate it in the more formal maximum likelihood fitting in Section 4.2.2.

We now discuss identification for three widely used copulas, namely the Gumbel-Hougaard, Frank and Cook-Johnson copulas. The form of the copula for each of these families appears in Table 3.3. The comparison of the resulting q-q plots is displayed in Figure 4.3. Because of the close agreement between nonparametric and parametric quantiles, the procedure suggests the use of the Gumbel-Hougaard copula. The quantiles based on Frank's copula are also close to the nonparametric quantiles although there is greater disparity at the higher quantiles, corresponding to high losses and expenses. We therefore identify both Frank's and Gumbel-Hougaard's as copulas that we will fit more formally in Subsection 4.2.2.
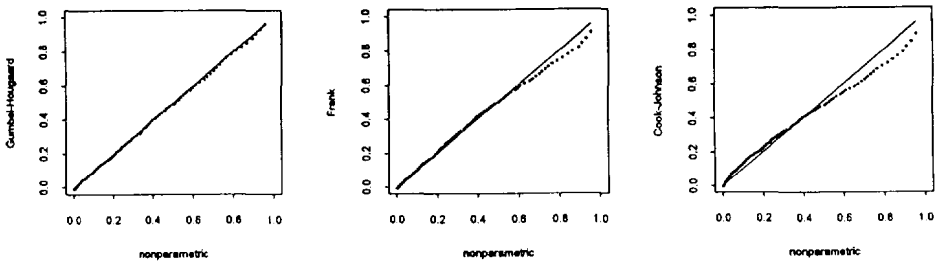
Figure 4.3 Quantile-Quantile (Q-Q) Plots, corresponding to the parametric and nonparametric distribution estimates of the pseudo-observations defined in Section 3.2. The dotted lines correspond to the quantiles of nonparametric and parametric estimates of the Archimedean generator ϕ. The smoothed lines correspond to the case where the quantiles are equal.

4.2.2 Fitting a Copula Using Maximum Likelihood

Recall that our data consist of losses (X_1) and expenses (X_2) and that we also have available an indicator for censoring (δ), so that $\delta = 1$ indicates the claim is censored. Parameters were estimated using maximum likelihood procedures that were programmed using the SAS procedure IML. In the development of the likelihood equation, we use the following partial derivatives: $F_1(x_1,x_2) = \partial F(x_1,x_2)/\partial x_1$, $F_2(x_1,x_2) = \partial F(x_1,x_2)/\partial x_2$, and $f(x_1,x_2) = \partial^2 F(x_1,x_2)/\partial x_1 \partial x_2$. Similarly, the first partial derivatives for the copula will be denoted by C_1 and C_2; the second mixed partial derivative by C_{12}.

Using a one-parameter copula and Pareto marginals, we have a total of five parameters to estimate; two each for the marginals and one for the dependence parameter. To develop the likelihood, we distinguish between the censored and uncensored cases. If the loss variable is not censored, then $\delta = 0$ and the contribution to the likelihood function is

$$f(x_1,x_2) = f_1(x_1)f_2(x_2)C_{12}\big(F_1(x_1),F_2(x_2)\big). \tag{4.1}$$

If the loss variable is censored, then $\delta = 1$ and the joint probability is given by $\text{Prob}\big(X_1 \geq x_1, X_2 \leq x_2\big)$ $= F_2(x_2) - F(x_1,x_2)$. Thus, the contribution to the likelihood when the observation is censored is

$$f_2(x_2) - F_2(x_1,x_2) = f_2(x_2)\big[1 - C_2\big(F_1(x_1),F_2(x_2)\big)\big]. \tag{4.2}$$

Combining equations (4.1) and (4.2), for a single observation, the logarithm of the likelihood function is

$$\log L(x_1,x_2,\delta) = (1-\delta)\log f(x_1,x_2) + \delta\big\{\log f_2(x_2) + \log\big[1 - C_2\big(F_1(x_1),F_2(x_2)\big)\big]\big\}. \tag{4.3}$$

The parameter estimates are then determined by maximizing the likelihood for the entire data set:

$$\sum_{i=1}^{n}\log L\big(x_{1i},x_{2i},\delta_i\big).$$

Results of the maximum likelihood estimation fitting the Gumbel-Hougaard copula, whose partial derivatives are derived in Appendix A, are summarized in Table 4.2. Here, we see that the parameter estimates of the marginal distributions are largely unchanged when we compare the univariate to the bivariate estimation. Standard errors are smaller in the bivariate fit, indicating greater precision of the parameter estimates. The estimate of the dependence parameter is significantly different from one; the estimate of α is approximately thirteen standard errors above one. This provides strong statistical evidence that losses and expenses are not independent. Using Table 3.3 for Gumbel-Hougaard copula, we can convert the dependence parameter into a more familiar measure of association. Thus, the parameter value of $\hat{\alpha} = 1.453$ corresponds to an approximate Spearman's correlation measure of 31%. A 95% confidence interval for α is therefore given by $\hat{\alpha} \pm 1.96 * se(\hat{\alpha}) = 1.453 \pm 1.96(0.034) = (1.386, 1.520)$. This corresponds to a 95% con-

fidence interval of (28%, 34%) for the Spearman's correlation.

Table 4.2. Bivariate Data Parameter Estimates
Using Gumbel-Hougaard's Copula with Pareto Marginals

	Parameter	Bivariate Distribution		Univariate Distribution	
		Estimate	Standard Error	Estimate	Standard Error
Loss (X_1)	λ_1	14,036	1,298	14,453	1,397
	θ_1	1.122	0.062	1.135	0.066
ALAE (X_2)	λ_2	14,219	1,426	15,133	1,633
	θ_2	2.118	0.153	2.223	0.175
Dependence	α	1.453	0.034	Not Applicable	Not Applicable

Results of the maximum likelihood estimation fitting the Frank copula are summarized in Table 4.3. Here, the behavior of parameter estimates and standard errors is similar to the fit using the Gumbel-Hougaard copula. For our parameterization of Frank's copula, the case of independence corresponds to $\alpha = 0$. In our case, the parameter estimate of $\hat{\alpha} = -3.158$ corresponds to an approximate Spearman's correlation measure of 32%, which is close to the estimate using the Gumbel-Hougaard copula.

Table 4.3. Bivariate Data Parameter Estimates
Using Frank's Copula with Pareto Marginals

	Parameter	Bivariate Distribution		Univariate Distribution	
		Estimate	Standard Error	Estimate	Standard Error
Loss (X_1)	λ_1	14,558	1,390	14,453	1,397
	θ_1	1.115	0.065	1.135	0.066
ALAE (X_2)	λ_2	16,678	1,824	15,133	1,633
	θ_2	2.309	0.187	2.223	0.175
Dependence	α	-3.158	0.174	Not Applicable	Not Applicable

It is difficult to compare the fit of the two copulas directly because they are non-nested models. However, we did compute Akaike's Information Criteria (AIC) for each model, given by AIC = (-2 ln (maximized likelihood) + 2(5))/1500. The results are 15.02 and 15.06, for the Gumbel-Hougaard and Frank copula models, respectively. The smaller AIC value for the Gumbel-Hougaard model indicates that this model is preferred.

4.3 Uses of the Bivariate Fit

This section describes two applications using the estimated bivariate distribution of losses and expenses.

4.3.1 Calculating Reinsurance Premiums

After having identified the joint distribution of (X_1, X_2), we can examine the distribution of any known function of X_1 and X_2 say $g(X_1, X_2)$. To illustrate, consider a reinsurer's expected payment on a policy with limit L and insurer's retention R. Then, assuming a pro-rata sharing of expenses, we have

$$g(X_1, X_2) = \begin{cases} 0 & \text{if } X_1 < R \\ X_1 - R + \dfrac{X_1 - R}{X_1} X_2 & \text{if } R \le X_1 < L \\ L - R + \dfrac{L - R}{L} X_2 & \text{if } X_1 \ge L \end{cases}.$$

The expected payment $E\, g(X_1, X_2)$ could be calculated using numerical integration when the joint density of losses and expenses is available. However, simulation is a simpler, numerical evaluation tool. The procedure for simulation is described in Section 3.3.

The idea with simulation is to generate a sequence of bivariate data (x_{1i}, x_{2i}) from the bivariate distribution model. The procedure for the Gumbel-Hougaard copula is summarized in algorithm 3.3, using equation (3.6) for equation (3.5). In the procedure, the inverse of the marginal distribution functions is needed. In the Pareto case, it is not difficult to verify that $F^{-1}(x) = \lambda\big((1-x)^{-1/\theta} - 1\big)$.

The simulation steps outlined in algorithm 3.3, using Gumbel-Hougaard copula, are repeated a large number of times. Let nsim be the number of simulations performed so that we generated the sequence of sample (x_{1i}, x_{2i}), i=1,...,nsim. Thus, the estimated value for the reinsurer's expected payment is given by

$$\hat{g}^*(L, R) = \frac{1}{nsim} \sum_{i=1}^{nsim} g(x_{1i}, x_{2i}),$$

with standard error

$$se\big(\hat{g}^*(L, R)\big) = \sqrt{\frac{\dfrac{1}{nsim} \sum_{i=1}^{nsim} g(x_{1i}, x_{2i})^2 - \hat{g}^*(L, R)^2}{nsim}}.$$

We performed a simulation study of size nsim=100,000; the results are summarized in Table 4.4.

	Table 4.4. Simulation-Based Reinsurance Premiums. Simulation Standard Errors are in Parenthesis.				
	Ratio of Insurer's Retention to Policy Limit (R/L)				
Policy Limit (L)	0.00	0.25	0.50	0.75	0.95
10,000	15,636 (640)	11,232 (480)	7,220 (320)	3,498 (160)	684 (32)
100,000	34,264 (655)	17,965 (493)	10,003 (328)	4,425 (164)	819 (33)
500,000	49,367 (733)	17,457 (544)	9,234 (359)	4,007 (179)	739 (36)
1,000,000	55,683 (818)	16,762 (597)	8,740 (390)	3,716 (193)	672 (38)

The results in Table 4.4 provide the adjusted premiums the reinsurer would have assessed to cover costs of losses and expenses according to various policy limits and ratios of insurer's retention to policy limit. On a crude basis, these results appear to make sense. For example, from the summary statistics in Table 4.1, the average policy limit is 559,098 with an average of losses plus expenses of 53,796. Without any reinsurance, our results indicate an adjusted premium of 49,367, with a standard error of 733, for a policy limit of 500,000. Furthermore, the results are intuitively appealing because as expected we observe (i) higher premium for larger policy limits, and (ii) lower premium when the ratio R/L is higher, i.e. insurer retains larger amount of losses.

	Table 4.5. Simulation-Based Ratios of Dependence to Independence Reinsurance Premiums				
	Ratio of Insurer's Retention to Policy Limit (R/L)				
Policy Limit (L)	0.00	0.25	0.50	0.75	0.95
10,000	0.80	0.95	1.02	1.07	1.10
100,000	0.89	1.24	1.36	1.44	1.50
500,000	0.92	1.31	1.40	1.47	1.52
1,000,000	0.93	1.31	1.39	1.47	1.53

Because it is common in practice to assume independence, we provide Table 4.5 which gives ratios of dependence to independence reinsurance premiums. The dependence assumption is based on the Gumbel-Hougaard estimation results while the independence assumption is based on the estimation results when the joint distribution is assumed to be the product of the marginals. In previous sections, we argue that the estimation results using dependence is statistically significant. The table above shows that substantial mispricing could result if the unrealistic assumption of independence between losses and expenses is made. A ratio below 1.0 from the table suggests an overvalued reinsurance premium; a ratio above 1.0 suggests undervalued premiums. According to the table, undervalued premiums result from higher retention by the reinsured. This undervaluation is more important for higher policy limits. The greatest overvalued premiums are for large retention levels and policy limits. This is intuitively plausible; pricing in the tail of the

distribution is very sensitive to model misspecification.

4.3.2 Estimating Regression Functions

As described in Section 1, the regression function is the most widely used tool for describing multi-variate relationships. To illustrate, in the context of losses and claims we will examine situations where it is useful to estimate the expected amount of expenses for a given level of loss. Copulas can help us understand the full joint distribution and thus be used to address some important applications described in Sections 2, 4.3.1, and 5. We can also use copulas to define regression functions, as follows.

To be specific, let is assume an Archimedean form of the copula as in Section 3.3 so that the conditional distribution of X_k given $X_1, \ldots, X_{k-1}$ is

$$F_k\left(x_k | x_1, \ldots, x_{k-1}\right) = \frac{\phi^{-1(k-1)}\left(c_{k-1} + \phi(F_k(x_k))\right)}{\phi^{-1(k-1)}(c_{k-1})},$$

where $c_k = \phi\{F_1(x_1)\} + \ldots + \phi\{F_k(x_k)\}$. Basic results from mathematical statistics show that the regression function can be expressed as

$$E(X_k | x_1, \ldots, x_{k-1}) = \int_0^\infty (1 - F_k(x | x_1, \ldots, x_{k-1})) \, dx + \int_{-\infty}^0 F_k(x | x_1, \ldots, x_{k-1}) \, dx.$$

In certain situations, this expression is convenient to evaluate. For example, assuming k=2 and using uniform marginals and Frank's copula, Genest (1987) gave the regression function

$$E\left(X_2 | X_1 = x\right) = \frac{(1 - e^{-\alpha}) x e^{-\alpha x} + e^{-\alpha}(e^{-\alpha x} - 1)}{(e^{-\alpha} - 1)(e^{-\alpha} - e^{-\alpha x})}.$$

In general, however, the calculation of the regression function is tedious. As an alternative, copulas are well-suited to the concept of "quantile" regression. Here, in lieu of examining the mean of a conditional distribution, one looks at the median or some other percentile (quantile) of the distribution. Specifically, define the pth quantile to be the solution x_p of the equation:

$$p = F_k(x_p | x_1, \ldots, x_{k-1}).$$

For the case k=2, we have

$$p = F_2(x_p | X_1 = x_1) = C_1\left(F_1(x_1), F_2(x_p)\right). \tag{4.5}$$

The first partial derivative for the case of the Gumbel-Hougaard copula is derived in Appendix A. For the case of Frank copula, the first partial derivative is given by

$$C_1(u, v) = \frac{e^{\alpha u}(e^{\alpha v} - 1)}{e^\alpha - 1 + (e^{\alpha u} - 1)(e^{\alpha v} - 1)}. \tag{4.6}$$

For a complete derivation of partial derivatives of the Frank's copula, see also Frees, et al. (1996).

32

Thus, for a specified proportion p and amount of loss x_1, we can find the percentile of the corresponding expenses by solving equation (4.5). For the case of the Gumbel-Hougaard copula, we use equation (A.2) to get the pth percentile of the expense level which is given by

$$x_p = F_2^{-1}(v_p),$$

where v_p is the solution to the following equation:

$$\left(\frac{\ln F_1(x_1)}{\ln C(F_1(x_1), v_p)}\right)^{\alpha-1} \frac{C(F_1(x_1), v_p)}{F_1(x_1)} = p.$$

For various percentiles, Figure 4.4 graphically displays the result of regression curves that provide estimates of expenses conditional on the amount of loss using the Gumbel-Hougaard copula. We superimposed these regression curves on the scatter plot of losses and expenses. This plot allows a manager to estimate expenses for a pre-specified loss amount. By providing several percentiles, the manager may choose the degree of conservatism that is appropriate for the business decision at hand.
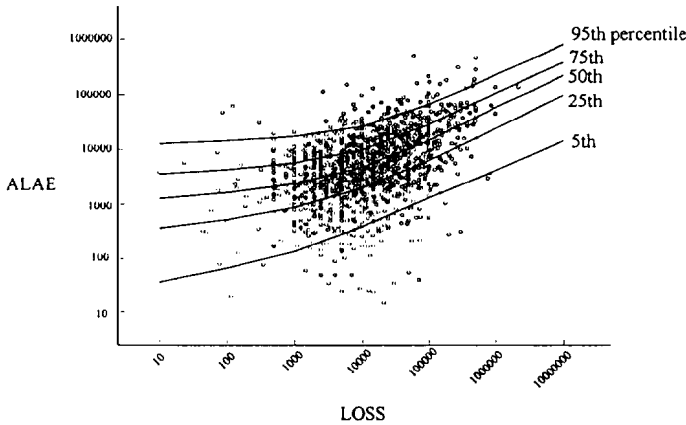
Figure 4.4 Scatter Plot of ALAE versus LOSS with quantile regression curves superimposed.

5. Additional Applications of Copulas

We now discuss three different subject areas useful to the actuary where copulas have been applied: stochastic ordering, fuzzy logic, and insurance pricing. Stochastic ordering refers to relationships among distribution functions of random variables. Fuzzy logic is an approach for dealing with uncertainty, analogous to probability theory. All actuaries are familiar with insurance pricing which involves premium calculation.

33

5.1 Stochastic Ordering

In actuarial science and the economics of decision making, the comparison of the attractiveness of various risks is usually of interest; therefore the subject of stochastic orderings is of prime importance. To illustrate, we say the random variable X_1 *stochastically dominates* the random variable X_2 with respect to a class of functions, say U, if for any function $u \in U$, we have

$$Eu(X_1) \geq Eu(X_2).$$

Often, u is the utility of wealth function that defines risk preferences of decision makers. As special cases, we consider the well-known classes of First (FSD) and Second (SSD) Stochastic Dominance. In FSD, the class U is taken to be the class of increasing utility functions. In SSD, U is the smaller class of increasing, concave utility functions, applicable to risk-averse decision makers. See Kaas, et. al. (1994), Heilman and Schroter (1991), and Gooavaerts, et. al. (1982).

Copulas are used in the analysis for the demand of insurance coverage studied by Tibiletti (1995). The problem is to find the optimal insurance coverage for a decision maker where a proportion of the wealth is uninsurable. Suppose that X_1 is the amount of uninsurable asset and X_2 is the insurable loss with $0 \leq x_2 \leq m$, i.e. m is the maximum value of the insurable asset. In a two-period model, define the final wealth to be

$$Z = X_1 + m - X_2 + \delta(X_2 - p). \tag{5.1}$$

Here, δ is the coinsurance rate and p is the premium rate. In this insurance set-up, the questions typically explored are: (a) what coinsurance rate δ maximizes E(Z), and (b) how this optimal coverage is affected whenever certain changes, called beneficial changes, increase expected utility. Tibiletti explored beneficial changes such as (a) a shift in the distribution of either X_1 or X_2, (b) a change in the dependence between X_1 and X_2, and (c) a change in both (a) and (b). Because dependence comes into play, copulas provide a natural tool in this situation. Most of the theorems described and proved use conditions that involve a certain dependence ordering called *more concordance*. If (X_1, X_2) and (Y_1, Y_2) are pairs with associated copulas C_X and C_Y, then the pair (X_1, X_2) is said to be *more concordant* than (Y_1, Y_2) if $C_X(a, b) \geq C_Y(a, b)$ for all $(a, b) \in [0,1]^2$.

The *more concordant* ordering is just one of several types of dependence ordering used to order multivariate distributions. See Tchen (1980), Kimeldorf and Sampson (1987, 1989), and Metry and Sampson (1991) for alternative orderings. For example, another type is the so called *more regression dependent* ordering. Here, the random vector (X_1, X_2) is said to be *more regression dependent* than

(Y_1, Y_2) if for any $x_1' > x_1$, we have $\dfrac{\Pr\left(X_2 \le x_2 | X_1 = x_1'\right)}{\Pr\left(Y_2 \le y_2 | Y_1 = x_1'\right)} \ge 1$ whenever we have $\dfrac{\Pr\left(X_2 \le x_2 | X_1 = x_1\right)}{\Pr\left(Y_2 \le y_2 | Y_1 = x_1\right)} \ge 1$.

See Bilodeau (1989). Dependence ordering is particularly useful for determining the range of possible dependence in multivariate random variables. Other areas where stochastic orderings are useful include probability theory, reliability, and operations research, see Mosler and Scarsini (1991).

5.2 Fuzzy Sets

Like probability theory, fuzzy set theory deals with uncertainty. To start, let U be a non-empty set whose subsets are of primary interest; in fuzzy set theory, it is called the universe of discourse. A fuzzy set is defined to be a pair (E, μ_E) where $E \subset U$ and $\mu_E : U \to [0,1]$. For any x in U, $\mu_E(x)$ denotes the degree to which x belongs to the fuzzy set, almost like a probability.

Operations, such as unions and intersections, on fuzzy sets are common. To illustrate, if we have two fuzzy sets (A, μ_A) and (B, μ_B), then their union is the fuzzy set (C, μ_C) where $\mu_C(x) = \max\left[\mu_A(x), \mu_B(x)\right]$ and their intersection is the fuzzy set (D, μ_D) with $\mu_D(x) = \min\left[\mu_A(x), \mu_B(x)\right]$. Other operations on fuzzy sets are performed using triangular norms. Triangular norms were considered originally in the context of probabilistic metric spaces and provide the link to copulas.

Following Ostazewski (1993) and Klement (1982a, 1982b), a triangular norm, t-norm for short, T is a mapping $T:[0,1]^2 \to [0,1]$ with the following properties: (1) boundary condition: $T(x,1) = x$; (2) monotonicity: $T(x,y) \le T(u,v)$ whenever $x \le u, y \le v$; (3) commutativity: $T(x,y) = T(y,x)$; and (4) associativity: $T\left(T(x,y),z\right) = T\left(x, T(y,z)\right)$. In Schweizer and Sklar (1983), several results are given that relate triangular norms and copulas. Many copulas can serve as triangular norms as the following examples illustrate.

Example 5.1 **Frechet Bounds** $T(x,y) = \min(x,y)$ and $T(x,y) = \max(x + y - 1, 0)$.

Example 5.2 **Independence** $T(x,y) = xy$.

Example 5.3 **Frank** $T(x,y) = \ln\left[1 + \dfrac{(\eta^x - 1)(\eta^y - 1)}{\eta - 1}\right] / \ln \eta$ where $\eta > 0, \eta \ne 1$.

Therefore, by considering familiar copulas, as well as newly constructed copulas, we can define new operations that can be performed on fuzzy sets. Fuzzy set theory has been applied in specific areas of interest to many actuaries such as risk economics, interest theory, and underwriting/classification of risks.

See Ostaszewski (1993), LeMaire (1990), and Young (1993) for descriptions of these applications.

5.3 Insurance Pricing

The calculation of a premium to be assigned to an insurance risk X is a fundamental job of the actuary. Recently, Wang (1996) developed a principle that exhibits several desirable properties of a premium principle. Wang (1997) uses the concept of a distortion function and extended this, using copulas, to a portfolio of dependent risks.

Let X be an insurance risk with survival function $S(x) = Pr(X > x)$. A class of premium principles can be defined by $\pi(X) = \int_0^\infty g[S(z)]\,dz$, where g, called the distortion function, is increasing with g(0)=0 and g(1)=1. In the special case of the function $g(t) = t^{1/\eta}$ with $\eta \geq 0$, we have the class of *proportional hazard* (PH) *transforms*.

For the bivariate case, consider a random vector (X_1, X_2) with distribution function $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$, then the function $\tilde{F}(x_1, x_2) = g[F(x_1, x_2)]$ is another joint distribution function with marginals $g(F_1)$ and $g(F_2)$. The associated copula is therefore $\tilde{C}(u_1, u_2) = g[C(g^{-1}(u_1), g^{-1}(u_2))]$. Some illustrations of distortion functions follow.

Example 5.4 **Power Distortion Function** Here, we have $g(t) = t^{1/\eta}$, where $\eta \geq 0$. The independence structure is preserved under this distortion function. To see this, note that if $C(u_1, u_2) = u_1 u_2$, then $\tilde{C}(u_1, u_2) = g[C(u_1^\eta, u_2^\eta)] = u_1 u_2$.

Example 5.5 **Exponential Distortion Function** Here, we have $g(t) = \dfrac{1 - e^{-\eta t}}{1 - e^{-\eta}}$, where $\eta > 0$. Again, beginning with the independence copula $C(u_1, u_2) = u_1 u_2$, it is straightforward to show that we can generate Frank's family of copulas $\tilde{C}(u_1, u_2) = \ln\left[1 + \dfrac{(\eta^{u_1} - 1)(\eta^{u_2} - 1)}{\eta - 1}\right] / \ln \eta$.

Using the concept of copulas, Wang (1997) extends the PH-measure to a portfolio of risks. In many situations, for a portfolio of risks, the proposed measure is useful because individual risks are considered dependent. Wang justified dependence of individual risks by stating that they are "influenced by the same underlying market environment."

6. Summary and Conclusions

In analyzing the impact of future contingent events, actuaries are faced with problems involving multivariate outcomes. In this paper, we reviewed the problems of (1) estimating distributions of joint life-times of paired individuals, useful in the analysis of survivorship insurance protection, and (2) investigating mortality experience, for the actuary who needs to distinguish among causes of death. We introduced, and provided a solution for, the problem of dependence between an insurer's losses and expenses. Failures of ignoring dependencies may lead to mispricing. Thus, it is important for actuaries to be able to adequately model multivariate outcomes.

The tool used to study multivariate outcomes is the copula function; it couples univariate marginals to the full multivariate distribution. The biological frailty models and the mathematical Archimedean models can motivate copulas. A statistical mixture of powers model serves as a bridge between these two sets of families.

Because copulas are parametric families, standard techniques such as maximum likelihood can be used for estimation. Other statistical tools have been recently developed to help fit copulas. We described a graphical tool to identify the form of the copula. We discussed how copulas could be used to simulate multivariate outcomes, an important tool for actuaries. We also developed the connection between copulas and the regression function, a widely used tool for summarizing what we expect based on conditional distributions.

This article has focused on the connection between copulas and statistics, the theory of data. Yet, much of the development of copulas has historically arisen from probability theory. To recognize this connection, we briefly reviewed topics in applied probability theory pertaining to copulas that are of the greatest interest to actuaries: stochastic ordering, fuzzy set theory and insurance pricing.

Our knowledge of copulas has been rapidly developing recently. Many of the articles cited in this review paper were written in the 1990's. In another recent survey paper, Kotz (1997) cites three conferences in the last five years that were largely devoted to copulas. Copulas offer analysts an intuitively appealing structure, first for investigating univariate distributions and second for specifying a dependence structure. Copulas offer a flexible structure that can be applied in many situations. We hope that this article encourages actuaries to seek new applications for this promising tool.

Acknowledgements

37

Appendix A
Gumbel-Hougaard Copula and its Partial Derivatives

In this appendix, we derive the formulas needed to evaluate the likelihood in equation (4.3) in the case where we apply the Gumbel-Hougaard copula as given in Table 3.1. First, we re-express the Gumbel-Hougaard copula as follows:

$$\left[-\ln C(u,v)\right]^{\alpha} = (-\ln u)^{\alpha} + (-\ln v)^{\alpha}.$$

For simplicity, we denote $C(u,v)$ by C. Now, take the partial derivative with respect to u of both sides of the equation to get:

$$\frac{(-\ln C)^{\alpha-1}}{C}\frac{\partial C}{\partial u} = \frac{(-\ln u)^{\alpha-1}}{u}. \tag{A.1}$$

Solving for $\partial C/\partial u$, we have:

$$\frac{\partial C}{\partial u} = \left(\frac{\ln u}{\ln C}\right)^{\alpha-1}\frac{C}{u}. \tag{A.2}$$

Applying symmetry, we get:

$$\frac{\partial C}{\partial v} = \left(\frac{\ln v}{\ln C}\right)^{\alpha-1}\frac{C}{v}. \tag{A.3}$$

From equation (A.1), we can take the partial derivative with respect to v to get:

$$\frac{(-\ln C)^{\alpha-1}}{C}\frac{\partial^2 C}{\partial u \partial v} - \frac{\partial C}{\partial u}\frac{\partial C}{\partial v}\frac{(-\ln C)^{\alpha-1}}{C^2}\left[(\alpha-1)(-\ln C)^{-1} + 1\right] = 0.$$

Rearranging terms yields the second partial derivative of the Gumbel-Hougaard copula:

$$\frac{\partial^2 C}{\partial u \partial v} = \frac{1}{C}\frac{\partial C}{\partial u}\frac{\partial C}{\partial v}\left[(\alpha-1)(-\ln C)^{-1} + 1\right]. \tag{A.4}$$

Equations (A.2), (A.3), and (A.4) are used to evaluate the log-likelihood given in equation (4.3).

Family	General Form of Copula	Parameter Constraint	Kendall's Tau	Spearman's Rho
Ali-Mikhail-Haq	$uv\left[1 - \alpha(1-u)(1-v)\right]^{-1}$	$-1 \leq \alpha \leq 1$	$\left(\dfrac{3\alpha - 2}{\alpha}\right) - \dfrac{2}{3}\left(1 - \dfrac{1}{\alpha}\right)^2 \ln(1-\alpha)$	Complicated form
Cook-Johnson	$\left[u^{-\alpha} + v^{-\alpha} - 1\right]^{-1/\alpha}$	$\alpha \geq 0$	$\dfrac{\alpha}{\alpha + 2}$	Complicated form
Farlie-Gumbel-Morgenstern	$uv\left[1 + \alpha(1-u)(1-v)\right]$	$-1 \leq \alpha \leq 1$	$\dfrac{2}{9}\alpha$	$\dfrac{1}{3}\alpha$
Frank	$\dfrac{1}{\alpha}\ln\left[1 + \dfrac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{(e^{\alpha} - 1)}\right]$	$\alpha \neq 0$	$1 - \dfrac{4}{\alpha}\left[D_1(-\alpha) - 1\right]$	$1 - \dfrac{12}{\alpha}\left[D_2(-\alpha) - D_1(-\alpha)\right]$
Gumbel-Hougaard	$\exp\left\{-\left[(-\ln u)^{\alpha} + (-\ln v)^{\alpha}\right]^{1/\alpha}\right\}$	$\alpha \geq 1$	$1 - \alpha^{-1}$	No closed form
Normal (Bivariate)	$H\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$ where H is the bivariate normal distribution function with correlation coefficient α and Φ^{-1} is the inverse of a univariate normal distribution function.	$-1 \leq \alpha \leq 1$	$\dfrac{2}{\pi}\arcsin(\alpha)$	$\dfrac{6}{\pi}\arcsin(\alpha/2)$
Plackett	$\dfrac{1}{2}(\alpha - 1)^{-1}\Big\{1 + (\alpha - 1)(u + v) - \left[(1 + (\alpha - 1)(u + v))^2 + 4\alpha(1-\alpha)\right]^{1/2}\Big\}$	$\alpha \geq 0$	No closed form	$\dfrac{(\alpha + 1)}{(\alpha - 1)} - \dfrac{2\alpha \ln \alpha}{(\alpha - 1)^2}$

ANNOTATED BIBLIOGRAPHY [*]

ANDERSON, T.W. *An Introduction to Multivariate Statistical Analysis* New York: John Wiley, 1958.

BILODEAU, M. "On the Monotone Regression Dependence for Archimedean Bivariate Uniform," *Communications in Statistics - Theory & Methods A* 18 (1989): 981-988.
> This paper provides necessary and sufficient conditions for Archimedean copulas to exhibit "monotone regression dependence" property. This property is desired for testing independence because of the resulting monotone power function. Frank's family and Cook and Johnson's family of copulas are shown to possess the property.

BOWERS, N., GERBER, H., HICKMAN, J., JONES, D. AND NESBITT, C. *Actuarial Mathematics* Schaumburg, Ill.: Society of Actuaries, 1997.

CARRIERE, J. "Dependent Decrement Theory," *Transactions of the Society of Actuaries* Vol. XLVI (1994): 45-74.
> Copulas are used to model dependence in a multiple decrement framework. The problem of identifiability was briefly discussed. As an application, the paper uses US population data to investigate the effect of removing heart/cerebrovascular diseases as a cause of death when assumed to be dependent with other causes.

CARRIERE, J. "A Large Sample Test for One-Parameter Families of Copulas," *Communications in Statistics - Theory and Methods* 23(5) (1994), 1311-1317.
> This article provides an asymptotic test of whether or not identically and independently distributed bivariate data arise from a specified single parameter copula family.

CLAYTON, D. G. "A Model for Association in Bivariate Life Tables and its Appllication in Epidemiological Studies of Familial Tendency in Chronic Disease Incidence," *Biometrika* 65 (1978) 141-151.

COMMITTEES ON ACTUARIAL PRINCIPLES of the Society of Actuaries and the Casualty Actuarial Society *General Principles of Actuarial Science* April 30, 1997 Discussion Draft.

COOK, R.D. AND JOHNSON, M.E. "A Family of Distributions for Modelling Non-Elliptically Symmetric Multivariate Data," *Journal of the Royal Statistical Society* B 43 (1981): 210-218.
> The Cook and Johnson's family of copulas was first proposed in this paper. Designed for joint distributions that are not elliptically symmetric, the family includes, as special cases, the multivariate Burr, logistic, and Pareto distributions. Finally, the family was used to fit a uranium mining data set.

COX, D. R. AND OAKES, D. *Analysis of Survival Data* New York: Chapman and Hall, 1984.

DAVID, H. A. AND MOESCHBERGER, M. L. *The Theory of Competing Risks* New York: MacMillan Publications, 1978.

ELANDT-JOHNSON, R. C. AND JOHNSON, N. L. *Survival Models and Data Analysis* New York: John Wiley, 1980.

FRANK, M.J. "On the Simultaneous Associativity of F(x,y) and x+y-F(x,y)," *Aequationes Mathematicae* 19 (1979) 194-226.
> Frank's copula first appeared in this paper as a solution to a functional equation problem. That problem involved finding all continuous functions such that F(x,y) and x+y-F(x,y) are associative, defined in the paper. Associative functions are typically studies of concern in topological semigroups.

FREES, E. *Data Analysis Using Regression Models: The Business Perspective* Englewood Cliffs: Prentice Hall, 1996.

FREES, E., CARRIERE, J. AND VALDEZ, E. "Annuity Valuation with Dependent Mortality," *Journal of Risk and Insurance* 63 (1996): 229-261.
> Copulas are used to value annuities based on more than one life in order to capture the possible dependence of lives. The study shows a reduction of approximately five percent in annuity values when dependent mortality models are used compared to standard models of independence. The model was calibrated using data from a large insurer.

GALTON, F. "Regression Towards Mediocrity in Heredity Stature," *Journal of Anthropological Institute* 15 (1885) 246-263.

---

[*] Articles directly related to copulas have been annotated for the reader's convenience.

GENEST, C. "Frank's Family of Bivariate Distributions," *Biometrika* 74 (1987): 549-555.
Basic properties of the Frank's family of copulas are further investigated in this paper. Members of this family exhibit properties of stochastically ordered and likelihood ratio dependent distributions. Nonparametric estimators of the dependence parameter are recommended. Simulation procedures were used to evaluate performance of these estimators.

GENEST, C. AND MACKAY, J. "Copules Archimediennes et Familles de lois Bidimensionnelles Dont les Marges Sont Donnees," *The Canadian Journal of Statistics* 14 (1986): 154-159.
This paper studies Archimedean copulas. Limits of sequences of members from this special class of copulas are investigated. Further, conditions for which these copulas exhibit property of stochastically ordered distributions are provided. (Written in French).

GENEST, C. AND MACKAY, J. "The Joy of Copulas: Bivariate Distributions with Uniform Marginals," *The American Statistician* 40 (1986): 280-283.
The authors extend their previous work on Archimedean copulas by proving properties using basic calculus only. In particular, they prove a result that relates the corresponding Archimedean function to the familiar Kendall's tau measure of association.

GENEST, C. AND RIVEST L. "Statistical Inference Procedures for Bivariate Archimedean Copulas," *Journal of the American Statistical Association* 88 (1993): 1034-1043.
Assuming uniform marginals, this article introduces a procedure for estimating the function which determines an Archimedean copula. The estimate is fully nonparametric and is (pointwise) root-n consistent. Using a uranium exploration data set, the article illustrates how to use the new procedure to choose a copula.

GENEST, C., GHOUDI, K. AND RIVEST, L. "A Semi-Parametric Estimation Procedure of Dependence Parameters in Multivariate Families of Distributions," *Biometrika* 82(1995): 543-552.

GOOVAERTS, M.J., DE VLYDER, F., AND HAEZENDONCK, J. "Ordering of Risks: A Review," *Insurance: Mathematics and Economics* 1 (1982): 131-163.

GOOVAERTS, M.J., DE VLYDER, F. AND HAEZENDONCK, J. *Insurance Premiums: Theory and Applications* Amsterdam: North Holland, 1984.

GUMBEL, E. J. "Bivariate Exponential Distributions," *Journal of the American Statistical Association* 55 (1960): 698-707.

HADAR, J. AND SEO, T.K. "A Note on Beneficial Changes in Random Variables," *The Geneva Papers on Risk and Insurance: Theory* 17 (1992): 171-179.

HEILMAN, W.R. AND SCHROTER, K.J. "Ordering of Risks and their Actuarial Applications," in *Stochastic Orders and Decision under Risk* edited by Mosler, K. and Scarsini, M. Hayward, California: Institute of Mathematical Statistics, 1991.

HOUGAARD, P. "Life Table Methods for Heterogeneous Populations: Distributions Describing for Heterogeneity," *Biometrika* 71 (1984): 75-83.

HOUGAARD, P. "A Class of Multivariate Failure Time Distributions," *Biometrika* 73 (1986): 671-678.

HOUGAARD, P. "Modelling Multivariate Survival," *Scandinavian Journal of Statistics* 14 (1987): 291-304.
This reviews models for multivariate survival and discusses practical situations for which they would be relevant. The author does not specifically suggest copula models for survival analysis but provides models based on marginal distributions. Here, the joint survival distribution is specified together with a single parameter to measure dependence. One can therefore infer that the joint survival can be expressed in terms of copulas. The paper is therefore appropriate for discussion of situations for which copula models are suitable.

HOUGAARD, P., HARVALD, B., AND HOLM, N. V. "Measuring the Similarities Between the Lifetimes of Adult Danish Twins Born Between 1881-1930," *Journal of the American Statistical Association* 87 (1992): 17-24.

HUTCHINSON, T.P. AND LAI, C.D. *Continuous Bivariate Distributions, Emphasising Applications.* Adelaide, South Australia: Rumsby Scientific Publishing, 1990.
> As the title indicates, this monograph organizes and describes the many families of bivariate distributions that have appeared in the literature. In addition to applications, special attention is given to copula families. In lieu of developing material from the foundations, the monograph collects a wide variety of properties and potential applications about many distributions, with a citation of original sources.

JAGGER, C. AND SUTTON, C. J. "Death After Marital Bereavement - Is the Risk Increased?," *Statistics in Medicine* 10 (1991): 395-404.

JOE, H. "Parametric Families of Multivariate Distributions with Given Margins," *Journal of Multivariate Analysis* 46 (1993): 262-282.
> The author examines one-parameter bivariate families of copulas and describes some of their interesting properties. Whenever possible, these bivariate families were extended to the multivariate setting and corresponding properties studied. Several open problems are posed in the paper.

JOE, H. *Multivariate Models and Dependence Concepts* London: Chapman and Hall, 1997.
> An excellent, although highly technical, research monograph that describes recent results in non-normal multivariate models. Much of the monograph focuses on copula constructions. It also contains discussion of categorical, longitudinal, and time series data.

JOE, H. AND HU, T. "Multivariate Distributions from Mixtures of Max-Infinitely Divisible Distributions," *Journal of Multivariate Analysis* 57 (1996): 240-265.
> This article explores the procedure based on the mixture of powers to generate new families of copulas. The procedure is based on distributions that are called max-infinitely divisible. Using this procedure, several families with closed-form multivariate copulas can be obtained, but only copulas that can be used for modeling positive dependence. It was mentioned in the discussion that extensions to allow for negative dependence will follow in a subsequent paper.

JOHNSON, N. AND KOTZ, S. *Distributions in Statistics: Continuous Multivariate Distributions* New York: John Wiley, 1972.

JOHNSON, N., KOTZ, S. AND BALAKRISHNAN, N. *Discrete Multivariate Distributions* New York: John Wiley, 1997.

JOHNSON, M. AND TENENBEIN, A. "A Bivariate Distribution Family with Specified Marginals," *Journal of the American Statistical Association* 76 (1981): 198-201.
> This paper provides a procedure to construct families of bivariate distributions whose margins and measures of dependence are specified. The construction was based on the method of weighted linear combination. While not directly related to copulas, the procedure is well-suited for constructing bivariate families of copulas.

JOHNSON, R. AND WICHERN D. *Applied Multivariate Statistical Analysis* Englewood Cliffs: Prentice Hall, 1988.

KAAS, R., HEERWAARDEN, A.E. VAN, and GOOVAERTS, M.J. *Ordering of Actuarial Risks.* Amsterdam: North Holland, 1994.

KIMELDORF, G. AND SAMPSON, A. R. "Positive Dependence Orderings," *Annals of the Institute of Statistical Mathematics* 39 (1987): 113-128.

KIMELDORF, G. AND SAMPSON, A. R. "A Framework for Positive Dependence," *Annals of the Institute of Statistical Mathematics* 41 (1989): 31-45.

KLEMENT, E.P. "Construction of Fuzzy σ-Algebras Using Triangular Norms," *Journal of Mathematical Analysis and Applications* 85 (1982a): 543-565.

KLEMENT, E.P. "Characterization of Fuzzy Measures Constructed by Means of Triangular Norms," *Journal of Mathematical Analysis and Applications* 86 (1982b): 345-358.

KLUGMAN, S. AND PARSA, A. "Fitting Bivariate Loss Distributions with Plackett's Model," Paper presented at the 1995 CAS Ratemaking Seminar.

KLUGMAN, S., PANJER, H., VENTER, G. AND WILLMOT, G. *Loss Models: From Data to Decisions* Unpublished Monograph, 1997.

KOTZ, S. "Some Remarks on Copulas in Relation to Modern Multivariate Analysis," *Contemporary Multivariate Analysis and Its Applications* (1997): k.1-k.13.

KRZANOWSKI, W. J. *Principles of Multivariate Analysis: A User's Perspective* Oxford: Oxford University Press, 1988.

LEE, A. J. "Generating Random Binary Deviates Having Fixed Marginal Distributions and Specified Degrees of Association," *The American Statistician* 47 (1993): 209-215.

LEMAIRE, J. "Fuzzy Insurance," *Astin Bulletin* 20 (1990): 33-55.

LI, H., SCARSINI, M., AND SHAKED, M. "Linkages: A Tool for the Construction of Multivariate Distributions with Given Nonoverlapping Multivariate Marginals," *Journal of Multivariate Analysis* 56 (1996): 20-41.

MAGULURI, G. "Semiparametric Estimation of Association in a Bivariate Survival Function," *Annals of Statistics* 21 (1993): 1648-1662.
   This article considers bivariate survival data with no censoring of each risk. For the dependency between risks, a one parameter copula model is used. With nonparametric estimates of the marginal survival function, this paper derives the asymptotic lower bound for the information in the copula parameter.

MAKEHAM, W. M. "On an Application of the Theory of Decremental Forces," *Journal of the Institute of Actuaries* 18 (1874): 317-322.

MARSHALL, A.W. AND OLKIN, I. "A Multivariate Exponential Distribution," *Journal of the American Statistical Association* 62 (1967): 30-44.

MARSHALL, A.W. AND OLKIN, I. "Families of Multivariate Distributions," *Journal of the American Statistical Association* 83 (1988): 834-841.
   Bivariate distributions with marginals as parameters can be generated using the procedure of mixtures. If F and G are univariate distribution functions, the distribution $H(x) = \int F^{\bullet}(x) dG(\theta)$ can be generated. The paper studies properties of this mixture distribution and provides as examples familiar copulas such as Frank's, Cook and Johnson's, and Gumbel's. Extensions to the multivariate case are discussed.

MEESTER, S. G. AND MACKAY, J. "A Parametric Model for Cluster Correlated Categorical Data," *Biometrics* 50 (1994): 954-963.
   This article shows how to use copulas as a model for association in clustered or longitudinal data. The discussion of copulas is restricted to Frank's family, although extensions to other parametric families are indicated. Symmetric dependencies are considered which is natural for clustered data. The response are from the generalized linear model family and, in particular, may be categorical.

METRY, M. H. AND SAMPSON, A. R. "A Family of Partial Orderings for Positive Dependence Among Fixed Marginal Bivariate Distributions," in *Advances in Probability Distributions with Given Marginals: Beyond the Copulas* edited by Dall'Aglio, G., Kotz, S., and Salinetti, G. The Netherlands: Kluwer Academic Publishers, 1991.

MEYER, J. "Beneficial Changes in Random Variables Under Multiple Sources of Risk and their Comparative Statics," *The Geneva Papers on Risk and Insurance: Theory* 17 (1992): 7-19.

MOSLER, K. AND SCARSINI, M. (editors) *Stochastic Orders and Decision under Risk* Hayward, California: Institute of Mathematical Sciences, 1991.

NELSEN, R.B.. "Properties of a One-Parameter Family of Bivariate Distributions with Specified Marginals," *Communications in Statistics - Theory & Methods* 15 (1986): 3277-85.
   This paper explores properties of Frank's one-parameter family of bivariate distributions. In particular, it offers formulas to evaluate three nonparametric measures of correlation, namely Spearman's rho, Kendall's tau, and the less familiar medial correlation coefficient. It also provides procedure to generate observations from this bi-

variate distribution which may be used for simulation purposes.

OAKES, D. "A Model for Association in Bivariate Survival Data," *Journal of the Royal Statistical Society B* 44 (1982): 414-422.

   The author offers an alternative procedure to estimate the association parameter in the Cook and Johnson's family of bivariate distributions which has been used to model association in bivariate survival data. The paper examines the likelihood procedure used to estimate the parameter by earlier authors. As an alternative, it proposed the use of nonparametric procedures using Kendall's coefficient. The asymptotic variance of this estimator was evaluated.

OAKES, D. "Bivariate Survival Models Induced by Frailties," *Journal of the American Statistical Association* 84 (1989): 487-493.

   This article shows how Archimedean copulas arise naturally in the context of frailty models. The work emphasizes a special case of Marshall and Olkin (1989); bivariate survival times with a common latent mixing parameter. The paper introduces a "cross-ratio" function that provides a useful diagnostic tool for identifying the survival distribution based on bivariate survival data.

OAKES, D. "Multivariate Survival Distributions," *Journal of Nonparametric Statistics* (1994), 343-354.

   This article reviews properties of latent mixtures of survival distributions that give rise to frailty models.

OSTAZEWSKI, K. *Fuzzy Set Methods in Actuarial Science.* Schaumburg: Society of Actuaries, 1993.

PARKES, C. M., BENJAMIN, B., AND FITZGERALD, R. G. "Broken Heart: A Statistical Study of Increased Mortality Among Widowers," *British Medical Journal* i (1969): 740-743.

SCARSINI, M. "On Measures of Concordance," *Stochastica* 8 (1984): 201-219.

SCHWEIZER, B., "Thirty Years of Copulas," in *Advances in Probability Distributions with Given Marginals: Beyond the Copulas* edited by Dall"Aglio, G., Kotz, S., and Salinetti, G. The Netherlands: Kluwer Academic Publishers, 1991.

   This articles reviews the early development of copulas. The historical overview focuses on beginnings with a focus on probability theory. Properties of n-dimensional distribution functions are reviewed and their extensions to probabilistic metric spaces are discussed. Relationships to triangular norming functions are developed that provide bounds on functions of distribution functions. The paper also gives a useful introduction to the role of copulas for developing measures of dependence and their relationship with the dependence axioms of Renyi.

SCHWEIZER, B. AND SKLAR, A. *Probabilistic Metric Spaces.* New York: North Holland, 1983.

   A probabilistic metric space is a set, or space, of distribution functions combined with a function, or metric, that measures the distance between two elements of the set. This monograph develops probabilistic metric spaces. A special feature is that an entire chapter (six) is devoted to copulas. Further, the relationship between copulas and triangular norms is emphasized.

SCHWEIZER, B. AND WOLFF, E. F. "On Nonparametric Measures of Dependence for Random Variables," *The Annals of Statistics* 9 (1981): 879-885.

   This paper uses copulas to define three nonparametric measures of dependence for pairs of random variables. In particular, the familiar measures of dependence Pearson's correlation, Spearman's rho, and Kendall's tau are expressed in terms of copulas.

SEAL, H. L. "Studies in the History of Probability and Statistics XV: The Historical Development of the Gauss Linear Model," *Biometrika* 54 (1967): 1-24.

SEAL, H. L. "Multiple Decrements or Competing Risks," *Biometrika* 64 (1977): 429-439.

SHIH, J. H. AND LOUIS, T. A. "Inferences on the Association Parameter in Copula Models for Bivariate Survival Data," *Biometrics* 51 (1995): 1384-1399.

   This paper studies bivariate survival data where the risks may be dependent but do not preclude one another. Each of the two risks are subject to censoring by an independent mechanism. The dependency between risks is modeled using a one parameter copula function. The marginal survival functions are estimated (i) nonparametrically, using Kaplan-Meier estimators and (ii) parametrically. A two-stage estimation procedure is proposed; both finite and asymptotic properties are established. An AIDS data set illustrates the procedures.

SKLAR, A. "Fonctions de repartition a n dimensions et leurs marges," *Publ. Inst. Stat. Univ. Paris* 8 (1959): 229-231.

SKLAR, A. "Random Variables, Joint Distribution Functions and Copulas," *Kybernetika* 9 (1973): 449-460.
The author is one of the pioneers of copulas. In this paper, he proves elementary results that relates copulas to distribution functions and random variables. In particular, given a copula function C and an n-tuple distribution functions $(F_1,...,F_n)$, he establishes the existence of a probability space and a set of random variables $X_1,...,X_n$ defined over that space with C as the associated copula. In addition, the paper describes structural properties of a special type of copulas called associative copulas.

STIGLER, S. M. *The History of Statistics: The Measurement of Uncertainty Before 1900.* Cambridge, MA: Harvard University Press, 1986.

SUNGUR, E. A. "Dependence Information in Parameterized Copulas," *Communications in Statistics: Simulations* 19 (1990): 1339-1360.
This paper provides mechanisms for extracting the dependence information from parameterized copulas. The author defines various dependence functions to capture dependence information. Approximations to copulas using Taylor's expansion are discussed. Most of the derivations relate to two-dimensional copulas although a brief attempt to extend to three-dimensional copulas is made.

TCHEN, A. "Inequalities for Distributions with Given Marginals," *Annals of Probability* 8 (1980): 814-827.

TIBILETTI, L. "Beneficial Changes in Random Variables via Copulas: An Application to Insurance," *The Geneva Papers on Risk and Insurance: Theory* 20 (1995): 191-202.

VAUPEL, J. W., MANTON, K. G. AND STALLARD, E. "The Impact of Heterogeneity in Individual Frailty on the Dynamics of Mortality," *Demography* 16 (1979): 439-454.

WANG, S. "Premium Calculation by Transforming the Layer Premium Density," *ASTIN Bulletin* 26 (1996): 71-92.

WANG, S. "PH-Measure of Portfolio Risks," Chapter 9 of Unpublished Lecture Notes, *Risk Measures with Applications in Insurance Ratemaking and Actuarial Valuation,* University of Waterloo, 1997.

WARD, A. "Mortality of Bereavement," *British Medical Journal* i (1976): 700-702.

YOUNG, V. "The Application of Fuzzy Sets to Group Health Underwriting," *Transactions of the Society of Actuaries* XLV (1993): 551-590.

ZHENG, M. AND KLEIN, J. P. "Estimates of Marginal Survival for Dependent Competing Risks Based on an Assumed Copula," *Biometrika* 82 (1995): 127-138.
For bivariate survival data, it is well-known that without assumptions about the relationship between the competing survival times, the marginal survival functions are unidentifiable. This article assumes a known copula and introduces nonparametric estimators of the marginal survival functions. The estimates reduce to the Kaplan-Meier product limit estimator in the case of independent survival times. Further, bounds on the survival functions are provided when the copula is known only approximately.