A Note on Credibility Using a Varying Parameter Model

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Abstract

In this paper, a basic varying parameter regression model, where the regression coefficients vary over time, is employed to predict the future expected claim. It is shown that the resulting credibility formula for the prediction is of the updating type. The method of moments is employed to estimate the unknown variance parameters required in the formula. The credibility formula performs well compared to the basic credibility formula of Bühlmann-Straub when applied to the Hachemeister data.

Keywords: Credibility, longitudinal data model, method of moments, mixed model.

1 Introduction

A basic problem of credibility ratemaking is to establish an appropriate future expected claim or pure premium based on the past claim experiences. Let y_{it}, $i = 1, 2, \cdots, I$ and $t = 1, 2, \cdots, T$, be the observed claim experience for the ith subject in time period t. Assume that y_{it} are identically distributed for all subjects and time periods and the variation of performance for the ith subject is described by b_i. To estimate the future expected claim or pure premium $P_{i,T+1}$ of a risk for the ith subject with an unknown b_i, Bühlman and Straub model (Herzog, 1994) yielded the following linear credibility formula.

$$
\begin{aligned}
P_{i,T+1} &= \left(1 - \frac{T}{T + E[V(b_i)]/V[\mu(b_i)]}\right)\mu + \frac{T}{T + E[V(b_i)]/V[\mu(b_i)]}\bar{y}_i; \qquad (1) \\
&= (1 - \zeta_T)\mu + \zeta_T \bar{y}_i;
\end{aligned}
$$

where μ is the overall true mean, $V(b_i)$ is the variance of y_{it} given b_i, $\mu(b_i) = E(y_{it} \mid b_i)$ is the expected value of y_{it} given b_i, $\bar{y}_i$ is the sample mean of the ith subject over the observed time periods, and $\zeta_T = T/\{T + E[V(b_i)]/V[\mu(b_i)]\}$. The term ζ_T is said to be the credibility factor. Assuming y_{it} and b_i are normally distributed, it is well known that the pure premium is equal to the corresponding Bayesian estimator with a squared loss function. The unknown fixed parameters μ, the expected value $E[V(b_i)]$ of $V(b_i)$, and the variance $V[\mu(b_i)]$ of $\mu(b_i)$ in the formula need to be estimated using the observed samples.

Let $\xi_T = 1/\{T + E[V(b_i)]/V[\mu(b_i)]\}$. Notice that the credibility formula of $P_{i,T+1}$ in (1) can be written as

$$
\begin{aligned}
P_{i,T+1} &= (1 - \xi_T)\left[(1 - \zeta_{T-1})\mu + \zeta_{T-1}\frac{\sum_{i=1}^{T-1} y_{it}}{T-1}\right] + \xi_T y_{iT} \\
&= (1 - \xi_T)P_{i,T} + \xi_T y_{iT}, \qquad (2) \\
&\Rightarrow \; P_{i,T+1} - P_{i,T} = \xi_T(y_{iT} - P_{i,T})
\end{aligned}
$$

A linear credibility formula is said to be of the updating type if there is a sequence $\xi_1, \xi_2, \cdots, \xi_T$ of real numbers such that $P_{T+1} = (1 - \xi_T)P_T + \xi_T y_T$; see Gerber and Jones (1975). From equation (2), the Bühlman-Straub credibility formula is of the updating type where the premium adjustment from year T to year $T+1$ is proportional to the excess of claims over premium in year T.

Frees, Young, and Luo (1999), hereafter referred to as FYL, established the link between credibility ratemaking theory in actuarial science and the longitudinal data models or mixed model. FYL revealed that credibility formulas for the prediction of expected claims, such as those in Bühlman (1970), Bühlman-Straub (1972), Hachemeister (1975), and Jewel (1975), are the best linear unbiased predictors of the pure premiums under the longitudinal data models. With the link, it allows us to employed estimation techniques in the longitudinal data models to compute estimates of unknown parameters required in the credibility formula. It also allows us to examine other possible issues using the longitudinal models and to further utilize other statistical models for ratemaking.

Consider the basic one-way random-effects model, see Rao (1997),

$$y_{it} = b_i + \epsilon_{it}, \quad i = 1, 2, \cdots, I, \quad t = 1, \cdots, T_i, \tag{3}$$

where b_i is the random effect of the ith subject with mean β and standard deviation σ_b, and ϵ_{it} is the random error with mean 0 and standard deviation σ_ϵ. The parameter β is the overall true mean in this model. The random effect b_i and the error ϵ_{it} are assumed to be uncorrelated. The best linear unbiased prediction for the response is given by (1) with $E[V(b_i)] = \sigma_\epsilon^2$ and $V[\mu(b_i)] = \sigma_b^2$. That is, the basic linear credibility formula in (1) is the BLUP in the one-way random-effects model and is a special case of the longitudinal data models discussed in Laird, Lange, and Stram (1987) and FYI (1999). Notice that though the random errors ϵ_{it} vary over time, the random coefficient b_i does not. The longitudinal data models deal with the problem of parameter variation by modeling the effect b_i as random. However, it does not allow the effect to vary over time within each subject.

Rosenberg (1972) examined the problem of estimation in a repeated measurement model, in which the regression parameters vary according to a stationary stochastic process with a known covariance matrix, and derived the best linear unbiased estimators for prediction. A number of survey on varying coefficient regression models has been written including Nicholls and Pagen (1985), the book by Raj and Ullah (1981), and references therein. One can definitely further study the more sophisticated varying parameter models. In this paper, a formula for prediction is derived for a simple time varying model and is shown to be of the updating type. The covariance matrix in this paper is not assumed and variance parameters will need be estimated.

The organization of the paper is as follows. The credibility formula using a basic varying parameter model is introduced in section 2. Estimations of unknown parameters in the formula are derived via the method of moments in section 3. Empirical results for the Hachemeister data (1975) are presented in section 4, and finally summaries and discussion are made.

2 Varying Parameter Models

In this section, we will employ a simple varying parameter model to demonstrate that we can further establish a credibility formula utilizing statistical regression models. The resulting formula is shown to be of the updating type.

Consider the following regression structure such that the parameters vary over time according to a simple process described in (5),

$$y_{it} = b_{it} + \epsilon_{it}, \quad (4)$$

$$b_{it} = b_{i,t-1} + v_{it}, \quad (5)$$

where $i = 1, 2, \cdots, I$ and $t = 1, \cdots, T_i$. Assume that ϵ_{it} and v_{it} are independently normally distributed with mean 0 and variance σ_ϵ^2/c_{it} and σ_v^2, respectively. The constants $c_{it}, t = 1, 2, \cdots, T_i$, are assumed to be known. The coefficient parameters b_{it} for the $i$the subject are changing with time and follow the structure in (5). The initial conditions $b_{i0}, i = 1, 2, \cdots, I$, are assumed to be identically normally distributed with mean β and standard deviation σ_b. The parameter β is equivalent to the true overall mean μ in (1). If the standard deviation $\sigma_v = 0$, this varying parameter regression model is equal to the one in (3) with $b_i = b_{i0}$.

Let $Y_i'(t) = (y_{i1}, y_{i2}, \cdots, y_{it})$, $Y_i = Y_i(T_i)$, I be the identity matrix of a proper dimension, and J be a column vector of ones of a proper dimension. Define the matrix $C_i(t) = diag(c_{i1}, c_{i2}, \cdots, c_{it})$ and define the matrix A_t of dimension $t \times t$ to be

$$A_t = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 1 & 1 & \cdots & & 1 \end{pmatrix},$$

where the upper diagonal entries are zeroes and the diagonal and lower diagonal entries are ones. The equation (5) implies that $b_{it} = b_{i0} + \sum_{s=1}^{t} v_{it}$ and $y_{it} = b_{i0} + \sum_{i=1}^{t} v_{it} + \epsilon_{it}$. The covariance matrix of $Y_i(t)$, the observed data up to time t on subject i, is $\Omega_i(t) = \sigma_b^2 JJ' + \sigma_v^2 A_t A_t' + \sigma_\epsilon^2 C_i^{-1}(t)$. The covariance matrix of dimension $1 \times t$ between $y_{i,t+1}$ and $Y_i(t)$ is $\Omega_{12}(t+1) = \sigma_b^2 J' + \sigma_v^2 J' A_t'$. The conditional random variable of $y_{i,t+1}$ given $Y_i(t)$ has a normal distribution with a mean of $\beta + \Omega_{12}(t+1)\Omega_i^{-1}(t)[Y_i(t) - \beta J]$ and a covariance matrix of $\sigma_b^2 + (t+1)\sigma_v^2 + \sigma_\epsilon^2/c_{i,t+1} - \Omega_{12}(t+1)\Omega_i^{-1}(t)\Omega_{12}'(t+1)$. Under the normality assumption, the maximum a posteriori estimator of the future expected claim $P_{i,T_i+1} = E[y_{i,T_i+1} \mid Y_i(T)]$ is therefore given by

$$P_{i,T_i+1} = \beta + \Omega_{12}(T_i+1)\Omega_i^{-1}(T_i)(Y_i - \beta J). \tag{6}$$

Partition the covariance matrix $\Omega_i(T_i)$ for each subject i such that

$$\Omega_i(T_i) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix},$$

where Ω_{11} is the variance of y_{iT_i}, $\Omega_{12} = \sigma_b^2 J' + \sigma_v^2 J'A'$ is the covariance matrix between y_{i,T_i} and $Y_i(T_i-1)$, and Ω_{22} is the covariance matrix of $Y_i(T_i-1)$. The inverse matrix of $\Omega_i(T_i)$ is

$$\Omega_i^{-1}(T_i) = \begin{pmatrix} \Omega^{11} = (\Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21})^{-1} & -\Omega_{11}^{-1}\Omega_{12}\Omega^{22} \\ -\Omega_{22}^{-1}\Omega_{21}\Omega^{11} & \Omega^{22} = (\Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12})^{-1} \end{pmatrix}.$$

Note that the covariance matrix of dimension $1 \times T_i$ between y_{i,T_i+1} and Y_i is $\Omega_{12}(T_i+1) = (\sigma_b^2 + T_i\sigma_v^2, \Omega_{12})$ and that $Y_i' = (y_{iT_i}, Y_i'(T_i-1))$. Use the fact that $(I+AB)^{-1} = I_A(I+BA)^{-1}B$, the multiplication of the matrices $\Omega_{12}(T_i+1)$ and $\Omega_i^{-1}(T_i)$ in the right hand side of (6) becomes

$$\begin{pmatrix} (\sigma_b^2 + T_i\sigma_v^2 - \Omega_{12}\Omega_{22}^{-1}\Omega_{21})\Omega_{11}^{-1}(1 - \Omega_{12}\Omega_{22}^{-1}\Omega_{21})^{-1} \\ [1 - (\sigma_b^2 + T_i\sigma_v^2)\Omega_{11}^{-1}]\Omega_{12}\Omega_{22}^{-1}[I + \Omega_{21}\Omega_{11}^{-1}\Omega_{12}\Omega_{22}^{-1}\Omega_{11}^{-1}] \end{pmatrix}' \times \begin{pmatrix} y_{iT_i} - \beta \\ Y_i(T_i-1) - \beta J \end{pmatrix}$$

Let $d_T = \Omega_{12}\Omega_{22}^{-1}\Omega_{21}\Omega_{11}^{-1}$. The above equation is equal to

$$\frac{(\sigma_b^2 + T_i\sigma_v^2)\Omega_{11}^{-1} - d_T}{1 - d_T}(y_{i,T_i} - \beta) + \frac{1 - (\sigma_b^2 + T_i\sigma_v^2)\Omega_{11}^{-1}}{1 - d_T}\Omega_{12}\Omega_{22}^{-1}[Y_i(T_i-1) - \beta J].$$

Now the following result is readily obtained.

Result : The credibility formula in (6) is of the updating type. That is,

$$P_{i,T_i+1} = (1-\xi_T)P_{i,T_i} + \xi_T y_{iT_i}, \tag{7}$$

$$\text{where} \qquad \xi_T = (\sigma_b^2 - T_i\sigma_v^2 - \Omega_{12}\Omega_{22}^{-1}\Omega_{21})\Omega_{11}^{-1}(1-\Omega_{12}\Omega_{22}^{-1}\Omega_{21}\Omega_{11}^{-1})^{-1}.$$

When $\sigma_v^2 = 0$, ξ_t are exactly those in the equation (2). This result shows that one can calculate the pure premium using a weighted average of the pure premium P_{i,T_i} from the previous time and the newly observed claim y_{i,T_i}. That is, the adjustment from time T_i to time $T_i + 1$ is proportional to the excess of claims over premium in year T.

3 Estimation of Unknown Parameters

The calculation of the credibility formula in (7) requires estimations of the variance parameters σ_b^2, σ_v^2, and σ_ϵ^2. In this section, we provide estimates of the variance parameters using the method of moments that can be more easily computed than the maximum likelihood estimates for the varying parameter model.

The variance of the random error ϵ_{it} is σ_ϵ^2/c_{it}, where c_{it} is a known constant. Let $c_i = \sum_{t=1}^{T_i} c_{it}$ and $c = \sum_{i=1}^{I} c_i$. Define the weighted ith subject sample average $\bar{y}_i = \sum_{i=1}^{T_i} c_{it}y_{it}/c_i$ and the overall sample mean $\bar{y} = \sum_{i=1}^{I} c_i\bar{y}_i/c$. Recall that $y_{it} = b_{i0} + \sum_{i=1}^{t} v_{it} + \epsilon_{it}$ and the fact that $E(X^2) = [E(X)]^2 + \text{Var}(X)$. Let $a_i' = (c_{i1}, c_{i2}, \cdots, c_{iT_i})$ and $s_i = a_i' A_{T_i} A_{T_i}' a_i = \sum_{j=1}^{T_i}(\sum_{k=j}^{T_i} c_{ik})^2$. We first obtain the following second moments of $y_{it}, \bar{y}_i$, and $\bar{y}$:

$$\begin{aligned}
E(y_{it}^2) &= \beta^2 + \sigma_b^2 + t\sigma_v^2 + \sigma_\epsilon^2/c_{it}; \\
E(\bar{y}_i^2) = E\left[(a_i'Y_i/c_i)^2\right] &= \beta^2 + a_i'\text{Cov}(Y_i)a_i/c_i^2 \\
&= \beta^2 + \sigma_b^2 + \frac{s_i}{c_i^2}\sigma_v^2 + \frac{1}{c_i}\sigma_\epsilon^2; \\
E(\bar{y}^2) = E\left[(\sum_{i=1}^{I} c_i\bar{y}_i/c)^2\right] &= \beta^2 + \sum_{i=1}^{I} c_i^2\text{Var}(\bar{y}_i)/c^2
\end{aligned}$$

$$= \beta^2 + \frac{\sum_{i=1}^I c_i^2}{c^2}\sigma_b^2 + \frac{\sum_{i=1}^I s_i}{c^2}\sigma_v^2 + \frac{1}{c}\sigma_\epsilon^2.$$

The method of moments estimates of σ_b^2, σ_v^2, and σ_ϵ^2 can be computed by solving the following equations:

$$\begin{aligned}
\sum_{i=1}^{I}\sum_{t=2}^{T_i}(y_{it} - y_{i,t-1})^2 &= \sum_{i=1}^{I}\sum_{i=2}^{T_i}\left(\frac{1}{c_{it}} + \frac{1}{c_{i,t-1}}\right)\sigma_\epsilon^2 + \sum_{i=1}^{I}(T_i - 1)\sigma_v^2; \\
\sum_{i=1}^{I}\sum_{t=1}^{T_i} c_{it}(y_{it} - \bar{y}_i)^2 &= \sum_{i=1}^{I}(T_i - 1)\sigma_\epsilon^2 + \left(\sum_{i=1}^{I}\sum_{t=1}^{T_i} tc_{it} - \sum_{i=1}^{I}\frac{s_i}{c_i}\right)\sigma_v^2; \\
\sum_{i=1}^{I} c_i(\bar{y}_i - \bar{y})^2 &= (I-1)\sigma_\epsilon^2 + \left(c - \frac{\sum_{i=1}^I c_i^2}{c}\right)\sigma_b^2 + \sum_{i=1}^{I}\left(\frac{s_i}{c_i} - \frac{s_i}{c}\right)\sigma_v^2.
\end{aligned}$$

Solving the above equations for the unknown parameters can be easily programmed. Note that the above will yield the restricted maximum likelihood (REML) estimates of the longitudinal model with $\sigma_v = 0$; see FYL, 1999. The parameter β can be unbiasedly estimated by the overall weighted sample average $\bar{y}$. One can also use the weighted least squares estimates $\left(\sum_{i=1}^I J'\Omega_i(T_i)J\right)^{-1}\left(\sum_{i=1}^I J'\Omega_i(T_i)Y_i\right)^{-1}$ with the variance parameters in Ω_i replaced by the estimates computed from the method of moments. In the empirical results presented in the following section, the estimate of β is computed using weighted least squares algorithm.

4 Empirical Results

The claim data from private passenger auto insurance, bodily injury coverage in Hachemeister (1975) are used to demonstrate the performance of the varying parameter model in section 2. The data consists of information from $I = 5$ states in the U.S. during $T = 12$ quarters, from the third quarter of 1970 to the second quarter of 1973. The number of claims n_{it} and the severity or average loss per claim y_{it} from state i in time period t are observed. Since the observed responses are the average losses, the variance of the random error ϵ_{it} is σ_ϵ^2/n_{it}. The known constant c_{it} is therefore n_{it}, $c_i = \sum_{t=1}^{T_i} n_{it}$, and the grand total number of claims form all states $c = \sum_{i=1}^{I} n_i$.

The claim data from all 5 states were utilized in the estimations of unknown variance parameters in the formula (7). When computing the estimated pure premium $\widehat{P}_{i,T_i+1}$, the weights attached to observations $y_{i1}, \cdots, y_{iT_i}$ are the corresponding coordinates of the vector $(\sigma_b^2 J' + \sigma_v^2 J' A')\Omega_i(T_i)$. Though not shown technically, the computation results showed that later observations have larger weights.

Model	β	σ_ϵ	σ_b	σ_v	MSE
Longitudinal Data	1,662.60	10,062.55	281.95		32,710.38
Varying Parameter	1,527.85	5,326.63	173.36	108.94	8,465.04

The REML estimates of the longitudinal data model were used to estimated the unknown parameters in the Bühlmann-Straub credibility formula in (1). The above table presents the result from the longitudinal data model and the proposed varying parameter model. For each state i and each time period t, the estimated pure premiums $\widehat{P}_{i,t+1}$ were computed using data observed up to time t for both models. The mean sum of square of errors (MSE) is then defined to be $\sum_{i=1}^{I}\sum_{t=2}^{T_i}(y_{i,t+1} - \widehat{P}_{i,t+1})^2/60$. Note the large magnitude of the MSE is partially due to the data are large numerically. The varying parameter model appeared to yield better results in terms of mean sum of square of errors between the observed severities and predicted premiums.

5 Summary and Discussion

The purpose of this paper is to study and establish a credibility formula using the basic varying parameter model. The resulting formula is quite attractive computationally and proved to be of the updating type. The estimation method of the unknown variance parameters is not limited to the method of moments employed here. One definitely can study the maximum likelihood estimation that will not be simple to derive under the assumed time-varying structure of b_{it}.

With the longitudinal data interpretation of credibility models introduced in Frees, Young and Luo (1999), it inspired us to employ existing statistical models, such as the one in this article, for credibility ratemaking. The simple model of $y_{it} = b_{it} + \epsilon_{it}$ in (4) can be further extended to $y_{it} = x_{it} b_{it} + \epsilon$, where x_{it} is a vector of explanatory variables, to allow for the adjustment of the explanatory variables. Also the structure in (5) can be assumed to follow a time series model. Other issues to be further explored include possible relations between the evolutionary model in Jewell (1974) and the varying parameter model.

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