Assessing Regime Switching Equity Return Models

R. Keith Freeland, ASA, Ph.D.
Mary R. Hardy, FSA, FIA, CERA, Ph.D.
Matthew Till

Abstract

In this paper we examine time series model selection and assessment based on residuals, with a focus on regime switching models. We discuss the difficulties in defining residuals for such processes and propose several possible alternatives. We determine that a stochastic approach to defining the residuals is the only way to generate residuals that are normally distributed under the model hypothesis. The stochastic residuals are then used to assess the fit of several models to the S&P 500 log-returns. We then complement this analysis by performing an out-of-sample forecast for each model. Only two of the models have a greater than 5 percent probability for the dramatic downturn in the equities markets by the end of 2008.

1. Introduction

The purpose of this paper is to help practitioners and regulators more accurately quantify the potential impact of market risk on insurance products with equity-linked guarantees. As a practicing actuary, one step in managing this type of risk is the calculation of reserves to determine capital requirements. This requires the modeling of the equity returns linked to the product. In fact the reserves and capital requirements are only as good as the underlying equity return model. Therefore, one of the keys to successfully managing this risk is to effectively understand and model the deviations from the mean equity returns; this is especially true for the large deviations. Thus one of the highest priorities is to adequately model the tails of the equity returns distribution. These deviations are observed in the residuals of the model. Before we can have effective risk management it is crucial that we determine that our equity model has an appropriate fit in the tail region. This is most easily accessed by examining the distribution of the residuals. If the observed residuals are consistent with what we expected under the assumed model, then we can use the model to help us quantify our risk exposure. However, if the observed residuals are not consistent with the assumed model (particularly if they are more widely dispersed in the tails than is assumed under the model), then we must modify the model or look for a better model. One of the most successful ways to introduce sufficient variation into an equity returns model is to use a regime switching model. These models are well suited for fitting long economic time series, where the underlying state of the economy changes several times. The two regime version has one regime for stable economic conditions and a second regime for more turbulent economic conditions. A hidden Markov process models the switching between these regimes. While the switching process adds the much needed variation to the model, the standard residuals do not properly represent this additional variation.

The paper is organized as follows. Section 2 presents two common single regime models for equity prices; these are the independent lognormal model and the GARCH model. In Section 3 we present multiple regime models, which include the mixture ARCH model, regime switching lognormal model, regime switching draw down model and the regime switching GARCH model. Then in Section 4 we discuss the difficulties in defining meaningful residual for these models and propose a stochastic approach to defining the residuals. These stochastic residuals are normally distributed. Additionally we discuss how to use these residuals to assess the model assumption of normally distributed innovations. Section 5 then assesses the models using the S&P total return index. This is done by first examining the residuals to test for normality and then by performing an out-of-sample forecast. In particular we examine how the tails of each of these models compare to actual dramatic drop in the equities markets do to the global recession in 2008.

2. Single Regime Equity Return Models

In this section we review two single regime models for equities, which are the independent lognormal model and the GARCH(1,1) model. We let S_t denote the value of an equity index at time t, and let Y_t denote the log-return of the index, which is defined as

$$Y_t = \log\left(\frac{S_t}{S_{t-1}}\right).$$

Thus,

$$S_t = S_{t-1}\exp(Y_{t-1}) = S_0 \exp(Y_1 + Y_2 + \cdots + Y_t),$$

where S_0 is the arbitrary starting point of the index and $\exp(Y_1 + Y_2 + \cdots + Y_t)$ is the accumulation factor over the time period 0 to t.

Single regime models with normal innovations take the following form

$$Y_t \mid \Im_{(t-1)} = \mu_t + \sigma_t z_t \quad \textit{where} \quad z_t \quad \textit{are i.i.d.,} \qquad \textit{and} \quad z_t \sim N(0,1) \ \forall t, \qquad (1)$$

where μ_t and σ_t depend on the particular model and may be functions of the past observations $Y_1, Y_2, \ldots, Y_{t-1}$ and $\Im_t$ is the standard filtration.

In the case of the independent lognormal ILN model $\mu_t = \mu$ and $\sigma_t = \sigma$ are both constants. This simple model will often appear consistent with equity returns over short periods of time with limited volatility clustering. It is also a discretized version of geometric Brownian motion, which is an underlying assumption of the Black-Scholes model (Black and Scholes 1973).

There are many specifications of the GARCH model (see Engle (1995) and the references within). The simplest and most used form is called the GARCH(1,1), where $\mu_t = \mu$ is constant and

$$\sigma_t^2 = \alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2 + \beta \sigma_{t-1}^2.$$

For single regime models with the formulation specified in equation (1) the residuals are straightforward to calculate and are give by

$$r_t = \frac{Y_t - \mu_t}{\sigma_t}.$$

Under the model assumptions these residuals are independent identically distributed standard normal random variables and hence can be used to assess the assumption of normality.

3. Multiple Regime Equity Return Models

While single regime models often fit data reasonably well over short time periods, they usually fail over longer time horizons or over a different time period. The idea behind multiple regime models or regime switching models is that we have multiple models which describe different sections of our time period. Additionally some stochastic mechanism is used to switch between the models. Equity markets are often described in terms of volatility. In the simplest terms we could say that the market goes through periods of low volatility with shorter periods of high volatility.

The regime switching model with normal innovations takes the following form

$$Y_t \mid \Im_{t-1}, \rho_t = \mu_t(\rho_t, t) + \sigma_t(\rho_t, t)\, z_t \quad \textit{where } z_t \textit{ are i.i.d., and } z_t \sim N(0,1)\ \forall t, \tag{2}$$

where $\rho_t = 1, 2, \ldots, K$ indicates which of the K possible regimes the process is in at time t and $\Im_t$ is the standard filtration. The mean and standard deviations, $\mu_{\rho_t,t}$ and $\sigma_{\rho_t,t}$, depend on the current regime and the particular model in that regime which may be a function of the past observations $Y_1, Y_2, \ldots, Y_{t-1}$.

There are two common methods to specify the stochastic nature of the regime process ρ_t. The first is to have fixed probabilities of each regime occurring, which gives a mixture model. The second is to use a Markov switching process, where the probabilities at time t depend on the regime at time $t-1$.

Wong and Chan (2005) used the first approach to create a mixture of ARCH models, which they call the MARCH model. In particular Wong and Chan identified the MARCH(2;0,0;2,0) model for use with log-returns. This is a mixture of an ARCH(2) process and a random walk process and has the following structure.

$$\mu_{1,t} = \mu_1 \quad \textit{constant} \tag{3}$$

$$\sigma_{1,t}^2 = \alpha_{1,0} + \alpha_{1,1}(Y_{t-1} - \mu_1)^2 + \alpha_{1,2}(Y_{t-2} - \mu_1)^2 \quad \textit{ARCH(2) volatility} \tag{4}$$

$$\mu_{2,t} = \mu_2 \quad \textit{constant} \tag{5}$$

$$\sigma_{2,t} = \alpha_{2,0} \quad \textit{constant} \tag{6}$$

$$P(\rho_t = 1) = q = 1 - P(\rho_t = 2) \quad \textit{mixture model} \tag{7}$$

Wong and Chan found this model to fit the higher moment of historical data better than the regime switching lognormal model.

We will consider three specific regime switching models, which are sometimes called Markov switching models. In these models the processes movement between regimes is governed by a hidden Markov chain. The models we consider have two regimes and the transition probabilities will be denoted as

$$P(\rho_t = 2|\rho_{t-1} = 1) = p_{12} = 1 - P(\rho_t = 1|\rho_{t-1} = 1) \tag{8}$$

$$P(\rho_t = 1|\rho_{t-1} = 2) = p_{21} = 1 - P(\rho_t = 2|\rho_{t-1} = 2). \tag{9}$$

Note that when $p_{12} = p_{22}$ and $p_{11} = p_{21}$ then the probabilities do not depend on the regime of the previous period. This special case recovers the mixture model.

The regime switching lognormal model has a constant mean and variance in each regime. A two state regime switching lognormal model can be parameterized as

$$\mu_{1,t} = \mu_1 \quad constant \tag{10}$$

$$\sigma_{1,t} = \sigma_1 \quad constant \tag{11}$$

$$\mu_{2,t} = \mu_2 \quad constant \tag{12}$$

$$\sigma_{2,t} = \sigma_2 \quad constant \tag{13}$$

Hamilton (1989) first proposed the regime-switching framework. While his models also have two regimes, they are more complicated because the means are autoregressive.

Panneton (2002) proposed a modified version of the regime switching lognormal model, which includes a varying mean. The mean is defined as

$$\mu_{1,t} = \kappa_1 + \varphi_1 D_{t-1} \tag{14}$$

$$\mu_{2,t} = \kappa_2 + \varphi_2 D_{t-1} \tag{15}$$

where

$$D_{t-1} = \min(0, D_{t-2} + Y_{t-1}).$$

When the $\varphi_j's$ are negative they act as a market recovery mechanism. When negative returns are realized D_{t-1} will become negative, which will increase the mean of the process, which leads to higher returns or market recovery. Once the market has recovered D_{t-1} returns to zero and they process follows the long term mean.

The final model we present is the regime switching GARCH model of Gray (1996), which has a GARCH specification in each regime. The two regime version with a GARCH(1,1) process in each regime can be parameterized as

$$\mu_{1,t} = \mu_1 \quad constant \tag{16}$$

$$\sigma_{1,t}^2 = \alpha_{1,0} + \alpha_{1,1}\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2 \tag{17}$$

$$\mu_{2,t} = \mu_2 \quad constant \tag{18}$$

$$\sigma_{2,t} = \alpha_{2,0} + \alpha_{2,1}\varepsilon_{t-1}^2 + \beta_2\sigma_{t-1}^2 \tag{19}$$

$$\varepsilon_t = Y_t - \left(p_1(t)\mu_1 + \left(1 - p_2(t)\right)\mu_2\right) \tag{20}$$

$$\sigma_t^2 = p_1(t)\left(\mu_1^2 + \sigma_{1,t}^2\right) + \left(1 - p_1(t)\right)\left(\mu_1^2 + \sigma_{1,t}^2\right) - \left(p_1(t)\mu_1 + \left(1 - p_2(t)\right)\mu_2\right)^2. \tag{21}$$

where $p_j(t)$ denotes the probability that the process is in state j at time t. This model has 10 parameters, making it more complex than the regime switching lognormal and regime switching draw down model, which have six and eight parameters respectively.

4. Regime Switching Residual Analysis

We begin this section by exploring the difficulties of defining residuals for regime switching models. We then show how a stochastic approach can solve these problems and finally demonstrate how stochastic residuals can be used to assess regime switching models.

Given the current regime ρ_t the residuals can be defined by

$$r_{\rho_t,t} = \frac{Y_t - \mu_{\rho_t,t}}{\sigma_{\rho_t,t}}.$$

Unfortunately the regime process is hidden, so although we can identify the set of possible residuals, we do not directly observe the residual as we do in the single regime models.

While the regime is not directly observable, we can make some inferences about the current regime. By conditioning on the observed information $Y_t, Y_{t-1}, \dots, Y_1$, we can determine the probability that the process is each possible regime. These probabilities are denoted $p_j(t) = P(\rho_t = j | Y_t, Y_{t-1}, \dots, Y_1)$. See Hardy (2001) for details on calculating these probabilities. There are several ways that we can use these probabilities to generate residual sets.

First is to use the $p_j(t)'s$ to calculate the expected mean and variance, which in the two regime case are

$$E\left[E[Y_t|\rho_t]\right] = E\left[\mu_{\rho_t,t}\right] = \mu_{1,t}\, p_1(t) + \mu_{2,t}\, p_2(t)$$

and

$$\begin{aligned} var(Y_t) &= E\left[var(Y_t|\rho_t)\right] + var(E[Y_t|\rho_t]) \\ &= \sigma_{1,t}^2 p_1(t) + \sigma_{2,t}^2 p_2(t) + \mu_{1,t} p_1(t)(1 - p_1(t)) + \mu_{2,t} p_2(t)(1 - p_1(t)). \end{aligned}$$

Which can be used to calculate standardized residuals given by

$$r_t^* = \frac{Y_t - (\mu_{1,t} p_1(t) + \mu_{2,t} p_2(t))}{\sqrt{\sigma_{1,t}^2 p_1(t) + \sigma_{2,t}^2 p_2(t) + \mu_{1,t} p_1(t)(1 - p_1(t)) + \mu_{2,t} p_2(t)(1 - p_1(t))}}.$$

This approach tends to produce residuals that are too small. For all the data points which lie near the middle of the two means, the resulting residuals will be close to zero. This is a particular problem when the two means are far apart, since the true residual should be large.

Since the actual residual is not observable, the next method is to simply calculate the expected residual, which in the two regime case is given by

$$r_t^W = E\left[r_{\rho_t,t}\right] = r_{1,t}\, p_1(t) + r_{2,t}\, p_2(t). \tag{22}$$

We call these the weighted residuals. Again the points between the two means create a problem, since in this case we are taking a weighted average between a positive and negative residual. This causes the residual to be understated, leading to thinner tails in the residuals than we would see if the regime process were known.

In Hardy, Freeland and Till (2006), we suggested the use of indicator residuals, defined by

$$r_t^I = r_{1,t} I_{\{p_1(t)>0.5\}} + r_{2,t} I_{\{p_2(t)>0.5\}}$$

which we found to produce far better residuals than the weighted residual method. The advantage of this approach is that most of the residual values are equal to the actual unobserved residuals. However, when the approach picks the wrong residual, the true residual will likely have a much larger magnitude and possibly a different sign. Once again, this process tends to generate residuals that are thinner tailed than the true model residuals.

The final approach is a small modification of the indicator approach that we call the stochastic approach. We use the probabilities for each regime to generate a stochastic residual set. The two regime stochastic residuals can be defined as

$$r_{\tau_t,t} = r_{1,t} I_{\{p_1(t)>\gamma_t\}} + r_{2,t} I_{\{p_2(t)>1-\gamma_t\}}$$

where γ_t is a random number between 0 and 1 and $\tau_t = 1 I_{\{p_1(t)>\gamma_t\}} + 2 I_{\{p_2(t)>1-\gamma_t\}}$. Note that τ_t is the randomly chosen regime that is going to be used at time t . Obviously a different set of random numbers $\{\gamma_1, \ldots, \gamma_n\}$ will result in a different set of chosen regimes $\{\tau_1, \ldots, \tau_n\}$ and a different realization of the stochastic residuals $\left\{r_{\tau_1,1}, \ldots, r_{\tau_n,n}\right\}$. What is remarkable about these stochastic residuals is that they can be shown to be independent identically distribution standard normal random variables, under the regime switching model assumption. See Freeland, Till and Hardy (2009) for details.

Since we can generate multiple sets of stochastic residuals, we need a method to collect together the information. A simple method is to calculate the average of the resampled sets of stochastic residuals. Doing this simply results in the weighted residuals given in Equation 22, which are not very useful, and are not normally distributed. However a more useful result can be obtained by sorting the stochastic residuals in each set and then averaging of the order statistics across sets. This set of averaged ordered residuals is normally distributed under the regime switching model assumptions.

The stochastic residuals can also be used to generate a stochastic version of the Jarque-Bera statistic (Jarque and Bera, 1980). This portmanteau statistic measures the skewness and excess kurtosis. The stochastic JB statistic can be assessed in the following.

- Calculate 1,000 realizations of the JB statistic, which are then ordered.
- Repeat the process 100 times and calculate the average of each order statistic.
- The averaged stochastic JB statistic can then be plotted against the values which would be expected for a standard normal distribution.

If the points line on the 45^o line then the skewness and excess kurtosis seen in the stochastic residuals is the same as that of a standard normal distribution, which indicates that the data is consistent with the regime switching model assumption.

5. Assessing the Regime Switching Models

In this section we will assess the models of Section 3. First we will examine the residuals of these models. In particular we examine the QQ plots of the averaged stochastic residuals and the QQ plots of the averaged stochastic Jarque-Bera statistic. In the second part we will compare out-of-sample forecasts of the models for the S&P 500. In particular, we compare the likelihood of the large market drop in 2008 due to the global recession.

5.1 Residual Based Model Assessment

Figure 1
Normal-Plot of the Averaged Stochastic Residuals MARCH(2;0,0;2,0)

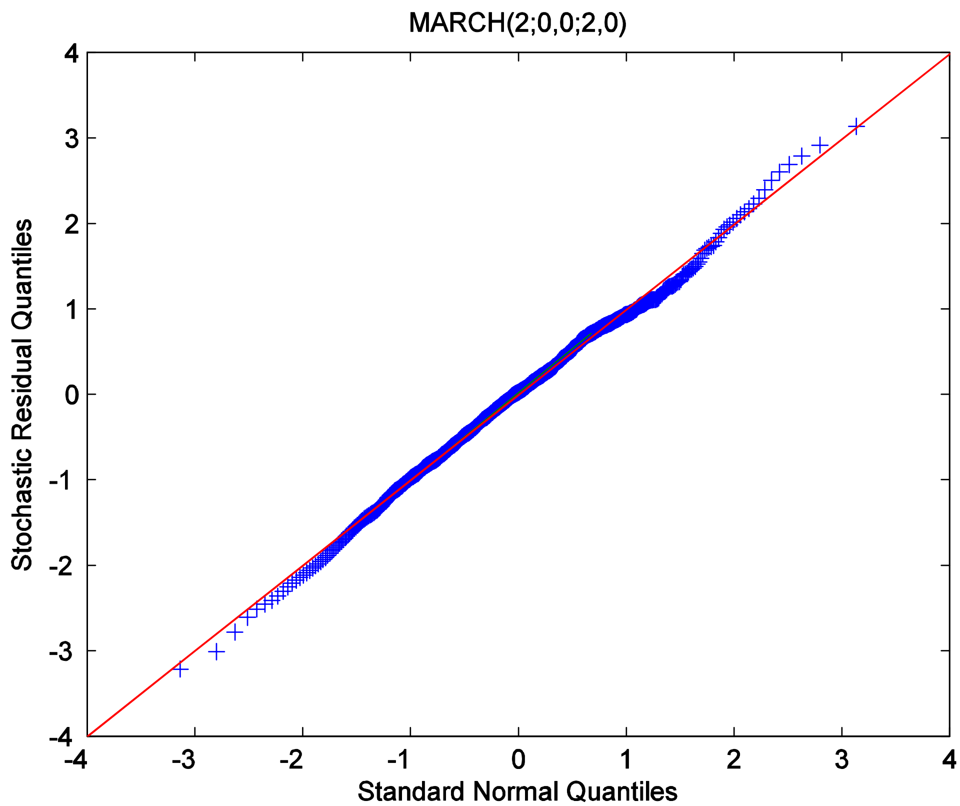


Figure 2
Normal-Plot of the Averaged Stochastic Residuals RSLN(2)

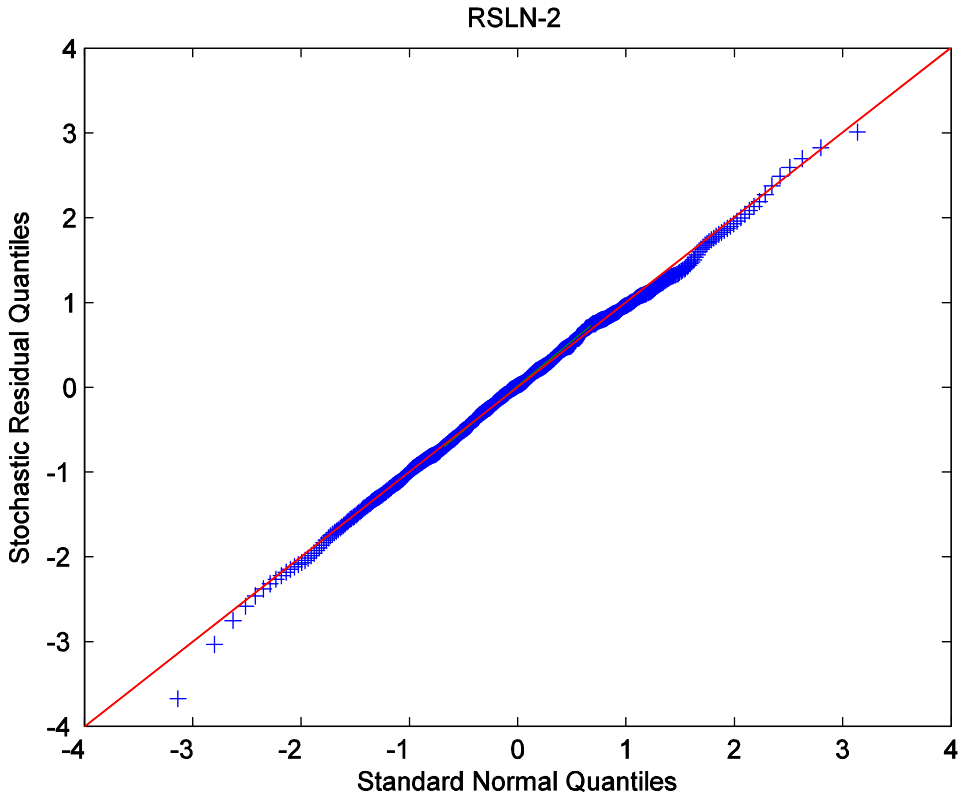


Figure 3
Normal-Plot of the Averaged Stochastic Residuals RSDD(2)

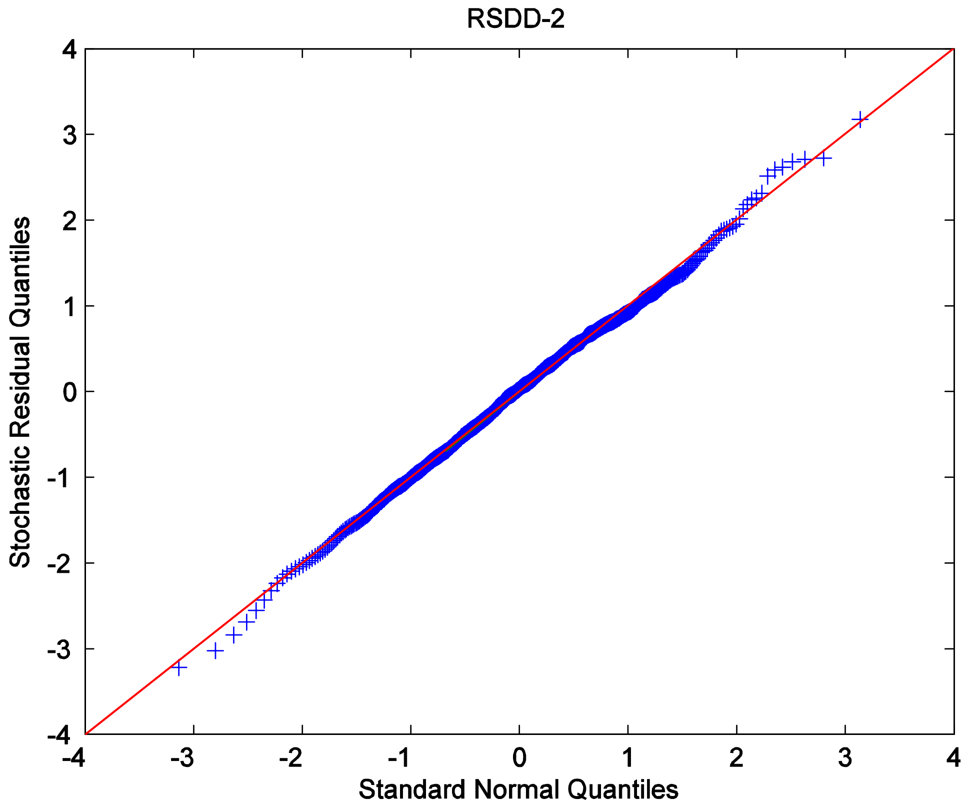


Figure 4
Normal-Plot of the Averaged Stochastic Residuals RSGARCH(2;1,1;1,1)

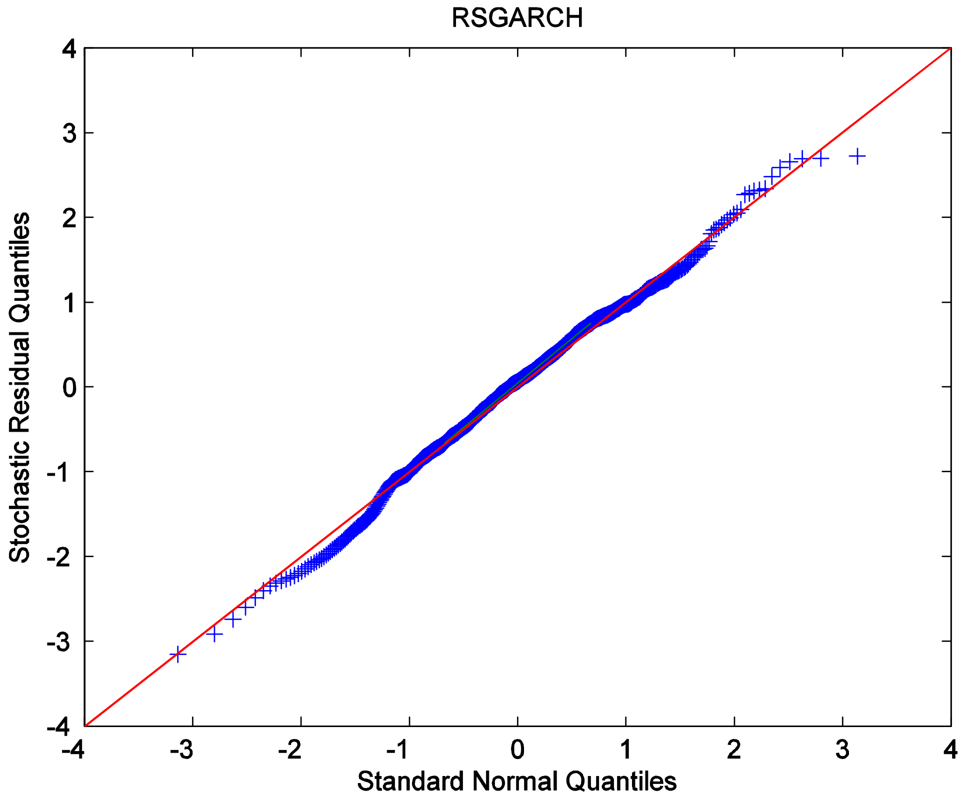


Figure 5
QQ-Plot of the Averaged Stochastic JB Statistic MARCH(2;0,0;2,0)

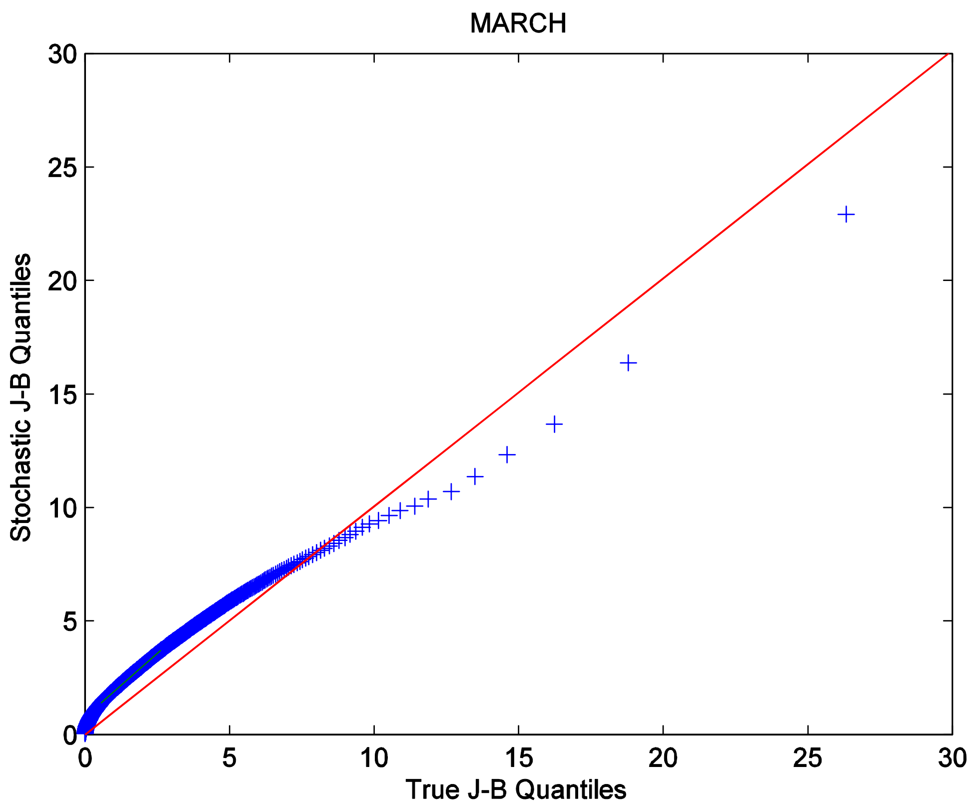


Figure 6
QQ-Plot of the Averaged Stochastic JB Statistic RSLN(2)

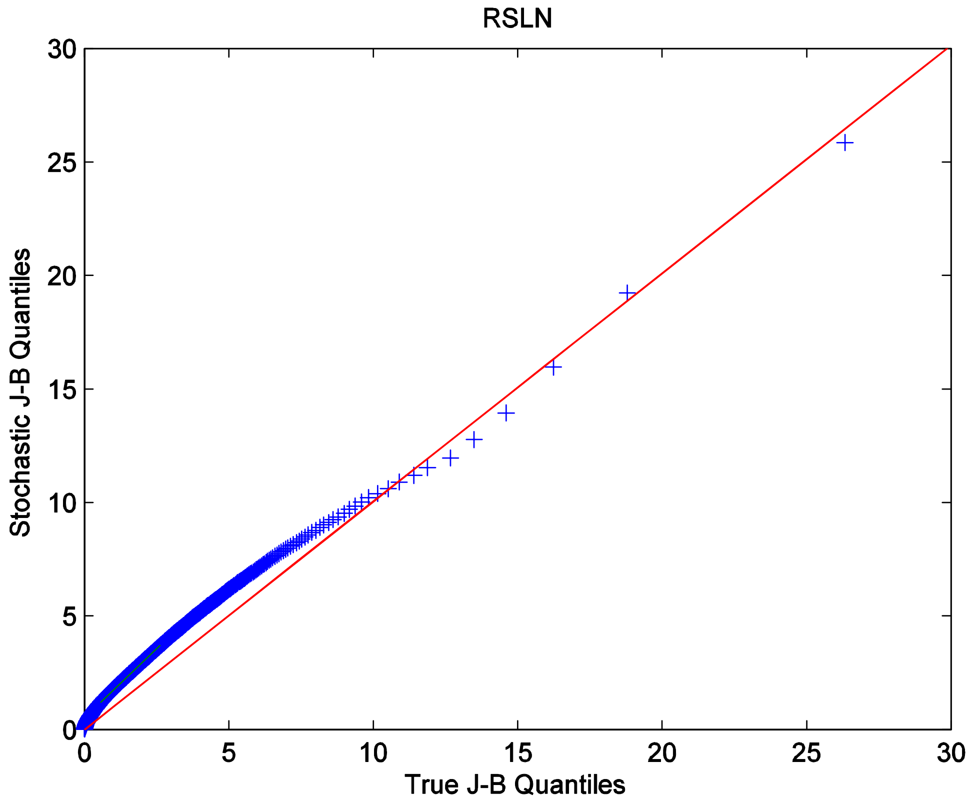


Figure 7
QQ-Plot of the Averaged Stochastic JB Statistic RSDD(2)

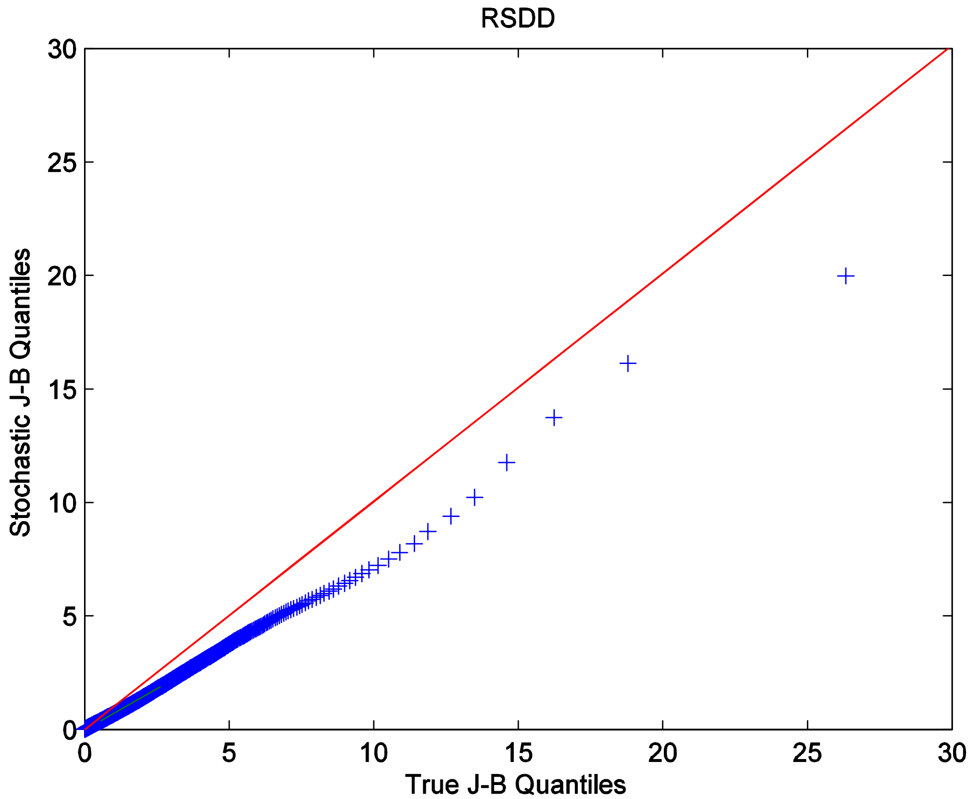
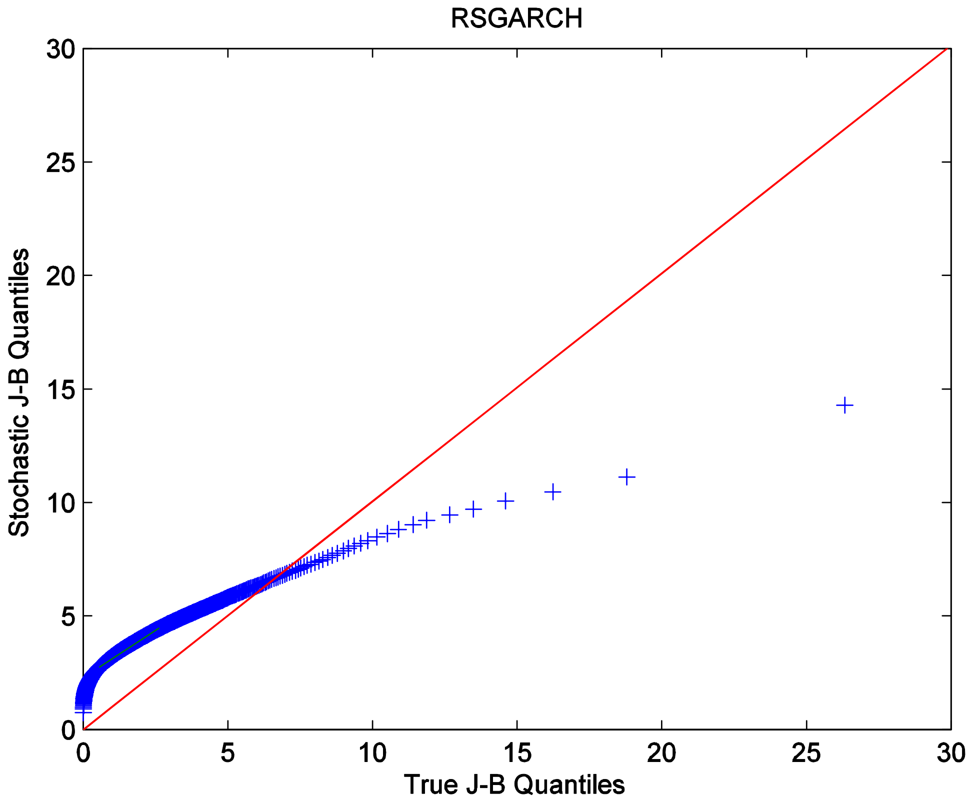


Figure 8
QQ-Plot of the Averaged Stochastic JB Statistic RSGARCH(2;1,1;1,1)



In this section, we look at both normal-quantile plots of the averaged stochastic residuals and QQ-plots of the averaged stochastic Jarque-Bera statistic. Plots are given for each of the models in Section 3.

The normal-quantile plots are given in Figures 1–4. Each of these plots shows a slight "S" shape, but tends to lie close to the 45° line. Of the four plots, the RSGARCH is the one with the most pronounced curves. Base on these plots alone, the models appear to have distributions similar to the normal distribution.

In Figures 5–8, we have given QQ-plots of the average Jarque-Bera statistic. This statistic compares the skewness and excess kurtosis that is found in the stochastic residuals to that which would be expected for a standard normal distribution. Ideally the plot should be a straight line lying on the 45° line.

For the MARCH model the line has an "S" shape. It starts above the 45° line and then crosses before 10. The plot for the RSLN model is similar except the line is straighter. It does not cross below the 45° line until after 10 and remains very close to the 45° line. The RSDD model has a straight line, but all of the points line below the 45° line. The RSGARCH model has the most curved line. It starts off almost vertically and crosses the 45° line before 10. Of the four models, the residuals of the RSLN model appear to most closely resemble the standard normal distribution.

5.2 Out-of-Sample Forecast

Figure 9
Fifty-One-Month Forecast from September 2004 to December 2008,
Percentiles Shown 5%, 10%, 50%, 90% and 95% ILN

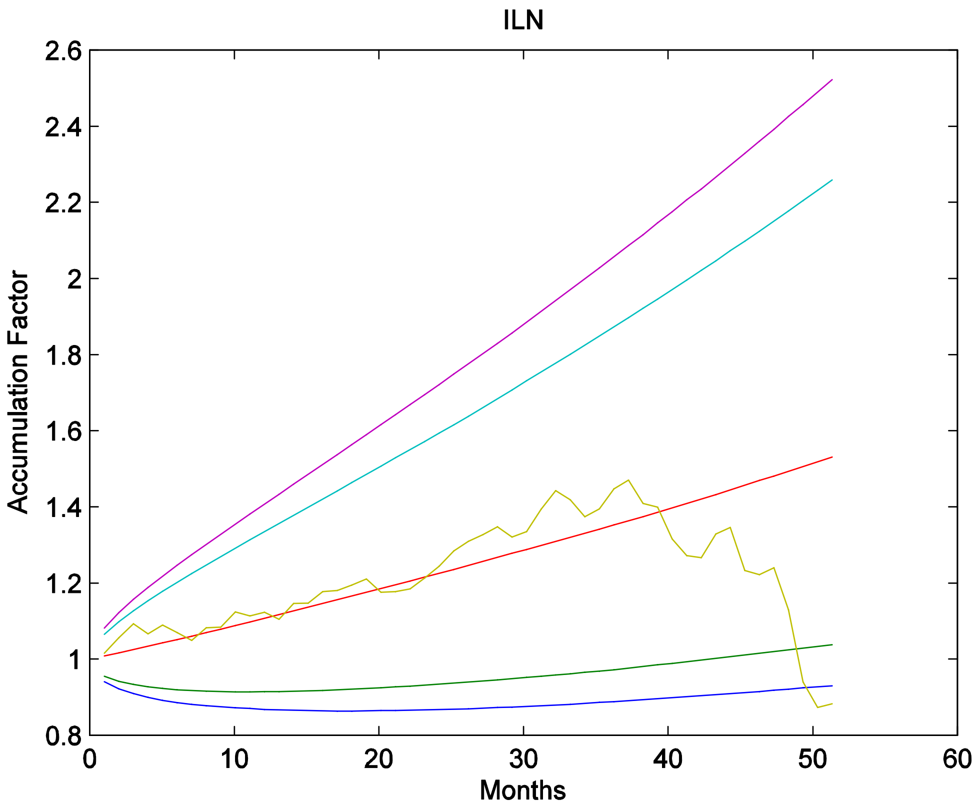
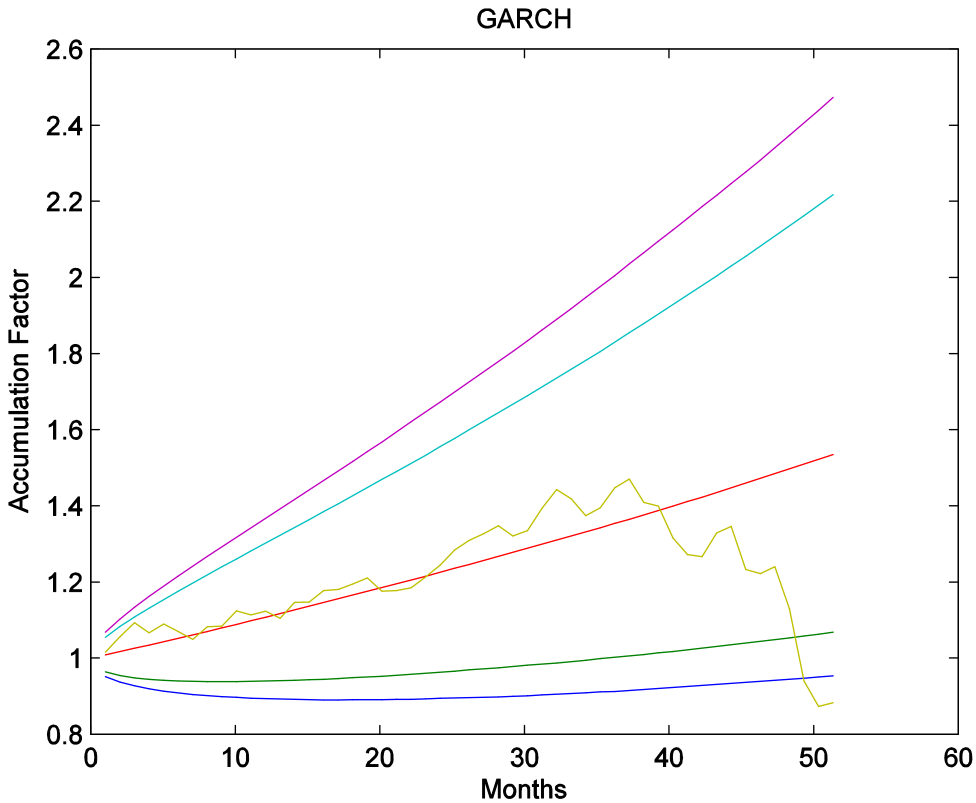


Figure 10
Fifty-One-Month Forecast from September 2004 to December 2008,
Percentiles Shown 5%, 10%, 50%, 90% and 95% RSLN(2)



In our 2006 paper, we described a number of models, each of which had advocates for use in actuarial economic capital calculations, particularly for variable annuity products. In this section, we see how some of these models performed, compared with the actual outcome of stock prices, which has been extraordinary in the past year. In particular, we are interested to see whether current equity prices were anticipated in any of these models.

We forecast each of the models from Sections 2 and 3. We take the parameter estimates from Hardy, Freeland and Till (2006), which were estimated using the S&P 500 data from January 1956 to September 2004. For the out-of-sample test, we forecast from the end of September 2004 until the end of December 2008. In 2008 the markets dropped dramatically. In eight of the 12 months the monthly log-returns were negative, in particular: January -6.19 percent, February -3.3 percent, June -8.8 percent, September -9.3 percent, October -18.4 percent and November -7.4 percent. Only one of the four positive returns, April 4.8 percent, was above average. As the economic numbers started to appear in the fall of 2008 it became clear that we were entering into a global recession. This time period is ideal for testing how well the models predicted the possibility of heavy losses in the equities market. The total accumulation over this time period is 0.8819. We are particularly interested in the probability of losses of this magnitude under our different models.

We begin by examining the forecast results from our single regime models, which can be seen in Figures 9 and 10. The plots show the $5^{th}, 10^{th}, 90^{th}$ and 95^{th} percentiles plus the actual accumulation over the period. For both models the actual accumulation drops below the 5^{th} percentile, with the final probabilities 3.43 percent and 2.95 percent, respectively, for the independent lognormal and GARCH(1,1) models.

Figure 11. Fifty-One-Month Forecast from September 2004 to December 2008, Percentiles Shown 5%, 10%, 50%, 90% and 95% MARCH(2;0,0;2,0)

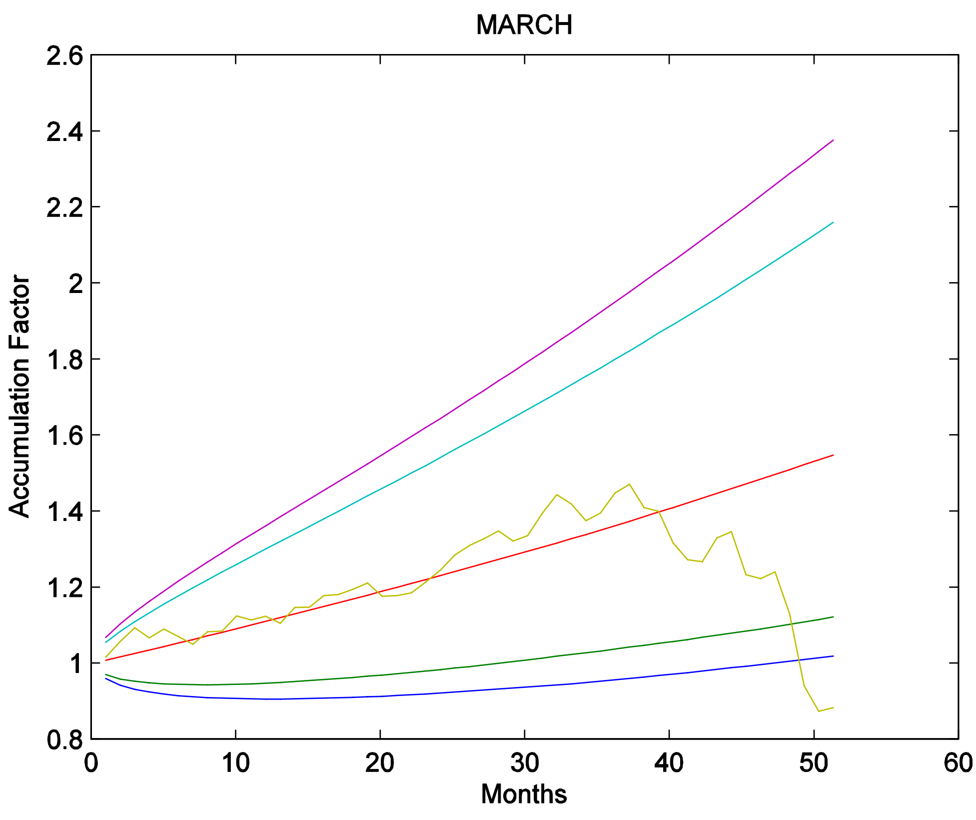


Figure 12

Fifty-One-Month Forecast from September 2004 to December 2008, Percentiles Shown 5%, 10%, 50%, 90% and 95% RSLN(2)

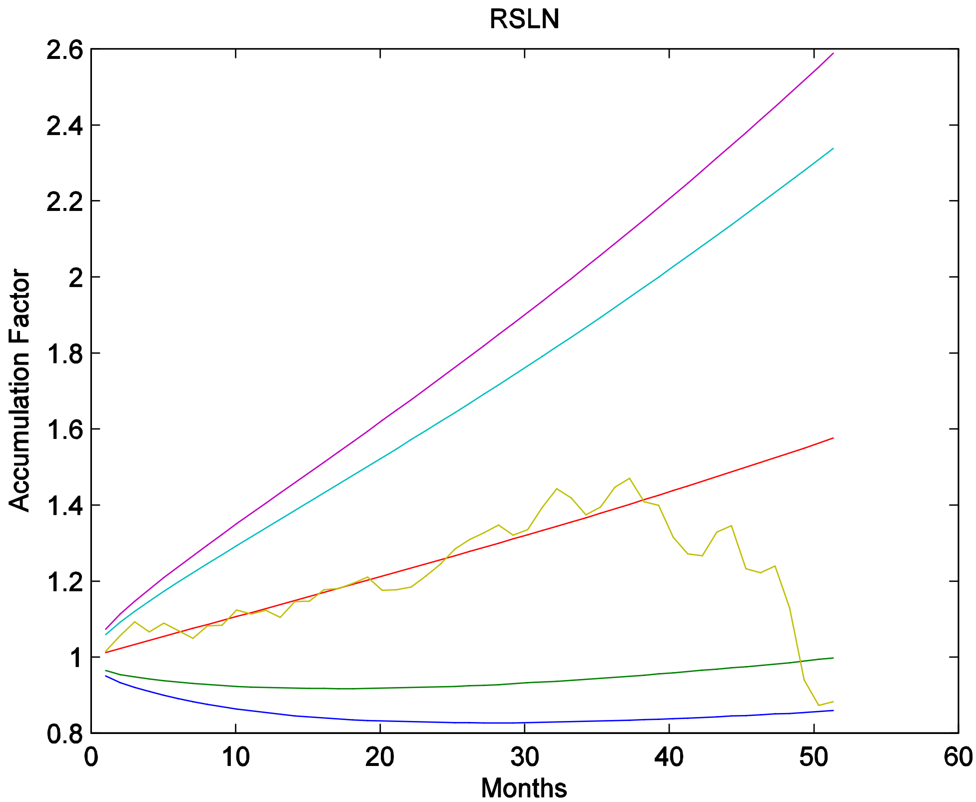


Figure 13

Fifty-One-Month Forecast from September 2004 to December 2008, Percentiles Shown 5%, 10%, 50%, 90% and 95% RSDD(2)

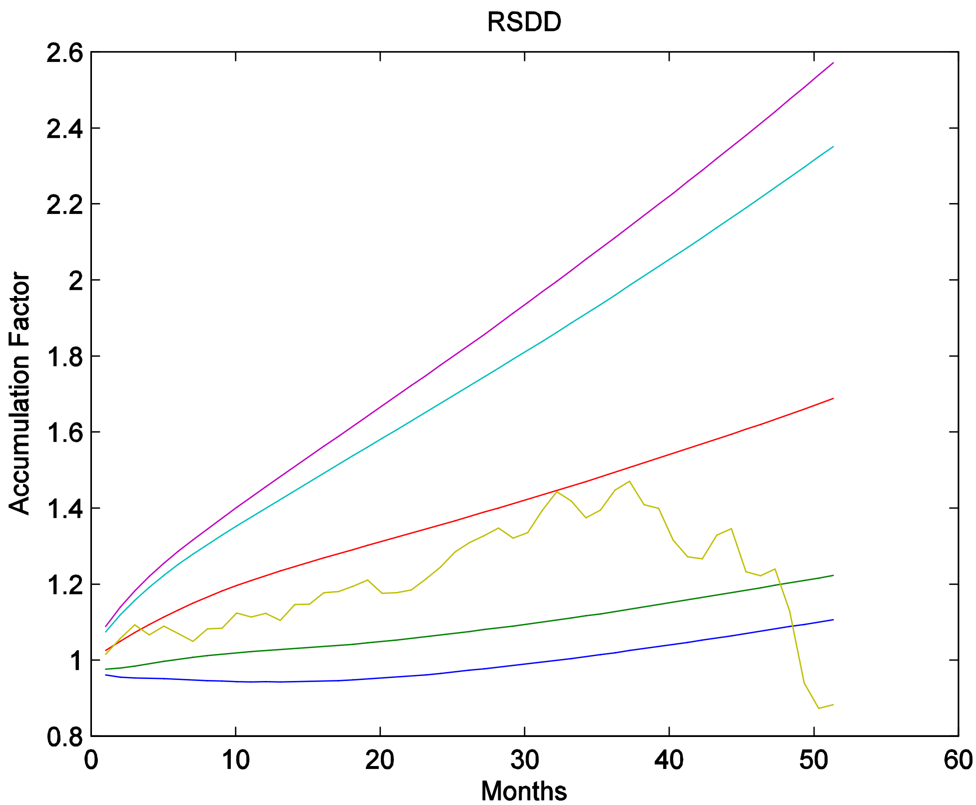
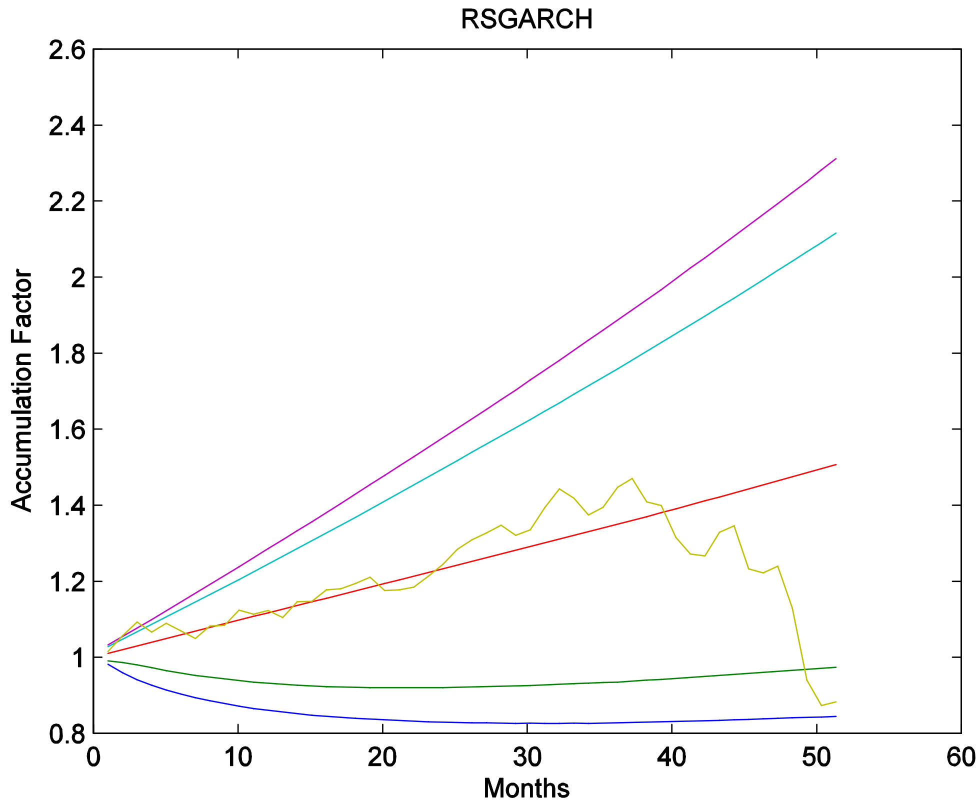


Figure 14

Fifty-One-Month Forecast from September 2004 to December 2008, Percentiles Shown 5%, 10%, 50%, 90% and 95% RSGARCH(2;1,1;1,1)



Next we look at the regime switching models. Since these models have more parameters and two sources of variation we would expect that they would be better able to reflect the left tail than the single regime models. However, we find the results to be mixed.

For the MARCH model we see an even worse representation of the left tail when compared to the single regime models. The resulting probability of reaching the observed accumulation by the end of 2008 is only 1.376 percent. This may be due to the fact the QQ-plot of the stochastic Jarque-Bera statistic shows that the residuals are not normally distributed under this model.The RSLN model performs better, with the observed accumulation remaining above the 5^{th} percentile. The resulting probability of reaching the observed accumulation by the end of 2008 is 5.65 percent. The results for the RSDD model are quite dramatic. Only two of the 51 accumulation points are above the median (November 2004 and December 2004). This model gives the smallest probability of reaching the observed accumulation by the end of 2008, 0.749 percent, and is the only model for which the experienced accumulation factor fell below the 1^{st} percentile of the model distribution. This may be due to the fact that all of the averaged stochastic JB statistics in Figure 13 fall below the 45^{o} line. The RSGARCH model is the most conservative model in the left tail and has probability of reaching the observed accumulation by the end of 2008 of 6.24 percent.

6. Conclusions

In this paper we illustrated the new stochastic residuals methodology for regime switching models. We showed how these residuals can be resampled and averaged to produce QQ-plots to test the model assumption of normality. Then we showed how these residuals can be used to generate stochastic Jarque-Bera statistics, which again can be averaged and plotted in a QQ-plot. The advantage of the stochastic Jarque-Bera statistic is that it tests the skewness and excess kurtosis of the results, which gives a better evaluation of tails of the distribution.

We demonstrated the performance of the different models in the severe out-of-sample test represented by the distribution of accumulation factors from September 2004 to December 2008, and concluded that only the RSLN and RSGARCH models indicated a probability of more than 5 percent of such poor returns over this period.

For future work we would like to see if it is possible to refine the method to see if the Jarque-Bera statistic can indicate specific problems in each tail.

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R. Keith Freeland, Ph.D., ASA, is an assistant professor in the Statistics and Actuarial Science Department of the University of Waterloo, Ontario, Canada. He can be reached at *rkfreeland@uwaterloo.ca*.

Mary Hardy, Ph.D., FSA, FIA, CERA, is CIBC professor of Financial Risk Management, in the Statistics and Actuarial Science Department of the University of Waterloo, Ontario, Canada. She can be reached at *mrhardy@uwaterloo.ca*.

Matthew Till is a graduate student in the Statistics and Actuarial Science Department of the University of Waterloo, Ontario, Canada. He can be reached at *matthew.till@hrsdc-rhdsc.gc.ca*.