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A JOINT-LIFE AND RESERVE PROBLEM

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The following problem is from the 1968 Society of Actuaries Part 4 Examination on Life Contingencies. It is an instructive problem because it interweaves joint-life functions with single-life reserves.

> A policy is issued on the joint lives of (x) and (y). Premiums are payable continuously until the time, t, of the first death. The death benefit is payable at the moment of such death, and is equal to
>
> $${}_t\bar{V}(\bar{A}_x)$$
>
> if (y) dies first, and
>
> $${}_t\bar{V}(\bar{A}_y)$$
>
> if (x) dies first. Determine the net annual premium for this policy in terms of single life and joint life net annual premiums.

We shall provide four solutions to this problem. Let π denote the net annual premium to be determined. For simplicity, we write d_tq_x for ${}_tp_x\mu_{x+t}dt$ and d_tq_y for ${}_tp_y\mu_{y+t}dt$. Applying the Equivalence Principle and assuming that (x) and (y) are independent lives, we have

$$\pi\bar{a}_{xy} = \int_0^\infty v^t {}_tp_x \, {}_t\bar{V}(\bar{A}_x)\, d_tq_y + \int_0^\infty v^t {}_tp_y \, {}_t\bar{V}(\bar{A}_y)\, d_tq_x.$$

Because of the symmetry between x and y, it is sufficient to evaluate just one of the two integrals on the right-hand side.

1 Reversionary Annuities

$$\int_0^\infty v^t {}_tp_x \, {}_t\bar{V}(\bar{A}_x)\, d_tq_y$$

$$
\begin{aligned}
&= \int_0^\infty v^t {}_tp_x \left[1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}\right] d_tq_y \\
&= \int_0^\infty v^t {}_tp_x \, d_tq_y - \frac{1}{\bar{a}_x}\int_0^\infty v^t {}_tp_x \bar{a}_{x+t} \, d_tq_y \\
&= \bar{A}_{\overset{1}{x}y} - \frac{1}{\bar{a}_x}\,\bar{a}_{y|x} \\
&= \bar{A}_{\overset{1}{x}y} - \frac{1}{\bar{a}_x}(\bar{a}_x - \bar{a}_{xy}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\pi\bar{a}_{xy} &= \bar{A}_{\overset{1}{x}y} - \frac{1}{\bar{a}_x}(\bar{a}_x - \bar{a}_{xy}) + \bar{A}_{x\overset{1}{y}} - \frac{1}{\bar{a}_y}(\bar{a}_y - \bar{a}_{xy}) \\
&= \bar{A}_{xy} - 2 + (\frac{1}{\bar{a}_x} + \frac{1}{\bar{a}_y})\bar{a}_{xy},
\end{aligned}
$$

or

$$
\begin{aligned}
\pi &= \bar{P}(\bar{A}_{xy}) - \frac{2}{\bar{a}_{xy}} + \frac{1}{\bar{a}_x} + \frac{1}{\bar{a}_y} \\
&= \bar{P}(\bar{A}_{xy}) - 2[\bar{P}(\bar{A}_{xy}) + \delta] + [\bar{P}(\bar{A}_x) + \delta] + [\bar{P}(\bar{A}_y) + \delta] \\
&= \bar{P}(\bar{A}_x) + \bar{P}(\bar{A}_y) - \bar{P}(\bar{A}_{xy}).
\end{aligned}
$$

2 Integration by Parts

$$
\begin{aligned}
&\int_0^\infty v^t {}_tp_x \, {}_t\bar{V}(\bar{A}_x) \, d_tq_y \\
&= \int_0^\infty {}_tE_x \, {}_t\bar{V}(\bar{A}_x) \, d_tq_y \\
&= {}_tE_x \, {}_t\bar{V}(\bar{A}_x) {}_tq_y|_0^\infty - \int_0^\infty {}_tq_y \, d\left[{}_tE_x \, {}_t\bar{V}(\bar{A}_x)\right] \\
&= -\int_0^\infty {}_tq_y \, d\left[{}_tE_x \, {}_t\bar{V}(\bar{A}_x)\right].
\end{aligned}
$$

To evaluate the differential in the last integral, we apply the method of integrating factors to the reserve differential equation

$$
d_t\bar{V} = \pi_t \, dt + \delta_t \bar{V} dt - (b_t - {}_t\bar{V})\mu_{x+t} dt
$$

to obtain

$$
d\left({}_tE_x \, {}_t\bar{V}\right) = {}_tE_x \pi_t \, dt - {}_tE_x \, b_t \, \mu_{x+t} dt,
$$

from which and the above it follows that

$$
\begin{aligned}
& \int_0^\infty v^t {}_tp_x \, {}_t\bar{V}(\bar{A}_x) \, d_tq_y \\
= & -\int_0^\infty {}_tq_y \, {}_tE_x \bar{P}(\bar{A}_x) dt + \int_0^\infty {}_tq_y \, {}_tE_x \, \mu_{x+t} dt \\
= & -\bar{P}(\bar{A}_x) \left[\bar{a}_x - \bar{a}_{xy}\right] + \bar{A}_{xy}^{2} \\
= & \bar{P}(\bar{A}_x)\bar{a}_{xy} - \bar{A}_{xy}^{1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\pi \bar{a}_{xy} &= \bar{P}(\bar{A}_x)\bar{a}_{xy} - \bar{A}_{xy}^{1} + \bar{P}(\bar{A}_y)\bar{a}_{xy} - \bar{A}_{x\overset{1}{y}} \\
&= [\bar{P}(\bar{A}_x) + \bar{P}(\bar{A}_y)]\bar{a}_{xy} - \bar{A}_{xy},
\end{aligned}
$$

or

$$
\pi = \bar{P}(\bar{A}_x) + \bar{P}(\bar{A}_y) - \bar{P}(\bar{A}_{xy}).
$$

3 Integration by Parts Again

It follows from the identity

$$
{}_tq_y = 1 - {}_tp_y
$$

that

$$
\begin{aligned}
& \int_0^\infty v^t {}_tp_x \, {}_t\bar{V}(\bar{A}_x) \, d_tq_y \\
= & -\int_0^\infty v^t {}_tp_x \, {}_t\bar{V}(\bar{A}_x) \, d_tp_y \\
= & -{}_tE_x \, {}_t\bar{V}(\bar{A}_x) {}_tp_y |_0^\infty + \int_0^\infty {}_tp_y \, d\left[{}_tE_x \, {}_t\bar{V}(\bar{A}_x)\right] \\
= & \int_0^\infty {}_tp_y \, d\left[{}_tE_x \, {}_t\bar{V}(\bar{A}_x)\right] \\
= & \int_0^\infty {}_tp_y \, {}_tE_x \bar{P}(\bar{A}_x) dt - \int_0^\infty {}_tp_y \, {}_tE_x \, \mu_{x+t} dt \\
= & \bar{P}(\bar{A}_x) \int_0^\infty v^t \, {}_tp_{xy} dt - \bar{A}_{xy}^{1} \\
= & \bar{P}(\bar{A}_x)\bar{a}_{xy} - \bar{A}_{xy}^{1}.
\end{aligned}
$$

Now the result follows from the end of last section.

4 Method of Undetermined Coefficients

Assume that the net annual premium for this policy is of the form

$$\pi = \alpha \bar{P}(\bar{A}_x) + \beta \bar{P}(\bar{A}_y) + \gamma \bar{P}(\bar{A}_{xy}),$$

where the coefficients α, β and γ are to be determined. By the symmetry between x and y, we have

$$\alpha = \beta$$

and hence

$$\pi = \alpha[\bar{P}(\bar{A}_x) + \bar{P}(\bar{A}_y)] + \gamma \bar{P}(\bar{A}_{xy}).$$

For an insurance policy with premiums prescribed by the right-hand side of this equation, let us examine the benefit payments, including surrender values, upon the first death. Should (x) die before (y), the total payment would be $\alpha[1 + {}_{T(x)}\bar{V}(\bar{A}_y)] + \gamma$. Should (y) die before (x), the total payment would be $\alpha[{}_{T(y)}\bar{V}(\bar{A}_x) + 1] + \gamma$. Thus, we have the following equations:

$$\begin{aligned} {}_{T(x)}\bar{V}(\bar{A}_y) &= \alpha[1 + {}_{T(x)}\bar{V}(\bar{A}_y)] + \gamma \\ {}_{T(y)}\bar{V}(\bar{A}_x) &= \alpha[{}_{T(y)}\bar{V}(\bar{A}_x) + 1] + \gamma. \end{aligned}$$

It follows from either equation that $\alpha = 1$ and $\gamma = -1$.

* * * * *

After he read the above, Professor Hans Gerber asked whether the independence assumption was necessary. In the method of undetermined coefficients, the independence assumption was not used. In the first method, we can avoid making the independence assumption by using the differential $d_t q_{xy}^{\;1}$ for ${}_t p_x d_t q_y$.

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