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ON THE DERIVATION OF DISCRETE INTERPOLATION FORMULAS

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INTRODUCTION

In recent years a great deal has been published in our actuarial journals on interpolation, and a great many different formulas have been proposed. Therefore, I wish to reassure the reader at the very outset by stating that no new formulas are presented in this paper, but merely a somewhat new method of deriving some of those already in print. This method is limited in its application to discrete interpolation formulas, by which is meant those which are to be used only to subdivide the intervals between given values into a specified number of equal parts. An example is the case frequently arising in actuarial work in which the values of some function depending on age are given for ages at intervals of five years and it is desired to interpolate to obtain the values by single integral years of age but there is no need for any values at fractional ages. Thus, the term is used in contradistinction to continuous interpolation formulas, of which the osculatory and tangential formulas are examples, since the assumptions made in deriving these formulas imply the existence of a continuous curve which yields values of the function being interpolated for all values of the independent variable. It is my purpose to show that the derivation of such formulas can, in many cases, be considerably simplified by employing an interesting analogy between interpolation and graduation formulas, with the help of certain mathematical devices.

The analogy between interpolation and graduation formulas, which is the cornerstone of the method, is not original with me, but is due to the brilliant Australian actuary, Mr. Hubert Vaughan, who presented it in a recent paper entitled "Some Notes on Interpolation" (*JIA* LXXII, 482). The mathematical devices which facilitate the application of Mr. Vaughan's principle to the derivation of discrete interpolation formulas were suggested in large part by the work of Professor I. J. Schoenberg, whose valuable work ("Contributions to the Problem of Approximation of Equidistant Data by Analytic Functions," *Quarterly of Applied Mathematics*, IV, 45 and 112) I have summarized elsewhere (*Journal of the American Statistical Association*, XLIII, 428).

ANALOGY BETWEEN INTERPOLATION AND GRADUATION FORMULAS

A discrete interpolation formula should be thought of not as an algebraic expression, but as a table of linear compound coefficients such as have appeared, for example, in the papers of Mr. Boyer (*RAIA* XXXI, 337) and Mr. Beers (*RAIA* XXXIII, 245 and XXXIV, 14). As an illustration* we may take the coefficients given in Table 1, which are those subdividing into five parts the intervals between four equally spaced given values, using Waring's (Lagrange's) formula applied centrally to three consecutive given values. This is, of course, equivalent to an ordinary central difference formula taken to second differences.

TABLE 1

COEFFICIENTS FOR SUBDIVIDING INTO FIVE PARTS THE INTERVALS BETWEEN EQUALLY SPACED GIVEN VALUES, USING ORDINARY CENTRAL DIFFERENCE INTERPOLATION TO SECOND DIFFERENCES

x	COEFFICIENTS OF u_x TO OBTAIN:				
	$u_{n-.4}$	$u_{n-.2}$	u_n	$u_{n+.2}$	$u_{n+.4}$
$n-1$	.28	.12	0	−.08	−.12
n	.84	.96	1	.96	.84
$n+1$	−.12	−.08	0	.12	.28

Let us now think of the successive coefficients by which a particular given value is multiplied in computing the successive interpolated values. For example, in the case of some function of age for which the given values are separated by intervals of five years, we may consider the successive coefficients by which the value for age 20 is multiplied in obtaining the interpolated values for ages 13 to 27, inclusive. Now, if we set down these coefficients in a single row, as follows:

$$-.12,\ -.08,\ 0,\ .12,\ .28,\ .84,\ .96,\ 1,\ .96,\ .84,\ .28,\ .12,\ 0,\ -.08,\ -.12,$$

it will be noted that their progression is very similar to that of the coefficients in a linear compound graduation formula. In fact, it will be found that they are precisely five times the coefficients of Woolhouse's graduation formula expressed in linear compound form. If some other interpolation formula is taken as the starting point, the result will be similar. Again it will turn out that the interpolation coefficients, after division by a constant to make their sum unity, can be used as the coefficients of a graduation formula, even though in general it will not be one of the recognized formulas to which names have been given.

* This illustration is due to Mr. Vaughan.

To express these ideas more precisely, a discrete interpolation formula for dividing the intervals between given values into m parts may be written in the form:

$$v_n = \sum_{k=-\infty}^{\infty} L_{n-km} u_{km} , \tag{1}$$

where the u's are the given values, the v's are the interpolated values, and the L's are the interpolation coefficients. In practice, of course, the summation is between finite limits, the coefficients L being regarded as zero outside a certain range.

Now, these coefficients, if divided by m, may be taken as the coefficients of a linear compound graduation formula. Thus,

$$V_n = \frac{1}{m} \sum_{t=-\infty}^{\infty} L_{n-t} U_t , \tag{2}$$

where the U's are the ungraduated, and the V's, the graduated values. In fact, as Mr. Vaughan shrewdly points out, if the series of ungraduated values is formed by placing the given values u_{km} in every mth position and inserting zeros in the remaining positions, then the graduation formula (2), omitting the factor $1/m$, will give the complete series of interpolated values v. For example, if the series:

$$\ldots , 0, 0, 0, 0, u_{15}, 0, 0, 0, 0, u_{20}, 0, 0, 0, 0, u_{25}, 0, 0, 0, 0, \ldots$$

is graduated by Woolhouse's formula, the smoothed series so obtained, after multiplication by five, is identical with that arrived at by replacing the zeros by the interpolated values for the intermediate ages computed by central difference interpolation to second differences. In general, we may write:

$$v_n = \sum_{t=-\infty}^{\infty} L_{n-t} U_t , \tag{3}$$

where

$$U_{km} = u_{km} \qquad (k = \ldots -1, 0, 1, \ldots)$$

$$U_n = 0 \qquad (n \neq km) .$$

Of course, in the computation of a particular interpolated value, only every mth coefficient L would actually be used, as the others are multiplied by zero terms in the series.

Alternatively, we could retain the factor $1/m$ and multiply the U's by m; and this form of the formula, while less convenient algebraically, has an interesting philosophical interpretation. If the extent of our information about a certain series of values consisted in knowing every mth value, we might very tentatively think of m consecutive terms of the series as being concentrated at the position of each given value, the intermediate terms being zero. Each given value would then need to be multiplied by m, since it would represent m values. The interpolation would then take the form of a graduation applied for the purpose of spreading out over the entire series the magnitudes which are temporarily concentrated at every mth position.

EQUIVALENT PROPERTIES OF GRADUATION AND INTERPOLATION FORMULAS

Since Mr. Vaughan has shown that a discrete interpolation formula is equivalent to applying a graduation formula to a specially constructed series, it follows that the various properties by which such interpolation formulas are usually described have as their counterparts certain specific properties of the equivalent graduation formulas. For example, if the interpolation formula reproduces the given values, this is equivalent to having

$$L_0 = 1 \; , \qquad L_{km} = 0 \qquad (k \neq 0) \, .$$

Obviously, in the case of smoothing (or "modified") interpolation formulas, this condition is not fulfilled.

If the number of terms in the interpolation formula (that is, the number of coefficients L actually used in the computation of each interpolated value) is h, then the maximum total number of coefficients is mh or $mh - 1$, according as h is odd or even. These two cases correspond to what have been termed midpoint and endpoint interpolation formulas, respectively. The subtraction of one in the latter case is due to the fact that symmetry requires the use of an odd number of coefficients L in computing the "interpolated" values at the points where the given values are situated. The case in which h is odd and m is even is peculiar, since the total number of coefficients is even, and, as a result, the subscripts of the L's are $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, etc., rather than integers. This means that the interpolated series does not contain terms corresponding to the given values, each of which is situated midway between two interpolated values.

VAUGHAN'S PRINCIPLE

Another important property of an interpolation formula is the order of differences to which it is correct, or, in other words, the degree of polynomial reproduced by it. The corresponding property of the equivalent graduation formula is not obvious, and its discovery represents Mr. Vaughan's distinctive contribution to the subject. This relationship is, as the reader will presently see, so fundamental in the theory of discrete interpolation that I propose naming it *Vaughan's principle*. In order to state the principle as precisely as our present purpose requires, it is necessary to define a concept not mentioned explicitly by Mr. Vaughan, and, in fact, introduced by Professor Schoenberg. The interpolation formula (1) is said to *preserve the degree r* if it has the property that if u_{km} is taken as a polynomial in km of degree not exceeding r, then v_n is a polynomial in n also of degree not exceeding r. This condition is weaker than the usual requirement of correctness to rth differences, or, in other words, of actually reproducing a polynomial of degree r. It means that if the values of a specified polynomial are used as given values in the formula, the resulting interpolated values are the corresponding values of a definite polynomial, which is of the same degree as the original polynomial, but not necessarily identical with it. However, the coefficients of the various powers of the independent variable in the second polynomial are definitely determined by those of the given polynomial.

It will be convenient to represent the graduation formula (3) in the symbolic form

$$v_n = G\,U_n\,,$$

where G is an operator given by

$$G \equiv \Sigma_t L_t E^{-t}, \tag{4}$$

the summation being over integral values of t or over values which are odd multiples of $\frac{1}{2}$, as the case may be. An operator of the form (4) will be called a linear compound operator. A particular example which we shall have occasion to use is the usual summation operator

$$[m] \equiv E^{-(m-1)/2} + E^{-(m-3)/2} + \ldots + E^{(m-1)/2}.$$

We can now state the following theorem:

A necessary and sufficient condition that a discrete interpolation formula for subdivision of the interpolation interval into m parts shall preserve the degree r is that the operator G be of the form $[m]^{r+1}H$, where H is any linear compound operator.

This is the first part of Vaughan's principle. In order to prove the sufficiency of the condition, we first note that $[m]\,(E^{1/2} - E^{-1/2}) \equiv E^{m/2} - E^{-m/2}$, or, in other words, $\delta[m] \equiv \delta_m$, where δ is the usual central difference operator and δ_m is a similar operator based on an interval of m units. Now suppose the interpolation formula is such that the operator G satisfies the condition stated in the theorem, and let u_{km} be taken as a polynomial in km of degree not exceeding r. Then

$$\delta^{r+1} v_n = \delta^{r+1} G\,U_n = \delta^{r+1}\,[m]^{\,r+1} H\,U_n = H\,\delta_m^{r+1}\,U_n\,.$$

The last of these expressions is zero for all values of n; for, if n is a multiple of m, $\delta_m^{r+1}U_n$ is an $(r+1)$th difference of u_{km}, a polynomial of degree r or less, while, for other values of n, it is an $(r+1)$th difference of a series consisting entirely of zeros. Therefore, $\delta^{r+1}v_n = 0$ for all values of n. It follows that v_n, since all its $(r+1)$th differences vanish, is a polynomial in n of degree not exceeding r.

To prove the necessity of the condition, suppose that the interpolation formula (1) does preserve the degree r. Differencing this formula gives

$$\delta^{r+1}v_n = \sum_{k=-\infty}^{\infty} \delta^{r+1}L_{n-km}u_{km} = \sum_{k=-\infty}^{\infty} (\delta^{r+1}L_{n-km})E^{-n+km}u_n\ .$$

Since the degree r is preserved, it follows that if u_n is any polynomial of degree r or less, v_n is also a polynomial of degree r or less, and hence $\delta^{r+1}v_n = 0$. It follows that the operator $\sum_{k=-\infty}^{\infty} (\delta^{r+1}L_{n-km})E^{-n+km}$ must give zero as the result when applied to *any* polynomial u_n of degree r or less. However, this can be the case only if this operator contains δ_m^{r+1} as a factor. The factor δ_m^{r+1} must appear, and not merely δ^{r+1}, since the operator in question is a sum of powers of E with exponents proceeding by steps of m. By giving n all possible values, we shall obtain m different operators of this form, each of which must contain δ_m^{r+1} as a factor. There will be only m different operators; since, for values of n differing by a multiple of m, the same operator will be obtained. It follows that the sum of the m operators contains δ_m^{r+1} as a factor. But this sum is

$$\Sigma_t(\delta^{r+1}L_t)E^{-t} \equiv \delta^{r+1}G\ ,$$

using the identity (4). Therefore,

$$\delta^{r+1}G \equiv \delta_m^{r+1}H$$

where H is a linear compound operator. In other words,

$$G \equiv \frac{\delta_m^{r+1}}{\delta^{r+1}}H \equiv [m]^{r+1}H\ .$$

This completes the proof.

In practice, we are not satisfied with mere preservation of a given degree, but require actual reproduction of polynomials of the specified degree: that is, correctness of the formula to the corresponding order of differences. This need is met by the following theorem, which constitutes the second part of Vaughan's principle:

A necessary and sufficient condition that a discrete interpolation formula shall be correct to rth differences is that it shall preserve the degree r and, at *the same time, the equivalent graduation formula shall be correct to rth differences.*

To prove the necessity of the condition, we first observe that if the interpolation formula has the property of reproducing a polynomial of degree r, then it certainly preserves the degree r. Moreover, the value of v_n given by the graduation formula (2) is the average of m "interpolated" values, represented by the m parts into which the summation can be broken up by taking only every mth term in each case, but starting at different places. Now, if the interpolation formula is correct to rth differences and if U_t is taken as a polynomial in t of degree r or less, then this polynomial will be reproduced, and the m "interpolated" values will be identical, each being equal to the value of the given polynomial for the argument n. Hence, their average has, of course, this same value. This means that if the graduation formula is applied to a series of values of a given polynomial of degree r or less, the graduated values will be identical with the given values. This is merely another way of saying that the graduation formula is correct to rth differences.

On the other hand, suppose the interpolation formula is merely known to preserve the degree r. This means that if the given values u_n are values of some specified polynomial in n of degree r or less, the formula will give as interpolated values the values of some other polynomial v_n, also of degree r or less. Now, if instead of taking $\ldots, u_{(k-1)m}, u_{km}, u_{(k+1)m}, \ldots,$ as given values, we take $\ldots, u_{(k-1)m+1}, u_{km+1}, u_{(k+1)m+1}, \ldots,$ this is equivalent to replacing n by $n + 1$ in the expression for the polynomial u_n. We are applying the same coefficients "L" to an expression of the same mathematical form, and the result must be a polynomial for "v" with the same coefficients, with n replaced by $n + 1$. Hence, both sets of given values will yield the same interpolated value v_n for a specified value of n. A similar observation applies if n is replaced by $n + 2$, $n + 3$, etc. In other words, if u_n is a polynomial in n of degree r or less, then the m interpolated values v_n, computed from the m possible sets of given values, are identical. It has already been shown that the graduated value, which is obtained by applying the equivalent graduation formula to the complete series of values of u_n, is the average of the m interpolated values. Since the latter are all identical, they must, therefore, be equal to the graduated value. But, if the hypothesis includes the statement that the graduation formula is correct to rth differences, the graduated value is the value of the "u" polynomial for the argument n. It follows that the interpolation formula also is correct to rth differences. Vaughan's principle is now completely established.

APPLICATION OF VAUGHAN'S PRINCIPLE

From an algebraic standpoint, correctness of the graduation formula to rth differences means that $V_n - U_n$ is an expression involving only $\delta^{r+1}U_n$ and higher differences of U_n; in other words, the operator $G - m$ has δ^{r+1} as a factor. Stirling's interpolation formula gives:

$$E^n \equiv 1 + n\mu\delta + \frac{n^2}{2}\delta^2 + \frac{n(n^2-1)}{6}\mu\delta^3 + \frac{n^2(n^2-1)}{24}\delta^4 + \frac{n(n^2-1)(n^2-4)}{120}\mu\delta^5 + \frac{n^2(n^2-1)(n^2-4)}{720}\delta^6 + \ldots$$

By means of this substitution, any linear compound operator may be expanded in terms of 1, $\mu\delta$, δ^2, $\mu\delta^3$, etc. In particular,

$$\left.\begin{aligned}\gamma_n \equiv E^n + E^{-n} \equiv 2 + n^2\delta^2 + \frac{n^2(n^2-1)}{12}\delta^4 \\ + \frac{n^2(n^2-1)(n^2-4)}{360}\delta^6 + \ldots\end{aligned}\right\} \quad (5)$$

On the other hand, Bessel's formula gives:

$$E^n \equiv \mu + n\delta + \tfrac{1}{2}(n^2 - \tfrac{1}{4})\mu\delta^2 + \tfrac{1}{6}n(n^2 - \tfrac{1}{4})\delta^3 + \tfrac{1}{24}(n^2 - \tfrac{1}{4})(n^2 - \tfrac{9}{4})\delta^4 + \tfrac{1}{120}n(n^2 - \tfrac{1}{4})(n^2 - \tfrac{9}{4})\delta^5 + \ldots,$$

whence

$$\delta_n \equiv E^{n/2} - E^{-n/2} \equiv n\delta + \frac{n(n^2-1)}{24}\delta^3 + \frac{n(n^2-1)(n^2-9)}{1920}\delta^5 + \ldots,$$

and

$$[n] \equiv \frac{\delta_n}{\delta} \equiv n + \frac{n(n^2-1)}{24}\delta^2 + \frac{n(n^2-1)(n^2-9)}{1920}\delta^4 + \ldots$$

If a discrete interpolation formula is to be correct to rth differences, the first few terms in the Stirling expansion of H must be such that $[m]^{r+1}H - m$ is divisible by δ^{r+1}. Thus, we must have

$$G \equiv [m](1 + \ldots)$$

for a formula correct to zeroth differences;

$$G \equiv \frac{[m]^2}{m}(1 + \ldots)$$

for a formula correct to first differences;

$$G \equiv \frac{[m]^3}{m^2}\left(1 - \frac{m^2-1}{8}\delta^2 + \ldots\right)$$

for a formula correct to second differences;

$$G \equiv \frac{[m]^4}{m^3}\left(1 - \frac{m^2-1}{6}\delta^2 + \dots\right)$$

for a formula correct to third differences;

$$G \equiv \frac{[m]^5}{m^4}\left[1 - \frac{5\,(m^2-1)}{24}\delta^2 + \frac{(m^2-1)(9m^2-1)}{384}\delta^4 + \dots\right]$$

for a formula correct to fourth differences; and

$$G \equiv \frac{[m]^6}{m^5}\left[1 - \frac{m^2-1}{4}\delta^2 + \frac{(m^2-1)(4m^2-1)}{120}\delta^4 + \dots\right]$$

for a formula correct to fifth differences.

An interesting illustration is the case of ordinary finite difference interpolation to rth differences, as exemplified by Newton's advancing difference formula or any of the other equivalent formulas. In the case of interpolation to rth differences, the number of coefficients in the linear compound expression for a particular interpolated value is $r + 1$; and, therefore, the total number of coefficients is $m(r + 1)$ if r is even, or $mr + m - 1$ if r is odd. On the other hand, the number of coefficients in the expanded form of $[m]^{r+1}$ is $m + r(m - 1) = mr + m - r$; and the number added by those terms of H which are determined by Vaughan's principle is r if r is even, or $r - 1$ if r is odd. This makes a total of precisely $mr + m$ if r is even, or $mr + m - 1$ if r is odd, as before. This result is to be expected, as these formulas are unique for their respective spans and the orders of differences to which they are correct. This means that by limiting the expressions given in the preceding paragraph to the terms which actually appear there, we obtain the graduation operators which are equivalent to ordinary finite difference interpolation. For example, ordinary fourth difference interpolation for the purpose of subdividing the interval into five parts is represented by

$$G \equiv \frac{[5]^4}{625}(1 - 5\,\delta^2 + 14\,\delta^4)\,.$$

MANIPULATION OF LINEAR COMPOUND OPERATORS

In applying Vaughan's principle to derive special formulas, certain general properties of linear compound operators will be needed. Consider an arbitrary linear compound operator

$$C \equiv \Sigma_t L_t E^{-t}.$$

The operator

$$C' \equiv \Sigma_t L_t E^t,$$

which is C applied in the reverse direction, will be called the *reflection* of C. We now introduce a special operator S, which is an operator on an operator, defined by the relation:

$$S(C) = \Sigma_t L_t^2 .$$

In other words, S operating on a linear compound operator gives a number, which is the sum of the squares of its coefficients. We shall need also another special operator F defined by:

$$F(C) = L_0 .$$

From these definitions it immediately follows that

$$S(C) = F(CC') . \tag{6}$$

If, as is usually the case, we are dealing with symmetrical linear compound operators (by which is meant that $L_t = L_{-t}$), we may write:

$$C \equiv L_0 + \Sigma_t L_t \gamma_t ,$$

$$S(C) = L_0^2 + \Sigma_t L_t^2 ,$$

where the summations are over positive values of t only. Also, in this case, $C' \equiv C$, and the relation (6) becomes:

$$S(C) = F(C^2) . \tag{7}$$

The following easily verified relations will also be found useful:

$$\left.\begin{aligned} \gamma_n \gamma_p &\equiv \gamma_{n+p} + \gamma_{n-p} \\ \gamma_n^2 &\equiv 2 + \gamma_{2n} . \end{aligned}\right\} \tag{8}$$

DERIVATION OF SPECIAL FORMULAS

As our first example, we shall derive Mr. Beers' 6-term, minimized fifth difference, reproducing formula, correct to fourth differences (*RAIA* XXXIII, 245). In this, as in subsequent examples, we shall take $m = 5$. Therefore, the total number of coefficients is $6 \times 5 - 1 = 29$. Applying Vaughan's principle, we find that

$$G \equiv \frac{[5]^5}{625} H ,$$

where

$$H \equiv 1 - 5\,\delta^2 + 14\,\delta^4 + k\,\delta^6 + l\delta^8.$$

We now wish to discover what restriction, if any, reproduction of the given values places on the values of k and l. Now, it is known that the interpolation formula associated with the graduation operator

$$\frac{[5]^5}{625}(1 - 5\,\delta^2 + 14\,\delta^4)$$

reproduces the given values. Therefore, the formula based on the operator $5^{-4}[5]^5(k\delta^6 + l\delta^8)$ must give zero for the "interpolated" values corresponding to the arguments at which the given values are situated; and this must be true no matter what the given values are. This operator may be written as $5^{-4}\delta_5^5(k\delta + l\delta^3)$. If this is expanded in powers of E, only the coefficients of those powers whose indices are multiples of five will enter into the calculation of the particular "interpolated" values under consideration. However, careful examination of this expression brings out that all such coefficients are zero. Hence, the requirement of reproduction of the given values is automatically satisfied, and does not impose any restriction on the values of k and l.

The constants k and l are to be determined so as to minimize the sum of the squares of the coefficients which express the fifth differences of the interpolated values in terms of the fifth differences of the given values. These coefficients are obtained from the relation

$$\delta^5 v_n = 5^{-4}\,\delta^5\,[5]^5 H\,U_n = 5^{-4}H\,(\,\delta_5^5 U_n)\,.$$

Hence, the sum of their squares is proportional to

$$S(H) = F(H^2)\,.$$

To find the values of k and l for which this expression is a minimum, we must equate to zero its partial derivatives with respect to these two quantities. Thus,

$$\frac{\partial}{\partial k}S(H) = 2F\left(H\,\frac{\partial H}{\partial k}\right) = 2F(H\,\delta^6) = 0$$

$$\frac{\partial}{\partial l}S(H) = 2F\left(H\,\frac{\partial H}{\partial l}\right) = 2F(H\,\delta^8) = 0\,.$$

Multiplying out the expressions $H\delta^6$ and $H\delta^8$, and recalling that $\delta^r \equiv (E^{1/2} - E^{-1/2})^r$ gives the equations:

$$-\,{}^6C_3 - 5\,{}^8C_4 - 14\,{}^{10}C_5 + k\,{}^{12}C_6 - l\,{}^{14}C_7 = 0$$

$${}^8C_4 + 5\,{}^{10}C_5 + 14\,{}^{12}C_6 - k\,{}^{14}C_7 + l\,{}^{16}C_8 = 0\,.$$

On substituting numerical values and combining, we have:

$$924k - 3432l = 3898$$

$$3432k - 12870l = 14266\,.$$

Solving these equations gives $k = 10.65$ and $l = 1.73$. Hence,

$$G \equiv \frac{[5]^5}{625}(1 - 5\,\delta^2 + 14\,\delta^4 + 10.65\,\delta^6 + 1.73\,\delta^8)\,.$$

Obtaining the expanded coefficients of $H/625$ and summing them in fives five times, we shall have the linear compound coefficients for interpolation.

It may be desired to experiment with various rounded values of $k/625$ and $l/625$ in order to find the combination which actually gives the smallest value of $S(H)$, using coefficients only to a specified number of decimal places. For this purpose, we may use the expression:

$$\begin{aligned} S(H) &= F(H^2) = F\,[1 - 10\,\delta^2 + 53\,\delta^4 + (2k - 140)\,\delta^6 \\ &\quad + (-10k + 2l + 196)\,\delta^8 + (28k - 10l)\,\delta^{10} + (k^2 + 28l)\,\delta^{12} \\ &\quad + 2kl\,\delta^{14} + l^2\,\delta^{16}] = 1 + 10\,{}^2C_1 + 53\,{}^4C_2 - (2k - 140)\,{}^6C_3 \\ &\quad + (-10k + 2l + 196)\,{}^8C_4 - (28k - 10l)\,{}^{10}C_5 \\ &\quad + (k^2 + 28l)\,{}^{12}C_6 - 2kl\,{}^{14}C_7 + l^2\,{}^{16}C_8 \\ &= 16859 - 7796k + 28532l + 924k^2 - 6864kl + 12870l^2. \end{aligned}$$

USE OF LAGRANGE MULTIPLIERS

Sometimes the derivation of a discrete interpolation formula can be appreciably shortened by the use of Lagrange multipliers, a device to which I have called attention in a previous paper (*RAIA* XXXIV, 33). An example is Mr. Beers' 6-term, minimized fourth difference, non-reproducing formula, correct to third differences (*RAIA* XXXIV, 14). Applying Vaughan's principle and using the same kind of reasoning as in the preceding example, we find that $G \equiv .008[5]^4H$, where $H \equiv 1 - 4\delta^2 + k\delta^4 + l\delta^6 + n\delta^8 + p\delta^{10} + q\delta^{12}$. As the formula is not required to reproduce the given values, we could now proceed immediately to find the values of k, l, n, p, and q for which $S(H) = F(H^2)$, a quantity proportional to the sum of the squares of the coefficients which express the fourth differences of the interpolated values in terms of the fourth differences of the given values, is a minimum. However, this would mean solving a set of five simultaneous equations in as many unknowns; and, moreover, the equations would run into big figures, involving binomial coefficients of degrees up to 24.

Therefore, we shall use a somewhat different approach, writing instead:

$$H \equiv a + b\gamma_1 + c\gamma_2 + d\gamma_3 + e\gamma_4 + f\gamma_5 + g\gamma_6 ,$$

and shall find the conditions on the coefficients a, b, c, etc., so that, when H is expanded in powers of δ, the first two terms will be $1 - 4\delta^2$. Expanding γ_1, γ_2, etc., by formula (5) gives:

$$\left.\begin{aligned} a + 2b + 2c + 2d + 2e + 2f + 2g &= 1 \\ b + 4c + 9d + 16e + 25f + 36g &= -4 \,. \end{aligned}\right\} \quad (9)$$

The procedure is, therefore, to equate to zero the partial derivatives with respect to a, b, c, etc., of the expression:

$$\tfrac{1}{2}F(H^2) - \lambda(a + 2b + 2c + 2d + 2e + 2f + 2g) \\ - \mu(b + 4c + 9d + 16e + 25f + 36g) \,.$$

Making use of the relations (8), we thus arrive at the equations:

$$\left.\begin{aligned} a &= \lambda & 2e &= 2\lambda + 16\mu \\ 2b &= 2\lambda + \mu & 2f &= 2\lambda + 25\mu \\ 2c &= 2\lambda + 4\mu & 2g &= 2\lambda + 36\mu . \\ 2d &= 2\lambda + 9\mu \end{aligned}\right\} \quad (10)$$

Substituting these values of a, b, c, etc. in the equations (9) and simplifying gives:

$$13\lambda + 91\mu = 1 \qquad 91\mu + 1137.5\mu = -4.$$

Solving these equations for λ and μ gives $\lambda = .2308$ and $\mu = -.0220$. Substituting these values in the equations (10), and computing the values of $.008a$, $.008b$, etc., to four decimal places, rounding them by trial in such a way that the equations (9) will hold exactly, we obtain the following values, in alphabetical order: .0018, .0018, .0015, .0010, .0005, $-.0004$, $-.0013$. Writing out the complete series (with $.008a$ in the middle and $.008g$ at each end) and summing in fives four times gives the interpolation coefficients published by Mr. Beers.

SPECIAL CONDITIONS ON THE COEFFICIENTS

In some instances, the specifications of the formula have the effect of imposing special conditions on the undetermined coefficients in the expression for the graduation operator. For example, the requirement of reproduction of the given values will not always be automatically satisfied as it was in the first example. This situation is illustrated by my own 5-term, minimized fourth difference, reproducing formula, correct to third differences (*RAIA* XXXIV, 26). Reasoning as in the previous cases, we find that $G \equiv .008[5]^4H$, where $H \equiv 1 - 4\delta^2 + k\delta^4 + l\delta^6 + m\delta^8$. Continuing as in the first example, we conclude that for reproduction of the given values, the expression $.008[5]^4(k\delta^4 + l\delta^6 + m\delta^8) \equiv .008\delta_5^4(k + l\delta^2 + m\delta^4)$ must have zero coefficients for those powers of E whose indices are multiples of 5. These terms in the expansion in powers of E are given by the expression $.008\delta_5^4(k - 2l + 6m)$. It follows that we must have $k - 2l + 6m = 0$, or $k = 2l - 6m$.

Subject to the other conditions mentioned, this formula minimizes the sum of the squares of the coefficients which express the fourth differences of the interpolated values in terms of the given values themselves. Since

$$\delta^4 v_n = .008\,\delta^4\,[5]^4 H\,U_n = .008\,\delta_5^4 H\,U_n\,,$$

we may, therefore, equate to zero the partial derivatives with respect to l and m of $S(\delta_5^4H) = F(\delta_5^8H^2)$, after replacing k by $2l - 6m$. This gives the equations:

$$F[\,\delta_5^8 H\,(2\,\delta^4 + \delta^6)\,] = 0 \qquad F[\,\delta_5^8 H\,(-6\,\delta^6 + \delta^8)\,] = 0\,,$$

which, after expansion and simplification, become:

$$1044l - 5048m = 640 \qquad 5048l - 24838m = 2842\,.$$

This gives $l = 3.46$ and $m = .59$, whence

$$G \equiv .008\,[5]^4(1 - 4\,\delta^2 + 3.38\,\delta^4 + 3.46\,\delta^6 + .59\,\delta^8)\,.$$

INTERLOCKING FORMULAS

While the interlocking interpolation formulas recently developed by Mr. White (*TASA* XLIX, 337) do not have to be considered as discrete formulas, nevertheless, the interlocking approach seems peculiarly appropriate in the derivation of a discrete formula. As an illustration we shall take Mr. White's interlocking analogue of Sprague's formula, for the case of $m = 5$. We shall take the interval between successive interlocking points the same as that between interpolated values. (This means that $h=.2$ in Mr. White's notation.) This is a 6-term formula correct to fourth differences, so that we must have $G \equiv .0016[5]^5H$, where $H \equiv 1 - 5\delta^2 + 14\delta^4 + k\delta^6 + l\delta^8$. The formula also reproduces the given values, but investigation readily brings out that this requirement is automatically satisfied.

As the formula is of the fifth degree, and there are three interlocking points in the neighborhood of each point of junction, there are eight consecutive interpolated (and given) values in and around each interpolation interval which lie on the same fifth degree curve. Therefore, in any set of five consecutive sixth differences of the interpolated values, two must be zero. If differences are taken centrally, these zero sixth differences will correspond to the two middle values in each interval. As these differences must vanish whatever the given values may be, it follows that $\delta^6 L_2$, $\delta^6 L_3$, $\delta^6 L_7$, $\delta^6 L_8$, $\delta^6 L_{12}$, $\delta^6 L_{13}$, and $\delta^6 L_{17}$ must vanish. Now,

$$\delta^6 G \equiv .0016\, \delta_5^5\, \delta H \equiv .0016\, \delta_5^5 (\delta - 5\delta^3 + 14\delta^5 + k\delta^7 + l\delta^9)\,.$$

Careful study reveals that if the expression in parentheses is expanded in powers of E, only the coefficients of $E^{1/2}$ and $E^{9/2}$ (and those of $E^{-1/2}$ and $E^{-9/2}$, which are, of course, the same) will affect the values of those $\delta^6 L$'s which are required to vanish; and further that the vanishing of these two coefficients is both necessary and sufficient for the vanishing of all seven $\delta^6 L$'s. This gives the equations:

$$156 - 35k + 126l = 0\,, \qquad l = 0\,.$$

Hence $k = 156/35$, $l = 0$, and

$$G \equiv .0016\,[5]^5 \left(1 - 5\delta^2 + 14\delta^4 + \frac{156}{35}\,\delta^6\right).$$

It is interesting to note that while this graduation operator might be expected on general principles to have 29 terms, it actually has only 27, since l turns out to be zero. This is always the case when the interval between interlocking points is the same as that between interpolated values as is borne out by the fact that s^2–h^2 will be found to be a factor of the coefficient of $\delta^4 u_{x+1}$ in Mr. White's formula (26).

DISCUSSION OF PRECEDING PAPER

AUBREY WHITE:

We should be grateful to the Papers Committee for allowing those of us who are still interested in the former actuarial subject of graduation theory to use this forum for our harmless play. Perhaps some future Wolfenden among the vital statisticians will someday rediscover in our transactions an important series of papers by Dr. Greville, laying the foundation for future progress in this field. Although I am not sure that these summation expressions will normally be as efficient in producing the required linear compound factors as the classical continuous curve, the primary value in Dr. Greville's paper would seem to lie in the creation of a new technique for research, out of which better forms and processes may emerge. I have no useful comments to offer in developing this approach, but I would like to refer to some incorrect statements regarding my interlocking curves which the original draft of this paper contained. At the same time, I wish to give full credit to Mr. D. C. Duffield, who first noticed the original errors, and whose sound suggestions for correction have, I believe, been adopted by Dr. Greville.

Mr. Duffield points out that my curves, when interlocking points are also interpolated points (as they should always be in practice), are fundamentally discrete interpolation formulas, so that my h will always be a multiple of $1/m$ in Dr. Greville's notation. Also, he notes that by their nature they must always involve less than the maximum range of given values in the expressions which determine the interlocking points.

This may be seen from general reasoning. Although a classical interpolation curve by its nature uses the same group of given values for determining each interpolated value (including the given values themselves), it is obvious that if two adjacent curves must invariably meet at certain fixed points, they must both use the same set of given values in the same proportions at those points. This can only be done if each drops the one given value not common to both curves. Therefore, the rule for an interlocking curve using h given values for noninterlocking points, with m points determined by each curve and with r interlocking points, is that the number of nonzero linear compound factors is a maximum of $mh-r$, less one if h is even and the given value is not an interlocking point, or if h is odd and the given value is such a point. In the interlocking analogue of Sprague's formula, h is even (6) and the given value is an interlocking point, so that the maximum is 27 factors for $m = 5$.

My minimum curves illustrate a peculiar variation of Dr. Greville's summation notation. As stated in *TASA* XLIX, 361, each may be expressed as

$$5\{5\}^{r+1}\, Du_{x+t}\,,$$

where

Du_{x+t} is zero unless t is a multiple of 5, and

$5\{5\}^{r+1}\, Du_{x+5m} = u_{x+5m}\,.$

The last line could have been relaxed somewhat, if perfect fit was not required. If I had used Dr. Greville's U_{kn} series, and defined D as an operator of the form $1 + a\delta_5^2 + b\delta_5^4 + \ldots$, my first condition would have been automatically satisfied. Then the second condition could have been relaxed to

(a) whatever D may be, the curves preserve the degree r;
(b) if the expansion of $(1 + \alpha\delta_5^2 + \beta\delta_5^4 \ldots)D$ (set equal to unity in my paper) differs from unity by powers of δ_5 greater than r, the curve will reproduce an rth degree curve;
(c) if this expansion is identically equal to unity, the curve will reproduce the given values.

It follows that if Dr. Greville's H operator were replaced by an operator of the form of D above, my minimum curves would represent the general expression in continuous form for the resulting interpolation operation. This also illustrates the essential unity of the several graduation processes, since the same smooth, powerful interpolation may be performed by applying either a summation operator or an interpolation formula to a series adjusted by means of a difference equation (or a linear compound operator if given value reproduction is not required).

I would also like to point out again that the so-called "interpolation" processes can be applied advantageously in any case where quinquennial totals are either the only available data or better-behaved than individual entries, without any need for a preliminary determination of central values. If v_n is a good series completing the special series U_n, then $\{5\}v_n$ should be an equally good substitute for the special series $\{5\}U_n$; and the latter is easily visualized as the histogram of data given in quinquennial sums. Mr. Beers gave a set of factors for this operation in his reply to the discussion of his paper on the six-term formula (*RAIA* XXXIV, 60) and Dr. Greville's present paper gives added emphasis to the appropriateness of this approach.

I wish that Dr. Greville or some other equally qualified analyst could devote some time to a critical analysis of the other finite difference graduation processes. Despite Mr. Spoerl's very thorough and scholarly paper on the difference equation, much remains to be done in exploring the philosophy of this method, and also much could be done in analyzing the criteria commonly used for judging graduations. It seems to be inevitable that adjusted average methods as such will lose their appeal as practical devices; and I note that Dr. Greville does not apply his new analysis to a single osculatory curve. I can foresee a considerable further reduction in the required reading, and I hope that some time in the future we will not have to ask a nonactuary to advise us when we have a real graduation job to do.

HUBERT VAUGHAN:

I am deeply impressed by the generosity of Dr. Greville, who has elevated a proposition of mine to the basis of a principle, embellished by his own sense of form and knowledge of recent research.

Under any circumstances I would esteem it an honour to be permitted to contribute to the discussions of this body.

Dr. Greville enunciates the principle in his own way, carefully scrutinizing the necessary and sufficient conditions. He supplies a mathematical device to facilitate the derivation of formulae and gives examples of its use.

Dr. Greville also introduces the concept of maintenance of degree and, in doing so, reduces the three steps of my demonstration to two. I had considered first the effect of simple summations, then the operand necessary for absolute reproduction, and thirdly the case of additional coefficients. The first and third steps are now taken up in the maintenance concept, which elucidates the part played by the summations, whatever the operand.

In working on this subject of interpolation I had a curious experience. Some years ago I set out to investigate the possibility of basing interpolation upon a smoothing test, but the work dragged on over years on account of war conditions and the impossibility of publication at the time. Unknown to me others were working in the same field and, as time passed, the matter in hand was covered by papers from several writers in *TASA* and *RAIA*. The main thesis of a projected paper was accordingly discarded and, had time, place and circumstance permitted, the residue would have been offered as a contribution to the discussion of American papers. As it was, a truncated paper eventually appeared in *JIA* LXXII as "Some Notes on Interpolation." As a further coincidence, Dr. Michalup of Venezuela informed me last year that my paper reached him exactly at the moment when he was working on the relation between interpolation and summation formulae (which had been the distinctive feature of my own efforts).

It is on the generality of this relation that Dr. Greville now seizes, perhaps because the growing number of good special formulae is becoming formidable and it is a relief to find a connection between them.

The existence of some relation between interpolation and the summation process has, of course, been known for over sixty years, and several early graduation formulae were deduced from the average of a set of interpolations; but the investigation seemed to become frozen at a certain point. This was no doubt due in the first place to the development of a type of summation formula ascertained without reference to interpolation. Then independent research restored the neglected outlook of De Forest, and actuarial thought came to be in terms of linear-compounds rather than summations. The powerful difference-equation method, absorbing much of the current interest, pushed the old enquiry even further back into the position of a mere episode in history.

However, on arranging the linear-compound coefficients of a finite-difference interpolation formula in suitable order, it turns out that they form a series divisible by a set of summations; and it now appears that not only some special averages of interpolations but each separate interpolation can be expressed as a summation formula. The application to interpolation is in fact wider than to graduation, because there are linear-compound graduation formulae that cannot be expressed in the old summation form, while in the case of interpolation we find that all formulae with certain properties must be so expressible. A summation formula then seems essentially a natural part of the theory of interpolation rather than of graduation. The possibility arises that we can reverse the old objective: in lieu of seeking a graduation expression by compounding interpolations, we can start with the summation form as a generalised interpolation formula, and use the principle of this paper to derive and classify interpolation methods. One advantage in research, as Dr. Greville remarks, is that only some two or three coefficients in the operand require determination to establish particular properties. It also weighed with me that recorded investigations into summation formulae provided much available data and that there was a standard of smoothing power ready-made. To use the principle in this way, it was necessary to provide a proof which would formulate it generally and specifically.

Though the form is discrete, it is the case that osculatory formulae can be expressed in the same way. Some such formulae have in fact been derived from the summation form, either by accident when experimenting with changes in the operand terms, or deliberately by considering the limit when the interval of interpolation becomes small. I did not get to the point of examining the infinite case beyond regarding it as a number of integrations applied to an operand and discarding coefficients that would become infinitesimal. Though this worked for particular cases, it is far from graceful mathematics. However, it suggests that the form could be modified to cover the case of a continuous curve, and one would expect to find such a limiting form related to that devised by Professor Schoenberg. As his work was previously unknown to me and has been set down for future study, I am not able at present to form any view on the point. Perhaps Dr. Greville can enlighten me.

An actual example of the use originally made of the principle may be interesting. Starting with the second-difference interpolation form, $5^{-3}[5]^3(1 - 3\delta^2)$, the operand was modified for third-difference smoothness (as was done for certain cases by J. R. Larus in *TASA* XIX). The result was to extend the operand by $(-2.305\delta^4 - .376\delta^6)$. This produced interpolation coefficients identical with those given by Dr. Greville in *RAIA* XXXIV, 25. Having now the best theoretical formula, the operand can (by a method published in *JIA* LXV) be modified for simplicity while retaining most of the virtue. The first modification indicated was to replace the above-mentioned extension by $(-2.4\delta^4 - .4\delta^6)$, but this turned out to be a "rediscovery" of Karup's formula (which incidentally is osculatory). The next attempt was $(-2.75\delta^4 - .5\delta^6)$, which provides a very simple interpolation form—it is Formula III of Mr. E. H. Wells (*RAIA* XXXIV, 43) given during the discussion of one of Mr. Beers' papers.

An interesting case was $m^{-4}[m]^4\left\{1 - \frac{1}{6}\left(1 - \frac{1}{m^2}\right)\delta_m^2\right\}$. This was chosen for experiment on account of its simplicity and because the operand is an approximate reciprocal to a difference-equation expression connected with minimizing second differences. It was noticed that the formula would be interlocking, and hence osculatory at the limit. The osculatory formula was obtained and turned out to be a very pleasing and simple one, *viz.*, Jenkins' fifth-difference modified formula. This unexpected result can be verified another way. Jenkins' formula, by the methods of the present paper, can be written $m^{-4}[m]^4(1 - \frac{1}{6}\delta_m^2)(1 + \frac{1}{6}\delta^2)$. Giving m the value 5 or any other number, the expansion of this will produce the interpolation coefficients for a corresponding subdivision. Now if we increase m so that the interval of interpolation becomes infinitesimal, δ^2 and $\frac{1}{m^2}$ both approach zero, so that at the limit the two summation forms coincide. We thus have a connecting link between the difference-equation method for

δ^2 (which at the limit becomes Henderson's difference-equation osculatory method), Jenkins' formula, and two summation forms, one of which represents Jenkins' formula at all ranges, while the other branches off as an interlocking formula.

Dr. Greville is judicious in writing his finite integral as from $-\infty$ to ∞, as this enables difference-equation interpolation to be included. For example a graduation formula corresponding to the difference-equation for minimizing on second differences may be written $5^{-2}[5]^4(4E^{-5} + 17 + 4E^5)^{-1}$ which is a combined summation and difference-equation formula similar to those of Henderson (*TASA* XXXIV). If we expand symmetrically the reciprocal of the expression in E's, this becomes a summation formula with an operand of infinite length, but the same conditions hold as for finite ranges, *e.g.*, the central coefficient of the expanded interpolation coefficients will be unity, and coefficients at intervals of 5 zero. The above summation form bears the same relation to the difference-equation interpolation as Woolhouse's graduation formula to an ordinary second-difference interpolation. It could be applied by using expanded coefficients (which Dr. Greville has, I think, published) or by factorizing into $(E^{-5} + 4)(4 + E^5)$, and might have an application in graduating population mortality.

I will close with the minor suggestion that to help students it might be well to mention specifically that when r is even, the graduation formula applied to a complete series will be correct to $(r + 1)$ differences, though as an interpolation formula it is correct only to r differences. The reason is that, in a graduation, terms are taken equally from each side so that the odd orders of differences cancel out. This does not of course affect the correctness of Dr. Greville's statement.

I hope the subject will be of some interest to members, because over the years I have obtained so much information from the *Transactions* and *Record* that any information I can give is a very small return.

HARWOOD ROSSER:

Dr. Greville disclaims any new formulas at the outset, stating that he is dealing here only with alternate derivations. However elegant his theory may be, not many actuaries will be deeply interested. Even someone who elected the advanced graduation option under Part 8 finds it heavy going.

But he and Vaughan, between them, have suggested a revolutionary thought—namely, interpolation by graduation formulas. Since actuaries require interpolations much oftener than graduations, and since the latter are often easier to perform, there seems to be a field for investigation here.

This discussion is devoted to emphasizing that point, chiefly by illustration, as lack of time prevents a more thorough examination.

Interpolation by Graduation

Dr. Greville notes that application of Woolhouse's graduation formula to the series:

$$\ldots, 0, 0, 0, 0, u_{15}, 0, 0, 0, 0, u_{20}, 0, 0, 0, 0, u_{25}, 0, 0, 0, 0, \ldots$$

and subsequent multiplication by five throughout, is equivalent to second-difference central difference interpolation on the series:

$$\ldots, u_{15}, u_{20}, u_{25}, \ldots$$

A numerical example is given in Table 2, starting from the figures in Table 1, Column 1.

TABLE 1

BASIC VALUES

x	u_x (1)	Δu_x (2)	$\Delta^2 u_x$ (3)	$3\delta'^2 u_x$ $=\frac{3}{25}(3)_{x-1}$ (4)	u'_x (1)−(4) (5)
35	311	− 80	165	....	....
40	231	85	190	19.8	211.2
45	316	275	150	22.8	293.2
50	591	425	90	18	573
55	1016	515	....	10.8	1005.2
60	1531	....	....	....	....

The following short-cuts make the process easier:

1. The w-column can be obtained by summation in fives twice of the u's and division by five. But it is quicker to use straight line interpolation between the quinquennial values.
2. 40% of the figures in Column 3 are copy work—*i.e.*, those where x ends in 2, 3, 7, or 8.
3. While it is instructive to compare Columns 4 and 7, Column 5 can be taken directly from Table 1, rendering the differencing of Columns 3 and 6 unnecessary.

There is a set of "G" formulas in the section headed "Application of Vaughan's Principle." The above example corresponds to the one correct to second differences, with $m = 5$. The one below it, to third differences, could be used by repeating the process in Column 3, and substituting $4\delta^2$ for $3\delta^2$ in Column 5. This would be more work, and would give a smoother result.

TABLE 2

INTERPOLATION BY WOOLHOUSE'S GRADUATION FORMULA

x	Crude u_x (1)	w_x (2)	$\frac{[5]}{5} w_x$ (3)	$\Delta^2(3)$ (4)	$3\delta^2(3) = 3(4)_{x-1}$ (or Table 1) (5)	Interpolated v_x (3)−(5) (6)	$\Delta^2 v_x$ (7)
35	311	311	....	....	....	....	....
36	0	295	....	....	....	....	....
37	0	279	279	6.6	....	....	....
38	0	263	263	6.6	19.8	243.2	6.6
39	0	247	253.6	6.6	19.8	233.8	6.6
40	231	231	250.8	6.6	19.8	231	6.6
41	0	248	254.6	6.6	19.8	234.8	3.6
42	0	265	265	7.6	19.8	245.2	10.6
43	0	282	282	7.6	22.8	259.2	7.6
44	0	299	306.6	7.6	22.8	283.8	7.6
45	316	316	338.8	7.6	22.8	316	7.6
46	0	371	378.6	7.6	22.8	355.8	12.4
47	0	426	426	6	22.8	403.2	1.2
48	0	481	481	6	18	463	6
49	0	536	542	6	18	524	6
50	591	591	609	6	18	591	6
51	0	676	682	6	18	664	13.2
52	0	761	761	3.6	18	743	−3.6
53	0	846	846	3.6	10.8	835.2	3.6
54	0	931	934.6	3.6	10.8	923.8	3.6
55	1016	1016	1026.8	3.6	10.8	1016	3.6
56	0	1119	1122.6	3.6	10.8	1111.8	....
57	0	1222	1222	....	10.8	1211.2	....
58	0	1325	1325	....	....	....	....
59	0	1428	....	....	....	....	....
60	1531	1531	....	....	....	....	....

TABLE 3

INTERPOLATION WITH SOME GRADUATION

x	Crude u'_x Table 1 (1)	w'_x (2)	Interpolated $v'_x = \frac{[5]}{5} w'_x$ (3)	$\Delta^2 v'_x$ (4)
40	211.2	211.2	....	....
41	0	227.6	....	....
42	0	244	244	7.9
43	0	260.4	260.4	7.9
44	0	276.8	284.7	8.0
45	293.2	293.2	316.9	7.8
46	0	349.2	357.1	8.0
47	0	405.1	405.1	6.0
48	0	461.1	461.1	6.2
49	0	517	523.1	6.0
50	573	573	591.3	6.2
51	0	659.4	665.5+	6.0
52	0	745.9	745.9	....
53	0	832.3	832.3	....
54	0	918.8	....	....
55	1005.2	1005.2	....	....

Interpolation with Some Graduation

A smoother result with no extra work, but at the sacrifice of reproduction of pivotal points, is illustrated in Table 3. This is somewhat analogous to modified osculatory interpolation. The procedure differs from that in Table 2 only in that the adjustment in Column 5 thereof is made at the beginning instead of at the end.

(AUTHOR'S REVIEW OF DISCUSSION)

THOMAS N. E. GREVILLE:

I wish to thank all three discussants for their contributions, and I am especially glad that Mr. Vaughan, to whom this subject owes so much, found it possible to prepare a discussion. Also, I particularly want to thank Mr. White for his kind remarks concerning the paper, and I am grateful to him and to Mr. Duffield for pointing out the incorrect statements concerning interlocking curves which appeared in the original draft of my paper.

Mr. White's discussion of the number of linear compound factors in an interlocking formula is to the point, and I am glad he included it in his discussion. However, it appears that he and I are not entirely in agreement as to the formula to be applied. To avoid possible confusion, it should be pointed out that by "the number of nonzero linear compound factors" he means the maximum number in the general formula for given values of m, h, and r. He does not mean to exclude those coefficients which may vanish because of the setting of other conditions (such as reproduction of the given values), or as an accidental result of the particular combination of characteristics chosen. With this understanding, it seems to me that the number of linear compound factors in an interlocking formula is $mh - r$ without any exception or qualification. If I am correct in this, then the portion of Mr. White's discussion which reads "less one if h is even and the given value is not an interlocking point, or if h is odd and the given value is such a point" should be deleted.

Referring specifically to the two cases cited by Mr. White as exceptions, I would point out that, if h is even, the given values would be points of junction of adjacent interpolation curves. In this case it would appear that either (1) the composite interpolation curve is discontinuous at these points and two distinct "interpolated" values are obtained at each such point, or (2) it is not intended to include in the final complete series values corresponding to these points. The deduction of one is not called for, it seems to me, on either of these assumptions.

In the other case, where h is odd, the given values would be the midpoints of the ranges of the respective curves. Therefore, if the given value is an interlocking point, it would have to lie on three curves: the one of which it is the midpoint and the two on either side. Thus, in a sense, there would be two interlocking points at this one point. This fact was apparently overlooked by Mr. White, as he has acknowledged in private correspondence. Here again the deduction of one should not be made.

In order to avoid possible confusion it may be well to point out that the letter h is used in an entirely different sense in the second and third paragraphs of Mr. White's discussion.

I was aware of the fact pointed out by both Mr. White and Mr. Vaughan that the same general principles can be applied to formulas of infinite range, such as Mr. White's "minimum curves" and the formulas based on a difference equation for minimizing on a specified order of differences which Mr. Vaughan and I have discussed elsewhere (*JIA* LXXII, 491–7 and *Boletim do Instituto Brasileiro de Atuária*, No. 2, 7–34). However, I have not fully explored this aspect of the matter and have encountered certain mathematical difficulties in trying to formulate a general treatment of the subject. Therefore, it seemed best to limit the present paper to formulas of finite range.

The situation is similar as regards another question raised by Mr. Vaughan: whether a similar method can be applied to continuous interpolation formulas. On the basis of Professor Schoenberg's work and of certain results which Mr. Vaughan has communicated to me in a letter, I am sure this can be done, but some further work is needed to put these results in a satisfactory form for publication.

It had not occurred to me to use the summation form of an interpolation formula for the purpose of actual computation, as Mr. Rosser does in his discussion. I had regarded the summation form merely as a mathematical device for deriving formulas required to satisfy certain prescribed conditions, and had supposed that the availability of calculating machines would make the linear compound form the most suitable one for computation. However, I must admit that Mr. Rosser has described a most efficient computing process which probably involves fewer actual computational steps than the linear compound method, although I suspect it might be somewhat more difficult to explain to a computing clerk.

I am afraid the ingenuity of Mr. Rosser's process might not be fully appreciated, as his very brief description does not adequately state the theoretical basis underlying his various steps. For example, it might not be clear to all readers why the figures in column (5) in his Table 2 are the same as those in column (4) of Table 1. The graduation operator in Woolhouse's formula is, in symbolic form, $\frac{[5]^3}{125}(1 - 3\delta^2)$. (It should be noted that my δ_5 is the same as Mr. Rosser's δ'.) Since $\delta_5 \equiv [5]\delta$, this can be written in the form $\frac{[5]^3}{125} - \frac{3}{25}\frac{[5]}{5}\delta_5^2$. Now $\delta_5^2 u_n$ is zero for all values of n not multiples of 5, so that column (4) of Table 1 gives all the nonzero values of this quantity. It follows that each value of $[5]\delta_5^2 u_n$ is the sum of five numbers, four of which are zero. This explains why the figures in column (5) of Table 2 are the same as those in column (4) of Table 1. As Mr. Rosser points out, this makes it possible to omit columns (3) and (4) of Table 2 and makes column (5) purely a copying job, as Table 1 has previously been computed. As column (7) is given for information only, and is not an essential part of the computation, his process can be made a very short and compact one.

The theory underlying Mr. Rosser's Table 3 is also very interesting. If we expand the operator $[5]^3$ in linear compound form, and then pick out the linear compound coefficients which multiply the nonzero values of U_n when the operator is applied to one of the given values, it is found that the operation performed on such a given value may be represented by $1 + \frac{3}{25}\delta_5^2$. Therefore, if we use as an operand the expression $1 - \frac{8}{25}\delta_5^2$, the graduation of the given values will be correct to third differences. The resulting operator $\frac{[5]^3}{125}(1 - \frac{8}{25}\delta_5^2)$ is the symbolic representation of the computation shown in Table 3. As this operator can be written in the form $\frac{[5]^3}{125}\left(1 - 3\frac{[5]^2}{25}\delta^2\right) \equiv \frac{[5]^3}{125}[1 - 3\delta^2(1 + 2\delta^2 + \cdots)]$, it follows from Vaughan's principle that this produces an interpolation correct to second differences. The graduating effect on the given values is indicated by the relation $v_n = (1 - \frac{8}{625}\delta_5^4)u_n$.