

# Investigation of Unfunded Liability in a Simplified Pension Fund Model

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## Abstract

This paper develops a simplified model for a pension fund, both deterministic and stochastic versions are presented. The probability distribution of the fund is derived incorporating the claims distribution and other parameters. Expected value and the variance of the fund are derived. Modifications required for the model when the interest rate is a moving average process are developed. A discussion of possible extension of the model is also presented.

## 1.0 INTRODUCTION

The savings for retirement have been an organized activity in all industrial nations. In modern societies, it is an essential feature providing tax savings and many other advantages. Moreover, government encourages such activities as it enhances the investment ability in the community. The main technical knowledge and management is provided by the actuaries and investment managers to these pension fund organizations.

There are basically two types of pension funds; these are either defined benefits schemes or defined contributions schemes. In the former the contributions are a proportion of the salary and the benefits are related to the salary at retirement normally in the form of an annuity. Additional benefits due to other contingencies such as disability, death could also be included. The accumulation schemes provide the accumulated contributions as a sum at retirement. Nowadays there are many other schemes such as "unit-linked" based on similar principles.

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**Key words:** *Pension fund, distribution, asymptotic values, moving average.*

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It is evident that normally pension funding for an individual is related to his survival probabilities and the investment return from the contributions. The technical aspect of funding and continuous monitoring in the form of valuations are carried out by actuaries using actuarial principles. So far very little work has been carried out on the stochastic aspects of pensions funds. This paper is aimed at providing some such investigations of pension fund behavior.

In this paper we primarily consider a pension fund for a large group of individuals so that as an approximation their survival probabilities can be neglected; this is more suited to the accumulation type pension schemes. We first consider a deterministic model and obtain some plausible relations. Thereafter a stochastic version of this process is developed. In both cases the fund commences with a known actuarial liability  $A$  and the objective is to maintain the liability at least at this level. Any unfunded liabilities below this level will be covered by making additional contributions as a fraction "k" of the difference between  $A$  and the current fund level. This is a normal method of covering unfunded liabilities. When the fund level exceeds the liabilities, it is the practice to credit the excess to the shareholders in some form in countries such as U.K., Australia, New Zealand, etc. Papers by Trowbridge (1952) and Dufresne (1988) consider similar treatments for pension funds. This paper derives, in the stochastic case the probability generating function of the fund level at time  $t$  and provides the expected value and its variance. Conditions for long term stability of the fund are derived and optimal values for "k" investigated. Possible modifications when the rate of return on the fund's investment follows a moving average,  $MA(1)$ , are also considered. In conclusion it also provides a discussion when survival probabilities are included. This investigation provides, as a preliminary attempt, important insights to pension fund behavior and conditions for long term stability to study the stochastic aspect of pension funds.

## 2.0 DETERMINISTIC MODEL

Consider a pension fund which accumulates the contributions to provide a lump sum for retirement purposes; this fund may have in its trust deed to provide defined benefits such as a pension, death benefit, etc. We will not consider in this paper any general defined benefit schemes as it involves many more contingencies. It is anticipated that modifications with such treatments be subsequently investigated.

We first treat the development of the fund in a continuous time basis and as a deterministic model. This will enable some insight to the process. We assume that the fund receives contributions at the rate of dollars  $c$  per unit time and that the effective rate of accumulation is  $i(t)$  at time  $t$ . The corresponding force of interest is  $\delta(t)$  where  $\exp\{\delta(t)\} = 1 + i(t)$ . For simplicity, we take these parameters as constants in the first instance. We assume that the main objective of the fund is to maintain its fund level to a minimum level  $A$  which level it maintained at the commencement. Let  $F(t)$  be the fund level at time  $t \geq 0$  with  $F(0) = A$ . Assume that the claim rate is  $\lambda$  per unit time.

In accordance with the funds objective if  $F(t) < A$ , the balance of  $A - F(t)$  which is the unfunded liability will be covered by additional contributions at the rate  $k\{A - F(t)\}$  where  $0 < k \leq 1$ . When  $F(t) > A$ , it is the practice by actuaries to credit the excess  $F(t) - A$  to the shareholders particularly in countries such as U.K., Australia, New Zealand, etc. This could be in the form of smaller contributions for a given period or increasing the benefits. As a consequence, it is reasonable to assume that the parameter  $k$  in  $k\{A - F(t)\}$  represents both cases.

In this paper, we assume that this additional contributions are made throughout. These assumptions are similar to Dufresne (1988). Considering the process in the time interval  $(t, t + dt)$  and using equation of value, we have

$$dF = F\delta dt + k(A - F)dt + cdt - \lambda dt \quad (2.1)$$

Hence, we get

$$\frac{dF}{dt} = (\delta - k)F(t) + c - \lambda + kA \quad (2.2)$$

We note that the relation (2.2) is comparable to classical Thiele equation (Gerber, 1995) with adjustments made for unfunded liability cover.

Using the initial condition

$$F(0) = A , \tag{2.3}$$

we have from (2.2) that

$$F(t) = (c - \lambda + kA)(\delta - k)^{-1} \exp\{(\delta - k)t - 1\} + A \exp(\delta - k)t . \tag{2.4}$$

Hence, assuming that  $\delta$  is constant, one gets

$$\frac{dF}{dt} \Big|_{t \rightarrow 0^+} = A\delta + c - \lambda \tag{2.5}$$

Thus, one can obtain the initial conditions for the necessity to provide extra funding using (2.5); this is  $A\delta \leq \lambda - c$ . However, as  $\lambda - c$  is the net claims per unit time, the earnings per unit time of the fund,  $A\delta$ , should be less than the net claims per unit time for the fund to require extra funding which is clearly an acceptable constraint.

This deterministic model thus provides a pedagogical introduction to pension fund modeling and also exhibits properties that are acceptable and plausible.

### 3.0 THE STOCHASTIC MODEL IN DISCRETE TIME

The deterministic model considered can be extended to develop the stochastic process governed by  $\{F(t)\}$ . As before, continuous time treatment can be investigated; in the first instance we consider the development of the process  $\{F(t)\}$  in discrete time;  $t = 0, 1, 2, \dots$  as it provides some further understanding of the process structure. Thus the relationship corresponding to (2.2) is given by

$$a^{-1}F(t + 1) = (1 - k)F(t) + c - X_t + n , \quad t = 0, 1, \dots . \tag{3.1}$$

where  $n = kA$ ,  $a = 1 + i(t)$  with  $i(t)$  constant in the interval and the claim  $X_t$ , in the interval  $(t, t + 1)$ , is independently and identically distributed.

Relations such as (3.1) are common on actuarial science, for example equation (7.8.2) in Bowers et al. (1986).

We are therefore motivated to consider a general relation of the type in (3.1). Let this relation be

$$p(t)F(t + 1) = q(t)F(t) + r(t) - X_t, t = 0, 1, 2, \dots \quad (3.2)$$

where  $p(t)$ ,  $q(t)$  and  $r(t)$  are functions of  $t$ . Let

$$g(s) = \sum_{\text{all } n} Pr(X_t = x)s^x; |s| \leq 1 \quad (3.3)$$

and

$$U_t(s) = \sum Pr(F(t) = j)s^j; |s| \leq 1. \quad (3.4)$$

Then using (3.2), one gets

$$Pr\{F(t + 1) = y\} = \sum_{\text{all } x} Pr\{F(t) = \frac{p(t)y + x - r(t)}{q(t)}\} \cdot Pr\{X_t = x\}$$

Hence

$$U_{t+1}(s) = s^{\alpha(t)}g(s^{-\beta(t)})U_t(s^{\gamma(t)}) \quad (3.5)$$

where

$$\alpha(t) = r(t)/p(t), \quad \beta(t) = 1/p(t)$$

and  $\gamma(t) = q(t)/p(t)$ . Thus in general, the recurrence relation (3.1) could be used recursively to find  $U_t(s)$  given the initial condition and hence the distribution of  $F(t)$ , the fund level at time  $t$ . Using (3.5) the corresponding recurrence relation for the generating functions for (3.1) is given by

$$U_{t+1}(s) = g(s^{-a})s^{a(c+n)} U_t(s^{a(1-k)}), t = 0, 1, 2, \dots \quad (3.6)$$

We use the fact that the initial fund level is  $A$  in (3.5) which gives

$$U_t(s) = s^{L(t)} \prod_{i=1}^t g(s^{m_i}) \quad (3.7)$$

$$t = 0, 1, 2, 3, \dots$$

where  $m_i = -a^i(1-k)^{i-1}$  and

$$L(t) = a(c+n) \sum_{j=1}^t \{a(1-k)\}^{j-1} + Aa^t(1-k)^t \quad (3.8)$$

It is more elegant to express (3.6) using cummulant generating functions. Thus, we have

$$\Psi_F(t, \phi) = \phi L(t) + \sum_{j=1}^t \Psi_X(m_j, \phi) \quad (3.9)$$

where

$$\Psi_F(t, \phi) = \ln U_t(\phi)$$

and

$$\Psi_X(\phi) = \ln g(\phi) \quad (3.10)$$

are the cummulant generating functions of  $F(t)$  and  $X_t$  respectively.

If we denote the  $r^{\text{th}}$  cummulants of  $F(t)$  and  $X_t$  by  $K_r(F)$  and  $K_r(X)$  respectively, then, using (3.9), we have

$$\begin{aligned} K_1(F) &= L(t) + \left( \sum_{i=1}^t m_i \right) K_1(X) \\ K_r(F) &= \left( \sum_{i=1}^t m_i^r \right) K_r(X), \quad r \geq 2 \quad . \end{aligned} \quad (3.11)$$

Therefore, from (3.11) the expected value and the variance of  $F(t)$  are given by

$$E\{F(t)\} = L(t) - a E(X_t) \sum_{j=1}^t \{a(1-k)\}^{j-1} \quad (3.12)$$

and

$$\text{Var}\{F(t)\} = \text{Var}(X_t) \left( \sum_{i=1}^t m_i^2 \right) \quad (3.13)$$

Now as  $t \rightarrow \infty$ , the values of  $E\{F(t)\}$  and  $\text{Var}\{F(t)\}$  are given by

$$E\{F(t)\} = a[c + Ak - E(X_t)]/[1 - a(1-k)] \quad (3.14)$$

and

$$\text{Var}\{F(t)\} = \frac{a^2}{[1 - a^2(1 - k)^2]} \quad (3.15)$$

provided that  $k > d$  where  $d$  is the discount rate.

We therefore conclude that the payment of more than the unfunded liability ( $k > d$ ) leads to stable values for the expected value and the variance of the fund  $F(t)$ . This is in agreement with Dufresne (1988).

We can easily obtain an upper limit for  $k$ ; using (3.14) and (3.15) this is given by

$$k > \text{Max}\{d, \{c - E(X_t)\}A^{-1}\} \quad (3.16)$$

Moreover, from (3.6)

$$F(t) - L(t) = \sum_{j=1}^t Z_j \quad (3.17)$$

where  $Z_j$  are independently distributed random variables with

$$\text{Pr}\{Z_j = i \mid m_j\} = p_i$$

and

$$\text{Pr}\{X_t = i\} = p_i$$

Also by (3.10), the distribution of  $F(t) - L(t)$  exists for all  $t$  and asymptotically its mean and variance are respectively

$$\begin{aligned} E \left\{ \sum_1^t Z_j \right\} &\Rightarrow -a(1 - r)^{-1} \\ \text{Var} \left\{ \sum_1^t Z_j \right\} &\Rightarrow aE(X)(1 - r)^{-1} + a[aV(X) + E(X)](1 - r^2), r = a(1 - k), \end{aligned} \quad (3.18)$$

provided that  $k > d$ .

#### 4.0 VARIABLE INTEREST RATE - A MOVING AVERAGE

Suppose now that the interest rate  $i(t)$  is considered as a stochastic process; a moving average  $MA(1)$  is a suitable assumption with  $a_t = a = 1 + i(t)$  and

$$a_t = \delta_1 + \epsilon_t - \theta\epsilon_{t-1} \quad \text{where } \theta > 1, \quad (4.1)$$

$\delta_1 = E(a_t) > 1$  in agreement with the deterministic version and  $\epsilon_t$  is a white noise with mean zero and variance  $\sigma^2$ .

Then

$$E(a_t^r) = \sum_{i=0}^r (2i^r) \delta_1^{r-2i} \frac{(2i)!}{i!2^i} (1 + \theta^2)^i \sigma^{2i} \quad (4.2)$$

for any integer  $r \geq 1$ .

Then from (3.11)

$$E\{F(t)|a_t\} = L(t) - aE(x) \sum_{j=1}^t \{a(1-k)\}^{j-1}$$

which from (4.2) provides

$$\begin{aligned} E\{F(t)\} &= \{(c + Ak) - E(x)\} \sum_{j=1}^t \sum_{i=0}^j (1-k)^{j-1} (2i^j) \frac{(2i)!}{i!2^i} \delta_1^{j-2i} (1 + \theta^2)^i \sigma^{2i} \\ &+ A(1-k)^t \sum_{i=0}^t (2i^t) \delta_1^{t-2i} \frac{(2i)!}{i!2^i} (1 + \theta^2)^i \sigma^{2i} \end{aligned} \quad (4.3)$$

Similarly, using (3.12) and the identity

$$\text{Var}\{F(t)\} = E[\{\text{Var}F(t)|a_t\}] + \text{Var}[E\{F(t)|a_t\}]. \quad (4.4)$$

the unconditional variance of  $F(t)$  can be derived.

Here, the optimal value of  $k$  that minimizes  $\text{Var}\{F(t)\}$  can be determined given the values of the other parameters. A numerical process is more suitable for this task as the expression for  $\text{Var}\{F(t)\}$  involves summation of many terms.



## 5.0 FURTHER WORK AND CONCLUSION

The development of the pension fund in the earlier sections has been on the basis of a large group of individuals whose individual mortality was not taken into account in the analysis. One could modify the above model for a single individual by incorporating his own survival probabilities. This could be done either by developing the joint distribution of the fund level and age or by examining the expected value of the fund level when his age is  $x$ . The latter is the method adopted by actuaries which, in fact, could be derived from the former.

Let  $F_x^*(t)$  = expected fund level at time  $t$  for an individual aged  $x$  with pension provision. Then

$$a^{-1}F_{x+1}^*(t+1) = (1-k)F_x^*(t) + c + n - X_t p_x \quad (5.1)$$

Analyzing the relation (5.1) may provide further insight to the process. Further investigations related to (5.1) and also having the liability level  $A$  as a random variable are expected to be completed in the near future.

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