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Some Recursive Formulas for Life Annuities

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ABSTRACT

We give some recursive formulas involving life annuities.

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1. INTRODUCTION

This note is motivated by problem 5.8 b in *Actuarial Mathematics* [1, p. 151], which gives a recursive formula involving continuous life annuity as follows:

$$\bar{a}_x = \bar{a}_{\overline{\omega-x}|} - \int_x^{\omega} e^{-\delta(y-x)}\,\bar{a}_y\,\mu_y\,dy \tag{1}$$

We prove its discrete counterpart:

$$\ddot{a}_x^{(m)} = \ddot{a}^{(m)}_{\overline{\omega-x}|} - \sum_{y=0}^{n(\omega-x)-2} {}_{1/m}q_{x+1/m}\;\ddot{a}_{x+(y+1)/m}\,v^{(y+1)/m} \tag{2}$$

We also derive the following recursive formulas for continuous and discrete term life annuities,

$$\bar{a}_{x:\overline{n}|} = \bar{a}_{\overline{\omega-x}|} - \int_x^{\omega} e^{-\delta(y-x)}\,\bar{a}_{y:\overline{n}|}\,\mu_y\,dy - \int_x^{\omega-n} e^{-\delta(y-x)}\,{}_nE_y\,dy \tag{3}$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{\overline{\omega-x}|} - \sum_{y=0}^{m(\omega-x)-2} {}_{1/m}q_{x+y/m}\;\ddot{a}_{x+(y+1)/m:\overline{n}|}\,v^{(y-1)/m} - \frac{1}{m}\sum_{y=0}^{m(\omega-x-n)-1} v^{n+y/m}\,{}_np_{x+y/m} \tag{4}$$

2. The Proofs

As noted in problem 5.8 b [1, p. 151], we can prove formula (1) by using the integrating factor $e^{-\delta y}$ to solve the following differential equation:

$$\frac{d\bar{a}_y}{dy} = (\mu_y + \delta)\bar{a}_y - 1 \tag{5}$$

where $x \le y \le \omega$, and $\bar{a}_y = 0$ for $\omega \le y$.

We use a similar technique to derive the formula (3), starting with the following differential equation involving n-year term life annuity [1, p. 129]:

$$\frac{\partial}{\partial y}\bar{a}_{y:\overline{n}|} = (\mu_y + \delta)\bar{a}_{y:\overline{n}|} - e^{-\delta y}[1 - {}_nE_y] \tag{6}$$

Applying the integrating factor $e^{-\delta y}$ to (6), we have

$$\frac{\partial}{\partial y} e^{-\delta y}\bar{a}_{y:\overline{n}|} = e^{-\delta y}\mu_y \bar{a}_{y:\overline{n}|} - e^{-\delta y}[1 - {}_nE_y] \tag{7}$$

Integrating formula (7) with respect to y, with $y = x$ to ω, and multiplying both sides by $e^{-\delta x}$, we obtain formula (3).

To prove formula (2), we use induction.

For $x = \omega - 1/m$, the left hand side of (2) is the m-thly life annuity-due of period $1/m$, and its value is $1/m$. The first term of the right hand side is the m-thly annuity-due of period $1/m$, and its value is also $1/m$. The second term of (2) is 0, as the limits of the summation are $y = 0$ and $y = -1$.

Now assume that (2) is true for some age $x = z$. Consider $x = z - 1/m$. The right-hand side of (2) is

$$\ddot{a}^{(m)}_{\overline{\omega - x + 1/m}|} - \sum_{y=0}^{m(\omega - z)-1} {}_{1/m}q_{z+(y-1)/m}\,\ddot{a}_{z+y/m}\,v^{(y+1)/m}$$

$$= \frac{1}{m} + v^{1/m}\ddot{a}^{(m)}_{\overline{\omega - z}|} - [\,{}_{1/m}q_{z-1/m}\,\ddot{a}^{(m)}_{z}\,v^{1/m} - \sum_{y=0}^{m(\omega - z)-2} {}_{1/m}q_{z+y/m}\,\ddot{a}_{z+(y+1)/m}\,v^{(y+1)/m}\,]$$

$$= \frac{1}{m} - {}_{1/m}q_{z-1/m}\,\ddot{a}^{(m)}_{z}\,v^{1/m} + v^{1/m}[\,\ddot{a}^{(m)}_{\overline{\omega - z}|} + \sum_{y=0}^{m(\omega - z)-2} {}_{1/m}q_{z+y/m}\,\ddot{a}_{z+(y+1)/m}\,v^{y/m}\,]$$

$$= \frac{1}{m} - {}_{1/m}q_{z-1/m}\,\ddot{a}^{(m)}_{z}\,v^{1/m} + v^{1/m}\ddot{a}^{(m)}_{z}$$

$$= \frac{1}{m} + v^{1/m}\ddot{a}^{(m)}_{z}[\,1 - {}_{1/m}q_{z-1/m}\,] = \frac{1}{m} + v^{1/m}\ddot{a}^{(m)}_{z}\;{}_{1/m}p_{z-1/m} = \ddot{a}^{(m)}_{z-1/m},$$

which is identical to the left-hand side of (2).

The proof of (4) is similar to (2), which we leave it to the interested readers.

References

1. Bowers N.L., Gerber H.U., Hickman J.C., Jones D.A. and Nesbitt C.J (1986) *Actuarial Mathematics*. Society of Actuaries, Schaumburg.