Risk Premiums and Their Applications

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Abstract:

In this paper we discuss some properties of the nth stop-loss order and their application in risk premium principles. We give a necessary condition and a sufficient condition for nth stop-loss orders. They are convenient tools to construct risk pairs with nth stop-loss orders.

The maintenance properties of nth stop-loss orders under the operation of compound, in the situation where counting variables N_1 and N_2 are not identical, are proved. The necessary condition for nth stop-loss orders is applied to the valuation of risk premium principles.

We show that exponential premium principles can differentiate between losses more finely than the net premium principles under some conditions.

Key Words:

nth stop-loss transform, nth stop-loss order.

1 Introduction

For an insurance company, each contract of insurance brings a risk with it. A claim may occur some time in the future and the amount of the claim is a nonnegative random variable which is called a risk. One of the main tasks of actuaries is to compare the attractiveness of different risks. This helps them to determine insurance premiums and to decide on the reinsurance needed. Another task of actuaries is to calculate the risk premiums. The basis of insurance is the hypothesis that claims can be compensated by fixed payments called premiums. Premiums are calculated by a premium calculation principle. The partial orders on a family of risks are called risk orders. The theorey of risk orders is a useful mathematical tool for comparing risks and risk premium principles.

From Bowers (1997), we know if the decision maker has decided on the fixed amount to be paid for insurance, also the expected claims is a fixed value, the stop-loss insurance will maximize the expected utility of the decision maker. Consequently, we concern more with the feature of the stop-loss insurance. The properties of nth stop-loss orders provide much more information for studying the stop-loss insurance, since the 1st stop-loss transforms are the stop-loss premiums.

This paper is based upon the works of Goovaerts et al. (1990) and Cheng and Pai (1999a). Many kinds of partial orders were discussed in Goovaerts et al. (1990).

The nth stop-loss order is one of them. In Cheng and Pai (1999a), the concept of stop-loss transforms was generalized to the nth stop-loss transforms. The maintenance properties of the nth stop-loss order under the individual risk model and the collective risk model were developed. In this paper, we first discuss the properties of the nth stop-loss order, later apply them to risk premium principles and ruin probabilities.

This paper is organized as follows. In Section 2, we introduce some definitions and results of Goovaerts et al. (1990) and Cheng and Pai (1999a). In Section 3, we continue the study by Cheng and Pai (1999a) on nth stop-loss orders. We give a necessary condition and a sufficent condition for nth stop-loss orders. They are convenient tools to construct risk pairs having nth stop-loss orders. The maintenance properties of nth stop-loss orders under the operation of compound, in the situation where counting variables N_1 and N_2 are not identical, are to be proved. In Section 4, the necessary condition for nth stop-loss orders will be applied in the valuation of risk premium principles. We will prove that exponential premium principles can differentiate between losses more finely than the net premium principles under some conditions.

2 nth Stop-Loss transform and Order

This article deals with risks to be insured, which are defined as non-negative random variables. Here we cite some definitions and results of Goovaerts et al. (1990) and Cheng and Pai (1999a).

Definition 1 (nth Stop-Loss Transform) Suppose loss random variable X is nonnegative with its distribution function being $F(x)$, its survival function being $\overline{F}(x) = 1 - F(x)$, and $E[X^n] < \infty$. Let

$$\Pi^{(n)}(u) = E[\{(X-u)_+\}^n], \quad u \geq 0,\ n = 1, 2, \cdots, \tag{1}$$

where

$$(x-u)_+ = \begin{cases} 0, & \text{for } x \cdot \ u, \\ x-u, & \text{for } x > u, \end{cases}$$

$$\Pi^{(0)}(u) = \overline{F}(u) = 1 - F(u). \tag{2}$$

As a function of u, $\Pi^{(n)}(u)$, $n = 1, 2, \cdots$, will have domain $[0, \infty)$. We call function $\Pi^{(n)}(u)$ the nth stop-loss transform of X.

Definition 2. **(nth Stop-Loss Order)** We say that X is less than Y in the meaning of the nth stop-loss order, denoted by $X <_{sl(n)} Y$, if

$$E[X^k] \cdot \quad E[Y^k], \quad k = 1, 2, \cdots, n-1, \tag{3}$$

and

$$\Pi_X^{(n)}(u) \cdot \quad \Pi_Y^{(n)}(u), \quad \text{for all } u \geq 0. \tag{4}$$

When $n = 0$, the formula (3) disappears and formula (4) becomes

$$\overline{F}_X(u) \cdot \quad \overline{F}_Y(u), \quad \text{for all } u \geq 0.$$

When $n = 1$, the formula (3) is trivial and formula (4) becomes

$$\int_u^\infty \overline{F}_X(x)dx \cdot \quad \int_u^\infty \overline{F}_Y(x)dx, \quad \text{for all } u \geq 0.$$

Definition 3. **(Weak nth Stop-Loss Order)** Let

$$- \;= \{H(x),\ x \geq 0 : \ H(x) \geq 0 \text{ monotonous decreasing and } \lim_{x \to \infty} H(x) = 0\ \}.$$

Suppose $H(x),\ G(x) \in -$. We say that $H(x)$ is less than $G(x)$ in the meaning of weak nth stop-loss order, denoted by $H <_{wsl(n)} G$, if

$$\Pi_H^{(n)}(u) \cdot \quad \Pi_G^{(n)}(u), \quad \text{for all } u \geq 0. \tag{5}$$

The following results are important and will be used in this paper.

Theorem 1.

$$\frac{d}{du}[\Pi_X^{(n)}(u)] = -n\Pi_X^{(n-1)}(u), \tag{6}$$

or

$$\Pi_X^{(n)}(u) = n\int_u^\infty \Pi_X^{(n-1)}(x)dx. \tag{7}$$

(see Cheng and Pai (1999a), Theorem 6)

Theorem 2. Let $n = 0, 1, 2, \cdots$ and $m > n$. Suppose risk $X <_{sl(n)} Y$. Then $X <_{sl(m)} Y$.

(see Goovaerts et al. (1990), Theorem 4.2.2)

Theorem 3. Suppose $u(x)$ is a utility function having $n-1$ continuous derivatives of alternating sign:

$$(-1)^{(k-1)}u^{(k)}(x) \geq 0,\ k = 1, 2, \cdots, n-1, \tag{8}$$

$$(-1)^{(n-1)}u^{(n)}(x) \geq 0, \text{ and non-decreasing in } x. \tag{9}$$

Let $U_n = \{u(x) : u(x)$ satisfies (8) and (9)$\}$, $w(x) = -u(-x)$, and $W_n = \{w(x) : w^{(k)}(x) = (-1)^{(k+1)}u^{(k)}(-x) \geq 0\}$. Then $X <_{sl(n)} Y$, if and only if

$$E[u(-X)] \geq E[u(-Y)], \text{ for all } u \in U_n,$$

if and only if

$$E[w(X)] \cdot \ E[w(Y)], \text{ for all } w \in W_n.$$

(see Cheng and Pai (1999a), Theorem 10)

Theorem 4. The nth stop-loss order is maintained under the summation of independent random variables. That is, if

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \cdots, k,$$

where k is a positive integer, then

$$\sum_{i=1}^{k} X_i <_{sl(n)} \sum_{i=1}^{k} Y_i, \quad n = 0, 1, 2, \cdots.$$

(see Cheng and Pai (1999a), Theorem 15)

3 Properties of nth Stop-Loss Orders

From Theorem 3, we can see that the nth stop-loss order can be characterized as the common preferences of a group of decision makers with increasingly regular utility functions $u(x) \in U_n$. We will continue the work of Goovaerts et al. (1990) and Cheng and Pai (1999a), to give more features of the nth stop-loss order.

Theorem 5 will be used to compare the differences of the net premium principle and the exponential premium principle in Section 4.

Theorem 5. (Necessary Condition) Suppose X , Y are not identically distributed risks. If $X <_{wsl(n)} Y$ and $E[X^{n+i}] < \infty$, then

$$E[X^{n+k}] < E[Y^{n+k}], \quad k = 1, 2, \cdots, i.$$

Proof

If $E[Y^{n+i}] = \infty$, the result is obvious. If $E[Y^{n+i}] < \infty$, we first show that for $k = 1$, we have $E[X^{n+1}] < E[Y^{n+1}]$. Indeed, let

$$g(u) = \Pi_X^{(n+1)}(u) - \Pi_Y^{(n+1)}(u).$$

From Definition 3 and Theorem 1 , we have: For all $u > 0$, $g(u) \cdot\ 0$, and

$$g'(u) = \frac{d}{du}[\Pi_X^{(n+1)}(u) - \Pi_Y^{(n+1)}(u)] = -(n+1)[\Pi_X^{(n)}(u) - \Pi_Y^{(n)}(u)] \geq 0.$$

Further more, there exists $u_\circ \geq 0$, such that

$$g'(u_\circ) = -(n+1)[\Pi_X^{(n)}(u_\circ) - \Pi_Y^{(n)}(u_\circ)] > 0.$$

(Otherwise we will have $F_X(u) = G_Y(u)$ differentiating $g'(u)$ n times.)

So the following inequality must be true:

$$g(0) = \Pi_X^{(n+1)}(0) - \Pi_Y^{(n+1)}(0) = E[X^{n+1}] - E[Y^{n+1}] < 0.$$

Applying the same method and the fact that $\Pi_X^{(n+j)}(u) \cdot \Pi_Y^{(n+j)}(u)$ for $j = 1, 2, \cdots$ and for all $u > 0$, we obtain the relation

$$E[X^{n+k}] < E[Y^{n+k}], \quad k = 2, \cdots, i. \ \blacksquare$$

A sufficient condition for the nth stop-loss order is given by Theorem 4.2.3 of Goovaerts (1990): $n+1$ sign changes in density functions implies the nth stop-loss order. Here we give another sufficient condition: n sign changes in distribution functions implies the nth stop-loss order.

Theorem 6. (Sufficient Condition) Suppose that for two risks X and Y there is a partition of $[0, \infty)$ into $n+1$ consecutive non-empty intervals (closed intervals containing only one point are acceptable) $I_0, I_1, \cdots, I_n$ such that

$$(-1)^{n+1-j}\{F_X(t) - F_Y(t)\} \cdot \ 0 \ \text{ on } I_j.$$

and the first n moments satisfy $E[X^i] = E[Y^i], \ i = 1, 2, \cdots, n$, then $X <_{sl(n)} Y$.

Proof

For convinence, we let n be an even number. When n is an odd number, we can apply the same method to arrive at the result. Let

$$h_i(t) = \Pi_Y^{(i)}(t) - \Pi_X^{(i)}(t), \quad i = 0, 1, \cdots, n,$$

then from Theorem 1, we have $h_i{}'(t) = -ih_{i-1}(t)$. We only need to show that

$$h_n(t) \geq 0, \quad \text{for all } t > 0. \tag{10}$$

First we know that

$$(-1)^j [F_Y(t) - F_X(t)] \cdot \;\; 0, \;\; j = 1, 2, \cdots, n,$$

and

$$h_1'(t) \cdot \;\; 0, \;\; h_1(t) \downarrow \text{ on } I_0,$$

$$h_1'(t) \geq 0, \;\; h_1(t) \uparrow \text{ on } I_1,$$

$$\vdots$$

$$h_1'(t) \cdot \;\; 0, \;\; h_1(t) \downarrow \text{ on } I_n.$$

On the other hand, from $h_n(0) = h_n(\infty) = 0$, we know that there exists $a_1 \in (0, \infty)$ such that $h_n'(a_1) = 0$. Using Rolle's theorem and repeating this process, we know that there exist $b_1 < b_2 < \cdots < b_{n-1}$ such that

$$h_1(0) = h_1(b_1) = \cdots = h_1(b_{n-1}) = h_1(\infty) = 0.$$

Combin the discussions above, the following conclusion must be true: There exist $c_1 \in I_1, \cdots, c_{n-1} \in I_{n-1}$ such that

$$h_1(t) \cdot \;\; 0 \quad \text{on } [0, c_1) = I_0{}^{(1)},$$

$$h_1(t) \geq 0 \quad \text{on } [c_1, c_2) = I_1^{(1)},$$

$$\vdots$$

$$h_1(t) \geq 0 \quad \text{on } [c_{n-1}, \infty) = I_{n-1}^{(1)}.$$

Repeat the same process, we finally have (10). ∎

We can see that the condition of Theorem 6 implies: $F_X(t) = F_Y(t)$ at least at n different points in $(0, \infty)$.

Theorem 5 and 6 are two useful tools to help us find out or construct the risk pairs which have nth stop-loss orders.

Compound risk was discussed in Theorem 16 of Cheng and Pai (1999a) where the counting variables N_1 and N_2 have identical probability distributions. Now we give another result where $N_1 <_{sl(1)} N_2$ but X_i and Y_i are two sequences of independent and identically distributed risks.

Theorem 7. (Compound Risks) Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be two sequences of independent and identically distributed risks, $N_j (j = 1, 2)$ be counting variables independent of X_i and Y_i. In the collective risk models, S_1 and S_2 are defined as

$$S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.$$

If $X_i <_{sl(n)} Y_i, \quad N_1 <_{sl(1)} N_2$, then we have $S_1 <_{sl(n)} S_2$.

Proof

According to Definition 2, we need to prove

$$E[S_1^i] \cdot \quad E[S_2^i] \quad i = 1, 2, \cdots, n-1, \tag{11}$$

and

$$\Pi_{S_1}^{(n)}(u) \cdot \quad \Pi_{S_2}^{(n)}(u), \text{ for all } u \geq 0. \tag{12}$$

First we prove (12). From Theorem 4, we have for all $u \geq 0$,

$$\begin{aligned}
\Pi_{S_1}^{(n)}(u) &= E[\{(S_1 - u)_+\}^n] \\
&= \sum_{k=0}^{\infty} E[\{(S_1 - u)_+\}^n \mid N_1 = k] \cdot \Pr(N_1 = k) \\
&\cdot \sum_{k=0}^{\infty} E[\{(S_2 - u)_+\}^n \mid N_1 = k] \cdot \Pr(N_1 = k) \\
&= \sum_{k=0}^{\infty} E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] \cdot \Pr(N_1 = k).
\end{aligned} \tag{13}$$

(Define $E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] = 0$ when $k = 0$.)

Let

$$w_1(k) = E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n].$$

It is obvious that $w_1(k) \cdot \quad w_1(k+1), \quad k = 0, 1, \cdots$. If

$$2w_1(k+1) \cdot \quad w_1(k) + w_1(k+2), \quad k = 0, 1, \cdots, \tag{14}$$

we can construct a convex function $w_2(t)$, such that $w_2(k) = w_1(k)$, and $w_2'(t) \geq 0$, and non-decreasing in t. Then from Theorem 3, we have $E[w_2(N_1)] \cdot \quad E[w_2(N_2)]$, and (13) becomes

$$\begin{aligned}
\Pi_{S_1}^{(n)}(u) \quad &\cdot \quad \sum_{k=0}^{\infty} E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] \cdot \Pr(N_1 = k) \\
&\cdot \quad \sum_{k=0}^{\infty} E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] \cdot \Pr(N_2 = k) \\
&= \quad \Pi_{S_2}^{(n)}(u).
\end{aligned}$$

Now we only need to show (14). Let $A_k = \sum_{i=1}^{k} Y_i$. (14) is equivalent to the following inequality

$$\begin{aligned}
&E[\{(A_k + Y_{k+1} - u)_+\}^n] + E[\{(A_k + Y_{k+2} - u)_+\}^n] \\
&\quad \cdot \quad E[\{(A_k - u)_+\}^n] + E[\{(A_k + Y_{k+1} + Y_{k+2} - u)_+\}^n],
\end{aligned}$$

and this follows directly if we look at the conditional distribution with $A_k = a$, $Y_{k+1} = y$, $Y_{k+2} = z$, and use the following inequality

$$(a + y - u)_+^n + (a + z - u)_+^n \cdot \quad (a - u)_+^n + (a + y + z - u)_+^n. \tag{15}$$

When $u \geq a$, (15) is obvious; when $u < a$, we can get (15) by using Binomial Theorem.

Applying the same method, we can prove (11). ∎

In the following Corollary, we generalized the result of Theorem 3.2.5 in Goovaerts et al. (1990) from stop-loss orders to nth stop-loss orders.

Corollary 8. (Conditional Compound Poisson Distribution) Let Λ_j be a non-negative structure variable, and N_j be an integer valued non-negative random variable. Their conditional distribution given $\Lambda_j = \lambda$ of N_j is Poisson(λ) distributed, $j = 1, 2$. Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be two sequences of independent and identically distributed risks, $N_j (j = 1, 2)$ be counting variables independent of X_i and Y_i. In the collective risk models, S_1 and S_2 are defined as

$$S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.$$

If $X_i <_{sl(n)} Y_i, \quad i = 1, 2, \cdots$, and $\Lambda_1 <_{sl(1)} \Lambda_2$, then $S_1 <_{sl(n)} S_2$.

Proof

In view of Theorem 7, we only need to know $N_1 <_{sl(1)} N_2$. From the proof of Theorem 3.2.5 of Goovaerts et al. (1990), $\Lambda_1 <_{sl(1)} \Lambda_2$ implies $N_1 <_{sl(1)} N_2$. ∎

4 The Application in Risk Premium Principles

Now we cite some concepts of risk premium principles in Goovaerts et al. (1990).

We make three assumptions.

1. If $X <_{sl(0)} Y$, then $\pi[X] \cdot \ \pi[Y]$, with equality only if $F_X = F_Y$.

2. If $P[X = c] = 1,\ 0 \cdot \ c$, then $\pi[X] = c$.

3. Let X, X' be risks such that $\pi[X] = \pi[X']$, $p \in [0,\ 1]$, then

$$\pi[pF_X + (1-p)F_Y] = \pi[pF_{X'} + (1-p)F_Y].$$

These assumptions lead to the Mean Value Principle. The premium is calculated from the formula

$$\pi[X] = f^{-1}(E[f(X)]),$$

for some suitable increasing continuous valuation function f. For example, $f(x) = -u(w-x)$ where $u(x)$ is a utility function and w is the wealth of the decision maker. We can narrow the class of premium principles even further by adding the fourth requirement of additivity.

4. A premium principle π is called additive if for independent risk X and Y, $\pi(X+Y) = \pi(X) + \pi(Y)$.

From Theorem 6.2.2 in Goovaerts (1990), we can see that by the four requirments mentioned above the set of feasible premium principles is reduced to the net premium principles $f(x) = x$ and the exponential principles $f(x) = e^{\alpha x}$.

For net premium principle, we can not distinguish the risk X and Y if $E[X] = E[Y]$ but $X <_{sl(1)} Y$ (that is $Var(X) < Var(Y)$ by Theorem 5), the situation is different if we use exponential principle, from the following theorem we can see that the exponential premium principle can differentiate between losses more finely than the net premium principle under some conditions .

Theorem 9. Let X and Y be two risks. If $E[X^k] = E[Y^k],\ k = 1, 2, \cdots, n-1$, and $X <_{sl(n)} Y$, then $\pi(X) < \pi(Y)$, under the exponential premium principle for the same α.

Proof

From Theorem 5, we know that $E[X^j] < E[Y^j],\ j = n, n+1, \cdots$. Consequently,

$$\begin{aligned}
\pi(X) &= \frac{1}{\alpha}\ln[E[e^{\alpha X}]] \\
&= \frac{1}{\alpha}\ln(1 + \alpha E[X] + \frac{\alpha^2}{2!}E[X^2] + \cdots + \frac{\alpha^n}{n!}E[X^n] + \cdots) \\
&< \frac{1}{\alpha}\ln(1 + \alpha E[Y] + \frac{\alpha^2}{2!}E[Y^2] + \cdots + \frac{\alpha^n}{n!}E[Y^n] + \cdots) \\
&= \pi(Y). \ \blacksquare
\end{aligned}$$

5 Concluding Remarks

The theory of partial orders of risks is interesting and useful in many fields. This paper discussed the properties of nth stop-loss orders. The necessary condition and the sufficient condition for the nth stop-loss order are convenient tools to construct risk pairs that can have nth stop-loss orders. The applications of these partial orders in evaluating existing risk premium principles and setting up new risk premium principles are worth further study.

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