Ruin Theory and Credit Risk*

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Abstract

This paper builds a new risk model for a firm which is sensitive to its credit quality. A modified Jarrow, Lando and Turnbull model (Markov Chain model) is used to model the credit rating. Recursive equations for finite time ruin probability and distribution of ruin time are derived. Coupled Volterra type integral equation systems for ultimate ruin probability, severity of ruin and joint distribution of surplus before and after ruin are also obtained. Some numerical results are included.

Keywords: Ruin theory, Credit rating, Markov Chain, Default probability, Default time, Severity of ruin, Recursive equation, Volterra type integral equation system.

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1 Introduction

In actuarial science, ruin theory has been a major area of interest for a long period of time. This is partly due to the fact that ruin probability has been used by insurance industries as a risk measure, and also partly because actuaries have developed elegant mathematics in this area over the past century. This makes the ruin theory both theoretical interest and practical importance. There is a large amount of literature on ruin theory, see, for example, Bühlmann (1982), Gerber (1979), Grandell (1991, 1997), Daykin, Pentikäinen and Pesonen (1994), Klugman, Panjer and Willmot (1998) and Rolski, Schmidli, Schmidt and Teugels (1999).

It is not difficult to see that ruin theory is closely related to default risk in finance. Credit risk and credit risk derivatives have recently attracted considerable interest in the finance society. J.P. Morgan's "Introduction to Credit Metrics" provides an overview on this subject. Also, see for example Das (1995), Das and Tufano (1996), Jarrow and Turnbull (1995), Duffie (1998), and Duffie and Singleton (1996) and the references therein. Jarrow, Lando and Turnbull (1997) proposed a Markov Chain model for valuing risky debts that explicitly incorporates a firm's credit rating as an indicator of the likelihood of default. Later Kijima and Komoribayushi (1998) did some further study on this model. Arvanitis, Gregory and Laurent (1999) built models for credit spreads, a Markov Chain was used to represent the credit rating dynamics.

More and more people have noticed that the interplay between finance and actuarial science is an interesting and productive research area. See for example, Gerber and Shiu (1994) and Embrechts (1999). The purpose of this paper is to build a new model which combines the credit rating model in finance, and ruin theory techniques. This model can be used in two ways: As an insurance risk model (where we build the credit rating classes in a ruin theory framework), and as a credit risk model (where we use some ruin probability techniques which have been developed in actuarial science to analyze the default probability, default time and severity of default).

We model the credit risk ratings using a Markov Chain which is similar to the model of Jarrow, Lando and Turnbull (1997) or Kijima and Komoribayashi (1998). By using the recursive method proposed in De Vylder and Goovaerts (1988) (which is also used in Sun and Yang (2000)), recursive equations satisfied by the finite time ruin probability and distribution of ruin time are obtained. Coupled Volterra type integral equation systems for ultimate ruin probability, distribution of severity of ruin and joint distribution of surplus before and after ruin are also obtained. For the purpose of illustration, we present some numerical results.

This paper is constructed as follows: Section 2 presents the problem formulation and the model. Section 3 derives the recursive equation satisfied by the finite time ruin probability. The time of ruin is discussed. Section 4 deals with the distribution of the severity of ruin and the distribution of surplus before and after ruin. Section 5 provides some numerical results. Some summary remarks and further research topics are given in the final section.

2 The model

In this paper we consider a firm which could be either a financial corporation or an insurance company. At the beginning of each time interval a rating agency will provide a credit rating to assess the firm's abilities in meeting its debt obligations (to pay possible claims in an insurance company case). We use a Markov Chain to model the dynamics of the firm's credit ratings, it is a modification of Jarrow, Lando and Turnbull (JLT) (1997) model. The only difference between our model and JLT model is that we only consider the non default rating states.

Let I_t be a time-homogeneous Markov Chain with a state space of $N = \{1, 2, \ldots, k\}$, where state 1 represents the highest credit class, and state k represents the lowest. In Moody's ratings, state 1 can be thought of as Aaa and state k as Caa, and in $S\&P$'s state 1 as AAA and state k as CCC.

Let

$$q_{ij} = P\{I_{t+1} = j \mid I_t = i\} \ , \quad i, j \in N \ , \quad t = 0, 1, 2, \ldots \tag{2.1}$$

be the one-step transition probabilities, the transition matrix of the Markov Chain I_t can then be written as

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1k} \\ q_{21} & q_{22} & \cdots & q_{2k} \\ \vdots & & & \\ q_{k1} & q_{k2} & \cdots & q_{kk} \end{bmatrix} . \tag{2.2}$$

Let u be the initial surplus of the firm, and X_n^i the portfolio change in the n^{th} time interval if the firm's credit rating in time interval n is of class i. The surplus of the firm at time n can then be written as

$$U_n = u + \sum_{m=1}^{n} X_m^{I_{m-1}}, \tag{2.3}$$

where we assume X_m^i $i = 1, \ldots, k$, $m = 1, 2, \ldots$ are independent random variables. We say that ruin or default occurred at time n if $U_n \leq 0$.

Let $T = \inf\{n; U_n \leq 0\}$, the stopping time, T, is called the ruin time. The probability of ruin before or at time n is defined as

$$\psi_{ni_0}(u) = P\{T \leq n \mid I_0 = i_0, U_0 = u\}. \tag{2.4}$$

Remarks:

1. In this paper we assume that I_t is a time-homogeneous Markov chain, this is only for simplifying notations. The model can be used in cases of non-homogeneous Markov chains. All the analysis in this paper works even if I_t is a non-homogeneous Markov chain.

2. We did not include the default state in the state space of I_t because, in practice, default is the most serious issue, so we would like to study it in detail. In our model, we can study default probability, default time and severity of default, assuming that we are able to estimate the transition matrix of the credit ratings.

3. The probability transition matrix, Q, can be estimated by using real data. We can also use some already available estimation results in literature, for example, from J. P. Morgan's home page, but conditional on non default.

We will assume that for any fixed $i = 1, \ldots, k$, X_m^i $m = 1, 2, \ldots$, are identically distributed. $X^1, \ldots, X^k$ are independent but have different distributions. For example, we may assume that X^1 only take positive values, X^k may take negative values with high probabilities. In our model, we assume that the rating at time t is given according to the information up to time t, for example the initial surplus at time t. The future rating is random and taking values according to the transition matrix. From now on, we will denote the distribution of X^i by $F_i(x)$.

3 Recursive formula for finite time ruin probabilities

Assume that at time 0, $I_0 = i_0$. Denote the ruin probability before or at time n given that the initial surplus is u and the initial state is i_0 by (2.4).

Let

$$\varphi_{ni_0}(u) = 1 - \psi_{ni_0}(u) \tag{3.1}$$

be the survival probability. We then have the following theorem:

Theorem 3.1 *$\varphi_{ni_0}(u)$ satisfies the following recursive equations:*

$$\begin{aligned} \varphi_{1i_0}(u) &= \bar{F}_{i_0}(-u) = 1 - F_{i_0}(-u) && (3.2)\\ \varphi_{ni_0}(u) &= \sum_{i=1}^{n} q_{i_0 i} \int_{-u-}^{\infty} \varphi_{(n-1)i}(u+y)d\,F_{i_0}(y) && (3.3) \end{aligned}$$

$n = 2, 3, \ldots$.

Proof:

$$\begin{aligned} \varphi_{1i_0}(u) &= P\{U_1 > 0 \,|\, U_0 = u, I_0 = i_0\} \\ &= P\{X_1^{i_0} > -u\} = \bar{F}_{i_0}(-u) \\ \varphi_{2i_0}(u) &= P\{U_1 > 0, U_2 > 0 \,|\, U_0 = u, I_0 = i_0\} \\ &= P\{X_1^{i_0} > -u, X_1^{i_0} + X_2^{I_1} > -u\} \\ &= \int_{-u-}^{\infty} P\{X_2^{I_1} > -u - y \,|\, X_1^{i_0} = y\} \; d\,F_{io}(y) \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} P\{X_2^{i} > -u - y\} d\,F_{i_0}(y) \\ &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} \varphi_{1i}(u+y) d\,F_{i_0}(y). \end{aligned}$$

Recursively, we have

$$\begin{aligned} \varphi_{ni_0}(u) &= P\{U_1 > 0, \cdots, U_n > 0 \;, |\, U_0 = u, I_0 = i_0\} \\ &= P\left\{ X_1^{i_0} > -u, \cdots, \sum_{i=1}^{n} X_i^{I_{i-1}} > -u \right\} \\ &= \int_{-u-}^{\infty} P\left\{ X_2^{I_1} > -u - y, \cdots, \sum_{i=2}^{n} X_i^{I_{i-1}} > -u - y \middle| X_1^{i_0} = y \right\} d\,F_{i_0}(y) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} P\Big\{ X_2^i > -u-y, \cdots, \sum_{i=2}^{n} X_i^{I_{i-1}} > -u-y \Big\} d\,F_{i_0}(y) \\
&= \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} \varphi_{(n-1)i}(u+y) d\,F_{i_0}(u) \ .
\end{aligned}
$$

□

In the above model, we have assumed that for any fixed $i \in \{1,\ldots,k\}$, X_n^i, $n = 1,2,\ldots$ are i.i.d. If we now assume that the business will expand, and the growth rate over each time interval is constant. We may use the following model to model the dynamic of the surplus process:

$$
U_n = u + \sum_{m=1}^{n} X_m^{I_{m-1}} (1+r)^{m-1}, \tag{3.4}
$$

with all other assumptions remaining the same as before, we have the following results:

Theorem 3.2 *Let $\tilde{\psi}_{ni_0}(u)$ be the ruin probability on or before time n for model (3.4) with an initial state i_0 and an initial surplus u, and let*

$$
\tilde{\varphi}_{ni_0}(u) = 1 - \tilde{\varphi}_{ni_0}(u) \ .
$$

Then $\tilde{\varphi}_{ni_0}(u)$ satisfies the following recursive equation:

$$
\begin{aligned}
\tilde{\varphi}_{1i_0}(u) &= \bar{F}_{i_0}(-u) \\
\tilde{\varphi}_{ni_0}(u) &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} \tilde{\varphi}_{(n-1)i}((u+y)(1+r)^{-1}) d\,F_{i_0}(y),
\end{aligned}
$$

$n = 2, 3, \ldots$.

Proof: Similar to the proof of Theorem 3.1.

Remark:

1. In Yang (1999), a discrete time insurance risk model with interest income effects was considered. Model (3.4) here is an analogous model but with a different interpretation.

In the following, we will discuss the ruin time (default time) distribution. For notational simplicity, starting from this section, we will only consider model (2.3). Corresponding results for model (3.4) can be obtained without difficulty. Let

$$G_{ni_0}(u) = P\{T = n \,|\, U_0 = u, I_0 = i_0\} \ .$$

Using the recursive method, we can obtain the following result:

Theorem 3.3 *The distribution of ruin time can be calculated using the following recursive equations:*

$$\begin{aligned} G_{1i_0}(u) &= F_{i_0}(-u) \\ G_{ni_0}(u) &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} G_{(n-1)i}(u+y) d\, F_{i_0}(y). \end{aligned}$$

Proof: Similar to the proof of Theorem 3.1.

Remarks:

1. By using our model to consider the default risk, we can not only obtain the default probability, but can also obtain the default time distribution. Furthermore, and as we will see in the next section, we can also obtain the distribution of the severity of default.

2. Let

$$\psi_{i_0}(u) = P\Big\{ \bigcup_{n=1}^{\infty} \Big\{ U_n \leq 0 \Big\} \,\Big|\, U_o = u, I_0 = i_0 \Big\}$$

be the ultimate ruin probability. Then, from definition, we have

$$\begin{aligned}
\psi_{i_0}(u) &= \sum_{n=1}^{\infty} G_{n_{i_0}}(u) \\
&= F_{i_0}(-u) + \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} \sum_{n=2}^{\infty} G_{(n-1)i}(u+y) d F_{i_0}(y) \\
&= F_{i_0}(-u) + \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} \psi_i(u+y) d F_{i_0}(y),
\end{aligned}$$

$i_0 = 1, 2, \ldots, k$.

Therefore, the ultimate ruin probability can be obtained from the coupled Volterra type integral equation system above.

Note that, since $G_{ni}(u+y) \leq 1$ and $F_{i_0}(y)$ is a distribution function, the convergences of the above summation and integration are obvious.

4 Distribution of surplus before and after ruin

In risk theory, actuaries are interested in the severity of ruin and the joint distribution of surplus process immediately before and after ruin. In our setup, we are also able to study these problems.

First, let

$$H_{i_0}(u, y) = P\{U_T \leq -y, T < \infty \mid I_0 = i_0, U_0 = u\}$$

be the distribution of the severity of ruin, where $y > 0$. Then

$$\begin{aligned}
H_{i_0}(u, y) &= P\{U_T \leq -y, T < \infty \mid I_0 = i_0, U_0 = u\} \\
&= \sum_{n=1}^{\infty} P\{U_T \leq -y, T = n \mid I_0 = i_0, U_0 = u\} \\
&= \sum_{n=1}^{\infty} P\{X_1^{i_0} > -u, X_1^{i_0} + X_2^{I_1} > -u, \cdots, X_1^{i_0} + \cdots + X_{n-1}^{I_{n-2}} > -u,
\end{aligned}$$

$$X_1^{i_0} + \cdots + X_n^{I_{n-1}} \leq -u - y\}$$

$$= \sum_{n=1}^{\infty} h_{ni_0}(u, y),$$

where

$$\begin{aligned} h_{ni_0}(u, y) &= P\{X_1^{i_0} > -u, \cdots, X_1^{i_0} + \cdots + X_{n-1}^{I_{n-2}} > -u \\ &\quad X_1^{i_0} + \cdots + X_n^{I_{n-1}} \leq -u - y\} \end{aligned}$$

and $h_{n_{i_0}}(u, y)$ can be calculated recursively by:

$$\begin{aligned} h_{1i_0}(u, y) &= P\{X_1^{i_0} \leq -u - y\} = F_{i_0}(-u - y) \\ h_{ni_0}(u, y) &= \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} h_{(n-1)i}(u + x, y) d\, F_{i_0}(x) \end{aligned}$$

$n = 2, 3, \cdots$.

Therefore, the distribution of the severity of ruin satisfies the following coupled Volterra type integral equation system:

$$H_{i_0}(u, y) = F_{i_0}(-u - y) + \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} H_i(u + x, y) d\, F_{i_0}(x)$$

$i_0 = 1, 2, \cdots, k$.

This system contains k unknown functions $H_1(u, y), \ldots, H_k(u, y)$ and k equations.

Next we will consider the joint distribution of surplus before and after ruin. Define

$$W_{i_0}(u, x, y) = P\{U_T \leq -y, U_{T-1} > x, T < \infty \,|\, I_0 = i_0, U_0 = u\}$$

where $x > 0$ and $y > 0$.

It is easy to see that

$$W_{i_0}(u, x, y) = \sum_{n=1}^{\infty} P\{U_n \leq -y, U_{n-1} > x, T = n \,|\, I_0 = i, U_0 = u\}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} P\{X_1^{i_0} > -u, \cdots, X_1^{i_0} + \cdots + X_{n-2}^{I_{n-3}} > -u, \\
&\qquad X_1^{i_0} + \cdots + X_{n-1}^{I_{n-2}} > -u + x, X_1^{i_0} + \cdots + X_n^{I_{n-1}} \leq -u - y\} \\
&= \sum_{n=1}^{\infty} w_{ni_0}(u, x, y) \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
w_{ni_0}(u, x, y) \ &= \ P\{X_1^{i_0} > -u, \cdots, X_1^{i_0} + \cdots + X_{n-2}^{i_{n-3}} > -u \\
&\qquad X_1^{i_0} + \cdots + X_{n-1}^{I_{n-2}} > -u + x, X_1^{i_0} + \cdots + X_n^{I_{n-1}} \leq -u - y\},
\end{aligned}
$$

and $w_{ni_0}(u, x, y)$ can be calculated recursively by:

$$
\begin{aligned}
w_{1i_0}(u, x, y) \ &= \ \begin{cases} F_{i_0}(-u - y) & \text{if} \quad u > x \\ 0 & \text{if} \quad u \leq x \end{cases} \\
w_{2i_0}(u, x, y) \ &= \ P\{X_1^{i_0} + X_2^{I_1} \leq -u - y, X_1^{i_0} > -u + x\} \\
&= \ \sum_{i=1}^{k} q_{i_0 i} \int_{-u+x-}^{\infty} F_i(-u - y - s) d\, F_{i_0}(s),
\end{aligned}
$$

for $n \geq 3$

$$
w_{ni_0}(u, x, y) = \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{\infty} w_{(n-1)i}(u + s, x, y) d\, F_{i_0}(s) \ .
$$

Therefore, let $\tilde{W}_{i_0}(u, x, y)$, $i_0 = 1, ..., k$, be the solutions from the following coupled Volterra type integral equation system:

$$
\tilde{W}_{i_0}(u, x, y) = F_{i_0}(-u - y) + \sum_{i=1}^{k} q_{i_0 i} \int_{-u}^{\infty} \tilde{W}_i(u - s; x, y) d\, F_{i_0}(s) \ .
$$

Then $W_{i_0}(u, x, y)$ is given by: when $u > x$,

$$
W_{i_0}(u, x, y) = \tilde{W}_{i_0}(u, x, y) - \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{-u+x} F_i(-u - y - s) d\, F_{i_0}(s),
$$

and when $u \leq x$

$$
W_{i_0}(u, x, y) = \tilde{W}_{i_0}(u, x, y) - \sum_{i=1}^{k} q_{i_0 i} \int_{-u-}^{-u+x} F_i(-u - y - s) d\, F_{i_0}(s) - F_{i_0}(-u - y).
$$

Remark:

1. In actuarial literature, the severity of ruin and the joint distribution of surplus before and after ruin have been studied by many authors. Gerber, Goovaerts and Kaas (1987) considered the distribution of the severity of ruin and an integral equation was obtained. In cases where the claims have an exponential-mixture or Gamma-mixture distribution, closed form solutions for the distribution of the severity of ruin were obtained. Later Dufresne and Gerber (1988) introduced the distribution of the surplus immediately prior to ruin in the classical compound Poisson risk model. Similar results to Gerber, Goovaerts and Kaas (1987) were obtained in that paper. Dickson (1992) used a different way to deal with the distribution of the surplus immediately prior to ruin. Using the relationship of various events, he found the relationship among the distributions of the surplus prior to ruin, after ruin, and the ruin probability. In the paper, Dickson used the distribution of the surplus after ruin and ruin probability to express the distribution prior to ruin, then the results of the distribution of the surplus after ruin and the ruin probability are used to obtain the results for the distribution of the surplus prior to ruin. Gerber and Shiu (1997,1998) examined the joint distribution of the time of ruin, the surplus immediately before ruin and the deficit at ruin. They showed that, as a function of the initial surplus, the joint density of the surplus immediately before ruin and the deficit at ruin satisfies a renewal equation.

5 Numerical results

In this section, we will present some illustrative numerical results. This example is chosen purely for illustration purpose. We use the one year credit rating transition matrix from “CreditMetrics - Technical Document”. However, we use the conditional probabilities on the non-default states rather than using the matrix directly.

$$Q = \begin{bmatrix} 0.9081 & 0.0833 & 0.0068 & 0.0006 & 0.0012 & 0 & 0 \\ 0,0070 & 0.9065 & 0.0779 & 0.0064 & 0.0006 & 0.0014 & 0.0002 \\ 0.0009 & 0.0227 & 0.9111 & 0.0552 & 0.0074 & 0.0026 & 0.0001 \\ 0.0002 & 0.0033 & 0.0596 & 0.8709 & 0.0531 & 0.0117 & 0.0012 \\ 0.0003 & 0.0014 & 0.0068 & 0.0781 & 0.8140 & 0.0893 & 0.0101 \\ 0 & 0.0012 & 0.0025 & 0.0045 & 0.0684 & 0.8805 & 0.0429 \\ 0.0027 & 0 & 0.0028 & 0.0162 & 0.0296 & 0.1401 & 0.8086 \end{bmatrix}.$$

We assume that the portfolio changes in each time intervals follows a shifted t distribution. That is

$$X^i \sim t(\alpha_i, n_i),$$

where α_i is the shift parameter (i.e. $X^i - \alpha_i \sim t(n_i),$), and n_i is the degree of freedom of the t distribution. In this example we let $n_1 = 19, \alpha_1 = 2; n_2 = 17, \alpha_2 = 1.5; n_3 = 15, \alpha_3 = 1; n_4 = 13, \alpha_4 = 0.5; n_5 = 11, \alpha_5 = 0; n_6 = 9, \alpha_6 = -0.5; n_7 = 7, \alpha_7 = -1.$ All the numerical results are plausible.

Table 1 gives the non-ruin probabilities calculated from the recursive formulas (3.2) and (3.3) for $u = 5$.

Table 1

n	$i_0 = 1$	$i_0 = 2$	$i_0 = 3$	$i_0 = 4$	$i_0 = 5$	$i_0 = 6$	$i_0 = 7$
1	1.00000	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998
2	1.00000	1.00000	1.00000	1.00000	0.99999	0.99998	0.99986
3	1.00000	1.00000	1.00000	0.99999	0.99998	0.99988	0.99910
4	1.00000	1.00000	0.99999	0.99999	0.99992	0.99934	0.99534
5	1.00000	1.00000	0.99999	0.99997	0.99964	0.99736	0.98360
6	1.00000	0.99999	0.99999	0.99990	0.99882	0.99221	0.95948
7	1.00000	0.99999	0.99997	0.99972	0.99698	0.98225	0.92310
8	1.00000	0.99998	0.99992	0.99934	0.99362	0.96667	0.87852
9	0.99999	0.99995	0.99984	0.99869	0.98847	0.94567	0.83067
10	0.99999	0.99991	0.99970	0.99769	0.98138	0.92028	0.78336
11	0.99998	0.99985	0.99949	0.99630	0.97243	0.89179	0.73880
12	0.99997	0.99977	0.99920	0.99448	0.96183	0.86155	0.69798
13	0.99996	0.99967	0.99880	0.99222	0.94987	0.83069	0.66115
14	0.99994	0.99954	0.99832	0.98954	0.93686	0.80011	0.62818
15	0.99991	0.99939	0.99773	0.98646	0.92310	0.77046	0.59875
16	0.99988	0.99921	0.99703	0.98301	0.90889	0.74217	0.57250
17	0.99984	0.99900	0.99623	0.97924	0.89446	0.71549	0.54909
18	0.99979	0.99876	0.99532	0.97519	0.88003	0.69056	0.52818
19	0.99974	0.99850	0.99432	0.97089	0.86576	0.66741	0.50947
20	0.99967	0.99821	0.99321	0.96639	0.85177	0.64601	0.49271

Table 1

21	0.99960	0.99789	0.99202	0.96174	0.83817	0.62629	0.47765
22	0.99952	0.99754	0.99074	0.95696	0.82502	0.60816	0.46409
23	0.99943	0.99716	0.98938	0.95209	0.81236	0.59151	0.45185
24	0.99934	0.99676	0.98795	0.94716	0.80022	0.57623	0.44079
25	0.99923	0.99632	0.98645	0.94219	0.78863	0.56220	0.43074
26	0.99912	0.99587	0.98489	0.93722	0.77757	0.54932	0.42161
27	0.99899	0.99538	0.98327	0.93226	0.76704	0.53748	0.41328
28	0.99886	0.99487	0.98160	0.92732	0.75704	0.52658	0.40567
29	0.99872	0.99434	0.97989	0.92243	0.74754	0.51654	0.39869
30	0.99857	0.99378	0.97814	0.91758	0.73852	0.50728	0.39227
31	0.99841	0.99320	0.97635	0.91281	0.72996	0.49873	0.38636
32	0.99825	0.99260	0.97454	0.90810	0.72184	0.49080	0.38089
33	0.99807	0.99198	0.97270	0.90347	0.71414	0.48346	0.37583
34	0.99789	0.99134	0.97083	0.89893	0.70683	0.47663	0.37113
35	0.99770	0.99068	0.96895	0.89447	0.69989	0.47028	0.36676
36	0.99750	0.99000	0.96706	0.89011	0.69330	0.56436	0.36267
37	0.99730	0.98930	0.96516	0.88583	0.68703	0.45882	0.35886
38	0.99708	0.98859	0.96324	0.88165	0.68107	0.45365	0.35528
39	0.99686	0.98786	0.96133	0.87757	0.67540	0.44879	0.35192
40	0.99663	0.98712	0.95941	0.87358	0.67000	0.44423	0.34876

The following table (Table 2) provides some values of the ruin time distribution when the initial surplus $u = 5$.

Table 2

n	$i_0 = 1$	$i_0 = 2$	$i_0 = 3$	$i_0 = 4$	$i_0 = 5$	$i_0 = 6$	$i_0 = 7$
1	1.2997E-10	9.6507E-10	7.0516E-09	5.1063E-08	3.7053E-07	2.7431E-06	2.1366E-05
2	1.8770E-10	4.8189E-09	1.9492E-08	2.0253E-07	2.1290E-06	1.6471E-05	1.2304E-04
3	8.7780E-10	2.4554E-08	8.0193E-08	9.5123E-07	1.2376E-05	1.0254E-04	7.7115E-04
4	4.4848E-09	1.3819E-07	4.1796E-07	4.9600E-06	6.7473E-05	5.4213E-04	3.7963E-03
5	2.3004E-08	6.8122E-07	2.0001E-06	2.2119E-05	2.8079E-04	2.0211E-03	1.1942E-02
6	1.0173E-07	2.5025E-06	7.3792E-06	7.4065E-05	8.4114E-04	5.2587E-03	2.4727E-02
7	3.5422E-07	6.8326E-06	2.0742E-05	1.8855E-04	1.8971E-03	1.0240E-02	3.7555E-02
8	9.7136E-07	1.4703E-05	4.6545E-05	3.8589E-04	3.4446E-03	1.6114E-02	4.6289E-02
9	2.1788E-06	2.6535E-05	8.7857E-05	6.7031E-04	5.3343E-03	2.1787E-02	4.9902E-02
10	4.1711E-06	4.2176E-05	1.4562E-04	1.0306E-03	7.3541E-03	2.6452E-02	4.9512E-02
11	7.0671E-06	6.1154E-05	2.1888E-04	1.4461E-03	9.3064E-03	2.9744E-02	4.6747E-02
12	1.0902E-05	8.2901E-05	3.0538E-04	1.8929E-03	1.1047E-02	3.1647E-02	4.2895E-02
13	1.5640E-05	1.0687E-04	4.0226E-04	2.3483E-03	1.2491E-02	3.2343E-02	3.8751E-02
14	2.1205E-05	1.3258E-04	5.0652E-04	2.7933E-03	1.3608E-02	3.2090E-02	3.4731E-02
15	2.7492E-05	1.5965E-04	6.1531E-04	3.2135E-03	1.4399E-02	3.1149E-02	3.1018E-02
16	3.4396E-05	1.8773E-04	7.2609E-04	3.5986E-03	1.4893E-02	2.9747E-02	2.7671E-02
17	4.1810E-05	2.1656E-04	8.3670E-04	3.9426E-03	1.5126E-02	2.8067E-02	2.4693E-02
18	4.9642E-05	2.4590E-04	9.4538E-04	4.2424E-03	1.5143E-02	2.6245E-02	2.2059E-02
19	5.7808E-05	2.7555E-04	1.0507E-03	4.4972E-03	1.4985E-02	2.4381E-02	1.9736E-02
20	6.6238E-05	3.0533E-04	1.1517E-03	4.7083E-03	1.4691E-02	2.2543E-02	1.7691E-02

Table 2

21	7.4875E-05	3.3510E-04	1.2474E-03	4.8779E-03	1.4295E-02	2.0776E-02	1.5891E-02
22	8.3668E-05	3.6469E-04	1.3374E-03	5.0089E-03	1.3826E-02	1.9106E-02	1.4306E-02
23	9.2578E-05	3.9400E-04	1.4213E-03	5.1048E-03	1.3308E-02	1.7548E-02	1.2910E-02
24	1.0157E-04	4.2290E-04	1.4990E-03	5.1693E-03	1.2760E-02	1.6108E-02	1.1679E-02
25	1.1062E-04	4.5129E-04	1.5703E-03	5.2058E-03	1.2198E-02	1.4787E-02	1.0593E-02
26	1.1969E-04	4.7909E-04	1.6353E-03	5.2179E-03	1.1632E-02	1.3580E-02	9.9331E-03
27	1.2878E-04	5.0621E-04	1.6942E-03	5.2088E-03	1.1073E-02	1.2481E-02	8.7835E-03
28	1.3785E-04	5.3258E-04	1.7470E-03	5.1815E-03	1.0526E-02	1.1484E-02	8.0302E-03
29	1.4690E-04	5.5815E-04	1.7942E-03	5.1389E-03	9.9973E-03	1.0580E-02	7.3612E-03
30	1.5590E-04	5.8286E-04	1.8358E-03	5.0833E-03	9.4891E-03	9.7610E-03	6.7657E-03
31	1.6485E-04	6.0668E-04	1.8722E-03	5.0169E-03	9.0036E-03	9.0199E-03	6.2346E-03
32	1.7372E-04	6.2957E-04	1.9037E-03	4.9418E-03	8.5421E-03	8.3492E-03	5.7600E-03
33	1.8252E-04	6.5151E-04	1.9306E-03	4.8597E-03	8.1049E-03	7.7418E-03	5

Table 3

y	$i_0 = 1$	$i_0 = 2$	$i_0 = 3$	$i_0 = 4$	$i_0 = 5$	$i_0 = 6$	$i_0 = 7$
0	1.2418E-02	3.2484E-02	7.3738E-02	1.4199E-01	2.2671E-01	3.0059E-01	3.4775E-01
0.1	9.8649E-03	2.7390E-02	6.5696E-02	1.3168E-01	2.1518E-01	2.9007E-01	3.4054E-01
0.2	8.0567E-03	2.2895E-02	5.6380E-02	1.1644E-01	1.9655E-01	2.7334E-01	3.2923E-01
0.3	6.5590E-03	1.9060E-02	4.8143E-02	1.0237E-01	1.7841E-01	2.5610E-01	3.1694E-01
0.4	5.3240E-03	1.5808E-02	4.0913E-02	8.9485E-02	1.6095E-01	2.3852E-01	3.0371E-01
0.5	4.3100E-03	1.3064E-02	3.4612E-02	7.7799E-02	1.4431E-01	2.2080E-01	2.8959E-01
0.6	3.4807E-03	1.0761E-02	2.9159E-02	6.7288E-02	1.2862E-01	2.0314E-01	2.7467E-01
0.7	2.8047E-03	8.8377E-03	2.4467E-02	5.7910E-02	1.1396E-01	1.8573E-01	2.5908E-01
0.8	2.2555E-03	7.2383E-03	2.0456E-02	4.9606E-02	1.0041E-01	1.6876E-01	2.4296E-01
0.9	1.8107E-03	5.9137E-03	1.7044E-02	4.2307E-02	8.7994E-02	1.5241E-01	2.2650E-01
1	1.4514E-03	4.8208E-03	1.4158E-02	3.5935E-02	7.6719E-02	1.3683E-01	2.0986E-01
1.1	1.1618E-03	3.9220E-03	1.1727E-02	3.0408E-02	6.6564E-02	1.2212E-01	1.9326E-01
1.2	9.2892E-04	3.1852E-03	9.6884E-03	2.5641E-02	5.7492E-02	1.0840E-01	1.7689E-01
1.3	7.4200E-04	2.5829E-03	7.9861E-03	2.1552E-02	4.9447E-02	9.5706E-02	1.6094E-01
1.5	4.7239E-04	1.6921E-03	5.3946E-03	1.5100E-02	3.6163E-02	7.3522E-02	1.3091E-01
2	1.5190E-04	5.8011E-04	1.9764E-03	5.9861E-03	1.5714E-02	3.5481E-02	7.1544E-02
2.5	4.9075E-05	1.9835E-04	7.1419E-04	2.3102E-03	6.5380E-03	1.6065E-02	3.5804E-02
3	1.6125E-05	6.8694E-05	2.5946E-04	8.8912E-04	2.6852E-03	7.0903E-03	1.7204E-02

The following Table 4 provides some values of the distribution of the surplus immediately after ruin for $u = 2$.

Table 4

y	$i_0 = 1$	$i_0 = 2$	$i_0 = 3$	$i_0 = 4$	$i_0 = 5$	$i_0 = 6$	$i_0 = 7$
0	6.9020E-03	1.9596E-02	4.8922E-02	1.0459E-01	1.8474E-01	2.6550E-01	3.2473E-01
0.1	1.5241E-04	7.2591E-04	3.2062E-03	1.2630E-02	3.8434E-02	8.1929E-02	1.3389E-01
0.2	1.2257E-04	5.9461E-04	2.6719E-03	1.0774E-02	3.3629E-02	7.3182E-02	1.2156E-01
0.3	9.8615E-05	4.8705E-04	2.2242E-03	9.1724E-03	2.9353E-02	6.5198E-02	1.1010E-01
0.4	7.9387E-05	3.9897E-04	1.8496E-03	7.7942E-03	2.5557E-02	5.7931E-02	9.9497E-02
0.5	6.3948E-05	3.2687E-04	1.5366E-03	6.6108E-03	2.2197E-02	5.1334E-02	8.9702E-02
0.6	5.1546E-05	2.6786E-04	1.2755E-03	5.5971E-03	1.9230E-02	4.5362E-02	8.0679E-02
0.7	4.1580E-05	2.1956E-04	1.0578E-03	4.7306E-03	1.6620E-02	3.9973E-02	7.2387E-02
0.8	3.3566E-05	1.8002E-04	8.7670E-04	3.9918E-03	1.4329E-02	3.5124E-02	6.4784E-02
0.9	2.7120E-05	1.4767E-04	7.2612E-04	3.3631E-03	1.2325E-02	3.0778E-02	5.7831E-02
1	2.1930E-05	1.2118E-04	6.0107E-04	2.8293E-03	1.0577E-02	2.6894E-02	5.1489E-02
1.1	1.7749E-05	9.9484E-05	4.9733E-04	2.3771E-03	9.0578E-03	2.3437E-02	4.5722E-02
1.2	1.4378E-05	8.1717E-05	4.1134E-04	1.9948E-03	7.7408E-03	2.0370E-02	4.0493E-02
1.3	1.1659E-05	6.7159E-05	3.4012E-04	1.6721E-03	6.6026E-03	1.7661E-02	3.5768E-02
1.5	7.6886E-06	4.5444E-05	2.3245E-04	1.1717E-03	4.7791E-03	1.3183E-02	2.7697E-02
2	2.7662E-06	1.7334E-05	8.9952E-05	4.7727E-04	2.0844E-03	6.1474E-03	1.4078E-02
2.5	1.0256E-06	6.7696E-06	3.5267E-05	1.9460E-04	8,9758E-04	2.7988E-03	6.9298E-03
3	3.9365E-07	2.7277E-06	1.4162E-05	8.0671E-05	3.8957E-04	1.2759E-03	3.4003E-03

The following Table 5 provides some values of the distribution of the surplus immediately after ruin for $u = 5$.

Table 5

y	$i_0 = 1$	$i_0 = 2$	$i_0 = 3$	$i_0 = 4$	$i_0 = 5$	$i_0 = 6$	$i_0 = 7$
0	3.0304E-03	9.5178E-03	2.6787E-02	6.5645E-02	1.3293E-01	2.1385E-01	2.8468E-01
0.1	1.6459E-06	1.4431E-05	8.3791E-05	6.2690E-04	3.7077E-03	1.2464E-02	2.8263E-02
0.2	1.4198E-06	1.2584E-05	7.2251E-05	5.4532E-04	3.2778E-03	1.1174E-02	2.5722E-02
0.3	1.2250E-06	1.0971E-05	6.2249E-05	4.7356E-04	2.8910E-03	9.9917E-03	2.3353E-02
0.4	1.0568E-06	9.5598E-06	5.3584E-05	4.1054E-04	2.5438E-03	8.9101E-03	2.1150E-02
0.5	9.1160E-07	8.3252E-06	4.6080E-05	3.5529E-04	2.2328E-03	7.9235E-03	1.9105E-02
0.6	7.8598E-07	7.2443E-06	3.9587E-05	3.0694E-04	1.9551E-03	7.0263E-03	1.7212E-02
0.7	6.7727E-07	6.2977E-06	3.3973E-05	2.6471E-04	1.7077E-03	6.2128E-03	1.5463E-02
0.8	5.8316E-07	5.4689E-06	2.9123E-05	2.2789E-04	1.4880E-03	5.4778E-03	1.3853E-02
0.9	5.0171E-07	4.7435E-06	2.4938E-05	1.9587E-04	1.2934E-03	4.8160E-03	1.2376E-02
1	4.3123E-07	4.1092E-06	2.1331E-05	1.6806E-04	1.1217E-03	4.2223E-03	1.1024E-02
1.1	3.7029E-07	3.5549E-06	1.8226E-05	1.4398E-04	9.7055E-04	3.6917E-03	9.7925E-03
1.2	3.1763E-07	3.0714E-06	1.5556E-05	1.2317E-04	8.3799E-04	3.2193E-03	8.6742E-03
1.3	2.7220E-07	2.6501E-06	1.3265E-05	1.0521E-04	7.2207E-04	2.8004E-03	7.6630E-03
1.5	1.9932E-07	1.9656E-06	9.6185E-06	7.6494E-05	5.3320E-04	2.1045E-03	5.9349E-03
2	9.0347E-08	9.1560E-07	4.2625E-06	3.3972E-05	2.4402E-04	9.9927E-04	3.0226E-03
2.5	4.0801E-08	4.2289E-07	1.8875E-06	1.5018E-05	1.1014E-04	4.6452E-04	1.4976E-03
3	1.8707E-08	1.9783E-07	8.5021E-07	6.7398E-06	5.0187E-05	2.1707E-04	7.4402E-04

6 Conclusive remarks and further research topics

In this paper we have built a new ruin theory model which incorporate the firm's credit risk. Recursive equations have been derived for the finite time probability of ruin and the distribution of ruin time. Coupled Volterra type integral equation systems for ultimate ruin probability, distribution of the severity of ruin and the joint distribution of surplus process before and after ruin have been obtained. As an insurance risk model, this model, to the best of our knowledge, is the first to incorporate the credit ratings. This model can also serve as a credit risk model. As a credit risk model, we have, by using some actuarial science techniques, provided some detailed study on the default risk. Some numerical result have been presented.

There are many problems which can be considered further. Currently the model is a simple one. If we assume the rating process I_n depends on the surplus process U_n, then the problem becomes very difficult mathematically. The portfolio change process X_m could be discussed further, especially some practical issues. Up to now we have only considered the discrete time model. The ideas in this paper can be used in other risk models, we will address this problem in future research.

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