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The Gompertz Distribution-Estimation of Parameters

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ABSTRACT. In this paper, we derive estimates for the Gompertz distribution based on maximum likelihood and order statistics. The maximum likelihood estimates are determined directly from the underlying distribution while the BLUE's and BLIE's are achieved through a transformation from the Extreme Value distribution. We also include examples to demonstrate the procedures.

1. INTRODUCTION

Most computations that are done within Actuarial Science entail using discrete death rates associated with incremental ages. It's not that continuous death rates cannot be used, it's just that discrete values tend to be more convenient to work with. Of the various continuous probability distributions, the one most closely associated with Actuarial Science may be the Gompertz distribution. In this paper, we will take a closer look at the Gompertz, starting with the motivation behind Benjamin Gompertz proposing this particular distribution, then develop procedures to estimate the parameters for this distribution, and then to end with a brief example of the procedures presented in this paper.

Within Actuarial Science, the force of mortality is defined as

$$\mu_x = -\left(\frac{1}{l_x}\right)\frac{dl_x}{dx}$$

where l_x is the number of individuals living at time x and μ_x is the force of mortality at time x. The force of mortality is analogous to failure rate which is encountered in reliability theory, where the failure rate is defined as

$$h(t) = -\left(\frac{1}{S(t)}\right)\frac{dS(t)}{dt}$$

and $S(t)$ is the survival function. The notation, $h(t)$ will be used throughout the development to avoid any confusion that may arise with the normally reserved definition of the greek letter μ in the statistical sciences.

Benjamin Gompertz (1825), through his observations while valuing life annuities, realized that if a constant force of mortality was in effect for each age, then regardless of age, the annuities would take on the same value whether they were valued at age 20 or at age 65. However, in practice, this was not the case. The price of an annuity was considerably less expensive for a person aged 65 than for a person aged 20. Gompertz felt that this discrepancy was due to the erroneous assumption that the mortality rate was constant. He conjectured

> "...that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or an increased inability to withstand destruction."

From this philosophy he proposed that

> If the average exhaustions of a man's power to avoid death were such that at the end of equal infinitely small intervals of time, he lost equal portions of his remaining power to oppose destruction which he had at the commencement of those intervals, then at age x his power to avoid death, or the intensity of his mortality might be denoted by aq^x, a and q being constant quantities;...

Traditionally, in Actuarial literature, a is denoted by B and q is denoted by c, where the parameter B is attributed to chance, and the parameter c is attributed to "...an increased inability to withstand destruction." Now, the "intensity" of mortality may be denoted as $h(t; B, c) = Bc^t$ where the parameter space is traditionally defined as $\Omega = \{(B, c) : B > 0; c > 1\}$ and the variable $t \in \Re^+$.

With the parameter space Ω defined, it can be seen that the Gompertz distribution possesses the property of increasing failure rate (IFR). If we would allow c to vary, we see that if $c = 1$, the rate of mortality will be constant, and for $c < 0$, the Gompertz distribution will possess the property of decreasing failure rate (DFR). This is consistent with Gompertz's philosophy. As time progresses, it is a logical assumption that the rate of deterioration would increase. Similarly, holding the parameter B constant at a positive value will ensure that for each unit of time, there will be a positive "chance" of death, as opposed to an unanticipated situation (like an organ transplant) where, due to chance, an individual's life will somehow be prolonged.

It follows that the probability density function (pdf) and the cumulative distribution function (CDF) may be stated as

$$f_G(t; B, c) = Bc^t \exp\left[-\frac{B}{\ln c}(c^t - 1)\right], \quad 0 \leq t < \infty \tag{1}$$

$$F_G(t; B, c) = 1 - \exp\left[-\frac{B}{\ln c}(c^t - 1)\right], \quad 0 \leq t < \infty. \tag{2}$$

Another form that is also encountered, and which will prove to be useful, is the *generalized Gompertz*, given by

$$f_g(t; B, c) = Bc^t \exp\left(-\frac{B}{\ln c}c^t\right), \quad -\infty < t < \infty \tag{3}$$

$$F_g(t; B, c) = 1 - \exp\left(-\frac{B}{\ln c}c^t\right), \quad -\infty < t < \infty \tag{4}$$

where the integration is allowed over the entire real line. In this paper we will show that in the case of the generalized Gompertz, transformations between the Extreme Value distribution and the 2-parameter Weibull distribution may be achieved from which some desirable consequences will follow. Namely the ability to calculate the best linear unbiased estimators(BLUE) and best linear invariant estimator(BLIE) based on the previous work of Mann(1969) and Lloyd(1952). We will develop for the Gompertz and the generalized Gompertz equations for the MLE's, moments and moment generating function. In contrast, the Gompertz will be shown, through appropriate transformation, to be a 3-parameter Weibull. Therefore, due to the complexity involved, no further development will be pursued in this regard.

Note that throughout this paper, the distribution described by equations (1) and (2) will be called the *Gompertz Distribution*, while the distribution described by equations (3) and (4) will be called the *generalized Gompertz*. To distinguish between the two, we denote the Gompertz by subscript G and the generalized Gompertz by the subscript g.

2. PROPERTIES

One of the most generally accepted and well known methods of estimating parameters for a distribution is with the method of moments. In this section, we develop the moment's and moment generating function(mgf) for the Gompertz, and state without development the results for the generalized Gompertz. In addition, we provide a transformation to the Extreme Value distribution as well as a transformation to the 2-parameter Weibull distribution.

2.1. Moments. The *first moment* for the Gompertz distribution is

$$E_G(x) = \int_0^\infty x \cdot Bc^x \exp\left[-\frac{B}{\ln c} \cdot (c^x - 1)\right] dx$$

and if we integrate by parts with $u = x$ and $dv = \frac{B}{\ln c} \exp\left[-\frac{B}{\ln c}(c^x - 1)\right] dx$, we have

$$E_G(x) = e^{\frac{B}{\ln c}} \int_0^\infty \exp\left(-\frac{B}{\ln c} e^{x \ln c}\right) dx$$

which is of the form

$$\int_0^\infty \exp(-ae^{nx})\, dx = \frac{-1}{n} Ei(-a)$$

where $a = B/\ln c$ and $n = \ln c$. If we make these substitutions, we then have

$$E_G(x) = e^{\frac{B}{\ln c}} \left(\frac{-1}{\ln c}\right) Ei\left(-\frac{B}{\ln c}\right) \tag{5}$$

where Ei is the exponential integral defined by $Ei(-a) = -a \int_1^\infty e^{-ax} \ln x dx$. Proceeding in a similar fashion, the first moment for the generalized Gompertz will be given by

$$E_g(x) = (\ln c)\Gamma\left[1 + \left(\frac{B}{\ln c}\right)^{\frac{1}{\ln c}}\right]$$

where $\Gamma(\alpha)$ is the Gamma Function defined by $\Gamma(\alpha) = \int_0^\infty w^{\alpha-1} e^{-w} dw$. For more discussion of the properties of the exponential integral, we refer the reader to Gradshteyn and Ryzhik(1980).

The *second moment* for the Gompertz distribution is

$$E_G(x^2) = \int_0^\infty x^2 \cdot Bc^x \exp\left[-\frac{B}{\ln c} \cdot (c^x - 1)\right] dx$$

and if we make the substitutions, $w = \frac{B}{\ln c} e^{x \ln c}$, $dw = Be^{x \ln c} dx$ and $x = \ln\left[w \ln\left(c^{\frac{1}{B}}\right)\right] / \ln c$, we see that the range on w becomes $\frac{B}{\ln c} < w < \infty$ and the we may now state the second moment as

$$E_G(w^2) = \frac{e^{\frac{B}{\ln c}}}{(\ln c)^2} \int_{\frac{B}{\ln c}}^\infty \left[\ln w + \ln\left(\ln c^{\frac{1}{B}}\right)\right]^2 e^{-w} dw. \tag{6}$$

The intricate computations involved in (5) and (6) render the method of moments intractable. In fact, within actuarial literature (see London, 1988), applying a Taylor Series expansion is generally utilized in conjunction with the method of moments so that (5) and (6) become more tractable. However, if we expand the range of integration from $\frac{B}{\ln c} < w < \infty$ to $[0, \infty)$, we see that we will obtain an upper bound for the true value. Furthermore, if it is discovered that the quantity, $(B/\ln c)$ is "close" to zero, the quantity obtained from integration will be "close" to the real value. With this approximation, we may now use the following equalities

$$\begin{aligned} \int_0^\infty e^{\mu x}(\ln x)^2 dx &= \frac{1}{\mu}\left[\frac{\pi^2}{6} + (C_{Euler} + \ln\mu)^2\right] \\ \int_0^\infty e^{\mu x}\ln x dx &= \frac{-1}{\mu}(C_{Euler} + \ln\mu) \end{aligned}$$

where C_{Euler} is Euler's constant, and is defined as $C_{Euler} = .5772\,1566$. Now we may bound the second moment from above and write

$$E_G(x^2) < \frac{e^{\frac{B}{\ln c}}}{\ln c^2}\left\{\pi^2 + C_{Euler}^2 + 2\ln(\ln c^{\frac{1}{B}}) - C_{Euler} + [\ln(\ln c^{\frac{1}{B}})]^2\right\}.$$

The second moment will exist and converge by virtue of the Dominated Convergence Theorem.

In the case of the generalized Gompertz, we will be able to achieve an exact value for the second moment, and will be defined as

$$E_g(x^2) = (\ln c)^2\,\Gamma\left[1 + 2\left(\frac{B}{\ln c}\right)^{\frac{1}{\ln c}}\right].$$

Now, we develop the *moment generating function.* The mgf is

$$\begin{aligned} E_G(e^{tx}) &= M_{G:x}(t) = \int_0^\infty e^{tx}Bc^x \exp\left[-\frac{B}{\ln c}(c^x - 1)\right]dx \\ M_{G:x}(t) &= e^{\frac{B}{\ln c}}\int_0^\infty B\exp\left[x(\ln c + t) - \frac{B}{\ln c}e^{x\ln c}\right]dx \end{aligned}$$

and with the substitutions $w = \frac{B}{\ln c}e^{x\ln c}$, $dw = Be^{x\ln c}dx$ and $x = \ln[w\ln(c^{\frac{1}{B}})]/\ln c$, the moment generating function becomes

$$M_{G:x}(t) = e^{\frac{B}{\ln c}}(\ln c^{\frac{1}{B}})^{\frac{t}{\ln c}}\int_{\frac{B}{\ln c}}^\infty e^{-w}w^{(\frac{t}{\ln c}+1)-1}dw. \tag{7}$$

We may write an exact expression for (7) as

$$M_{G:x}(t) = e^{\frac{B}{\ln c}}(\ln c^{\frac{1}{B}})^{\frac{t}{\ln c}}\Gamma\left(\frac{t}{\ln c} + 1\right)\left[1 - \Gamma\left(\frac{B}{\ln c}; \frac{t}{\ln c} + 1\right)\right]$$

where $\Gamma(\cdot;\cdot)$ is the incomplete Gamma function. Again, expanding the range of integration to $[0, \infty)$, the mgf may be bounded by

$$M_{G:x}(t) \le e^{\frac{B}{\ln c}}(\ln c^{\frac{1}{B}})^{\frac{t}{\ln c}}\,\Gamma\left(\frac{t}{\ln c} + 1\right).$$

With the moment generating function bounded, the existence of the moment generating function is guaranteed.

Proceeding in a similar manner for the generalized Gompertz, we arrive at an exact expression for the mgf, namely

$$M_{g:x}(t) = (\ln c^{\frac{1}{B}})^{\frac{t}{\ln c}} \Gamma\left(\frac{t}{\ln c} + 1\right)$$

2.2. Gompertz-Weibull Transformation. We may rewrite (2) as

$$F_G(t; B, c) = 1 - \exp\left(-\frac{e^{t \ln c} - 1}{\frac{\ln c}{B}}\right)$$

and comparing this to the 3-parameter Weibull

$$F_{W:3parm}(w; \theta, \gamma, \delta) = 1 - \exp\left[\frac{(w - \gamma)^{\delta}}{\theta}\right]$$

we see that with the substitutions $\theta = (\ln c)/B$, $\delta = \gamma = 1$ and $w = e^{t \ln c}$, we achieve a transformation from the Gompertz to the 3-parameter Weibull.

We now establish a transformation between the 2-parameter Weibull and the generalized Gompertz. The CDF for the the Weibull is given by

$$F_W(\tilde{t}; \beta, \theta) = 1 - \exp\left[-\left(\frac{1}{\theta}\right)^{\beta} e^{\beta \ln \tilde{t}}\right]$$

which when making the substitutions $t = \ln \tilde{t}$, $\beta = \ln c$ and $\left(\frac{1}{\theta}\right)^{\beta} = \frac{B}{\ln c}$, yields the CDF for the generalized Gompertz.

2.3. Gompertz-Extreme Value Transformation. Next, we establish a transformation between the Extreme Value distribution and the generalized Gompertz. Rewriting (4) with $\frac{B}{\ln c} = \exp\left[\ln\left(\frac{B}{\ln c}\right)\right]$, the CDF becomes

$$F_g(t; B, c) = 1 - \exp\left[-e^{t \ln c - \ln\left(\frac{\ln c}{B}\right)}\right].$$

Making the following substitutions $\sigma = \frac{1}{\ln(c)}$ and $\mu = \frac{\ln\left(\frac{\ln c}{B}\right)}{\ln c}$ the CDF becomes

$$F_g(t; B, c) = 1 - \exp\left(-e^{\frac{t-\mu}{\sigma}}\right) \tag{8}$$

and with $Z = \frac{T-\mu}{\sigma}$, equation (8) becomes $F_g(z) = 1 - \exp(-e^{z})$, which is the Extreme Value distribution, described in Gumbel(1958) as the second double exponential distribution.

3. ESTIMATORS

In this section we will develop MLE's and estimators based on order statistics. However, estimators based on the method of moments will not be presented due to the complexity of (5) and (6).

3.1. Complete Data. The likelihood function for the Gompertz with n observations is

$$L_G(B,c;\underline{x}) = \prod_{i=1}^{n}(Bc^{x_i})\exp\left[-\frac{B}{\ln c}(c^{x_i}-1)\right],\ i=1,2,\ldots n.$$

Differentiating the natural logarithm of the likelihood function with respect to B and c, we have

$$\begin{aligned}\frac{\partial \ln L_G}{\partial B} &= \frac{n}{B} - \frac{1}{\ln c}\sum_{i=1}^{n}(c^{x_i}-1)\\ \frac{\partial \ln L_G}{\partial c} &= \frac{1}{c}\sum_{i=1}^{n}x_i + \frac{B}{c(\ln c)^2}\sum_{i=1}^{n}(c^{x_i}-1) - \frac{B}{c(\ln c)}\sum_{i=1}^{n}x_i c^{x_i}.\end{aligned}$$

Setting these partial derivatives equal to zero, and solving the system of two equations for the two unknowns, yields the following solutions:

$$B = \frac{n\ln c}{\sum_{i=1}^{n}(c^{x_i}-1)}$$

$$\frac{\sum_{i=1}^{n}x_i c^{x_i}}{\sum_{i=1}^{n}(c^{x_i}-1)} = \overline{x} + \frac{1}{\ln c}$$

where $\overline{x}$ is the arithmetic mean.

The MLE's for the generalized Gompertz for complete data may be shown to take the form

$$B = \frac{n\ln c}{\sum_{i=1}^{n}c^{x_i}} \tag{9}$$

$$\frac{\sum_{i=1}^{n}x_i c^{x_i}}{\sum_{i=1}^{n}c^{x_i}} = \overline{x} + \frac{1}{\ln c}. \tag{10}$$

In the case of the generalized Gompertz, we can show that these estimators are unique and do indeed achieve a maximum. However, with respect to the Gompertz, these estimates will be unique and maximized only if

$$\left(\frac{\sum_{i=1}^{n}x_i c^{x_i}}{\sum_{i=1}^{n}(c^{x_i}-1)}\right)^2 \le \left(\frac{1}{\ln(c)}\right)^2 + \frac{\sum_{i=1}^{n}x_i^2 c^{x_i}}{\sum_{i=1}^{n}(c^{x_i}-1)}.$$

3.2. Censored Data. We develop MLE's for Type II censored data, where n denotes the total number of lives under investigation and r denotes the number of observed deaths. For Type II censoring, the likelihood function is

$$L_G(B,c;\underline{x}) = \frac{n!}{(n-r)!}\left\{\exp\left[-\frac{B}{\ln c}(c^{x_r}-1)\right]\right\}^{n-r}\cdot\prod_{j=1}^{r} Bc^{x_j}\exp\left[-\frac{B}{\ln c}(c^{x_j}-1)\right]$$

and differentiating the natural logarithm with respect to B and c, we have

$$\frac{\partial \ln L_G}{\partial B} = \frac{r}{B} - \frac{1}{\ln c}\left[\sum_{j=1}^{r}(c^{x_j}-1) + (n-r)(c^{x_r}-1)\right]$$

$$\begin{aligned}\frac{\partial \ln L_G}{\partial c} &= \frac{1}{c}\sum_{j=1}^{r}x_j + \frac{B}{c(\ln c)^2}\left[\sum_{j=1}^{r}(c^{x_j}-1) + (n-r)(c^{x_r}-1)\right]\\ &\quad -\frac{B}{\ln c}\left[\sum_{j=1}^{r}x_j c^{x_j-1} + (n-r)\cdot x_r c^{x_r-1}\right].\end{aligned}$$

Setting these partial derivatives equal to zero, and solving the system of equations yields the following solutions:

$$B = \frac{r\ln(c)}{\sum_{j=1}^{r}(c^{x_j}-1) + (n-r)(c^{x_r}-1)}$$

$$\overline{x}_r + \frac{1}{\ln c} = \frac{\sum_{j=1}^{r}x_j c^{x_j} + (n-r)x_r c^{x_r}}{\sum_{j=1}^{r}(c^{x_j}-1) + (n-r)(c^{x_r}-1)}$$

where $\overline{x}_r = \frac{\sum_{j=1}^{r}x_j}{r}$. The MLE's for the generalized Gompertz for complete data may be shown to take the form

$$B = \frac{r\ln(c)}{\sum_{j=1}^{r}c^{x_j} + (n-r)c^{x_r}} \tag{11}$$

$$\overline{x}_r + \frac{1}{\ln c} = \frac{\sum_{j=1}^{r}x_j c^{x_j} + (n-r)x_r c^{x_r}}{\sum_{j=1}^{r}c^{x_j} + (n-r)c^{x_r}}. \tag{12}$$

As in the previous section, estimators for the generalized Gompertz are unique and achieve a maximum. Again, the estimates for the Gompertz will also have this property when

$$\left(\frac{t^{**}}{t^{*}}\right)^2 \le \frac{\sum_{j=1}^{r}x_j^2 c^{x_j} + x_r^2(n-r)c^{x_r}}{t^{*}} + \left(\frac{1}{\ln c}\right)^2$$

where $t^{*} = \sum_{j=1}^{r}(c^{x_j}-1) + (n-r)(c^{x_r}-1)$ and $t^{**} = \sum_{j=1}^{r}x_j c^{x_j} + x_r(n-r)c^{x_r}$.

3.3. Estimators Based on Order Statistics. Using the work of Lloyd(1952) and Mann(1969), we develop estimators based on order statistics for the generalized Gompertz utilizing the transformation from the Extreme Value distribution.

Best Linear Unbiased Estimators(BLUE). We define $X_{1:n}, \ldots, X_{n:n}$ as the n order statistics from a sample of size n and let μ and σ denote the location and scale parameters from the distribution. Now, let

$$F(x;\mu,\sigma) = H\left(\frac{x-\mu}{\sigma}\right)$$

where, $H(\cdot)$ is a parameter-free distribution. Defining the following:

$$\begin{aligned} Z_i &= \frac{X_{i:n}-\mu}{\sigma} \\ E(X_{i:n}) &= \mu + \sigma E(Z_i) \\ \text{Cov}(X_{i:n}, X_{j:n}) &= \sigma^2 Cov(Z_i, Z_j) \end{aligned}$$

where

$$\begin{aligned} \mathbf{X} &= \begin{pmatrix} X_{1:1} \\ \cdot \\ \cdot \\ \cdot \\ X_{n:n} \end{pmatrix} \\ E(\mathbf{Z}) &= \mathbf{e} \\ \text{Cov}(\mathbf{Z}) &= \mathbf{V} \\ \mathbf{C} &= [\mathbf{1} \quad \mathbf{e}] \\ \mathbf{X} &= \mathbf{C}\boldsymbol{\beta} + \boldsymbol{\delta} \\ E(\boldsymbol{\delta}) &= \mathbf{0} \\ \text{Cov}(\boldsymbol{\delta}) &= \sigma^2\mathbf{V} \\ \boldsymbol{\beta} &= \begin{pmatrix} \mu \\ \sigma \end{pmatrix}, \end{aligned}$$

Lloyd(1952) developed the BLUE's for μ and σ and these estimators are given as

$$\hat{\mu}_1 = -\mathbf{e}^{\mathrm{T}}\mathbf{\Gamma}\mathbf{X} \tag{13}$$
$$\hat{\sigma}_1 = \mathbf{1}^{\mathrm{T}}\mathbf{\Gamma}\mathbf{X} \tag{14}$$

where $\mathbf{\Gamma} = \frac{\mathbf{V}^{-1}(\mathbf{1}\mathbf{e}^T - \mathbf{e}\mathbf{1}^T)\mathbf{V}^{-1}}{\mathbf{\Delta}}$ and $\mathbf{\Delta} = |\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}|$. The vectors $\mathbf{Z}$, $\mathbf{e}$ and $\mathbf{1}$ are $n \times 1$ column vectors, $\mathbf{V}$ is the variance-covariance matrix for $\mathbf{Z}$ and $\mathbf{C}$ is the design matrix. Through the appropriate transformation, we find that the BLUE's for the generalized Gompertz are

$$c_1 = \exp\left(\frac{1}{\sigma_1}\right) \tag{15}$$

$$B_1 = \frac{1}{\sigma_1}\exp\left(-\frac{\mu_1}{\sigma_1}\right). \tag{16}$$

The estimators for the generalized Gompertz Distribution , $\hat{c}_1$ and $\widehat{B}_1$, which are based on the BLUE's and BLIE's for the Extreme-Value distribution may now be calculated by substituting equations (13) and (14) into equations (15) and (16).

Best Linear Invariant Estimator(BLIE). Mann(1969) developed equations for the best linear invariant estimators for the Extreme Value distribution based on the BLUE's, which are given by

$$\hat{\sigma}_2 = \mathcal{D}\hat{\sigma}_1 \tag{17}$$

$$\hat{\mu}_2 = \hat{\mu}_1 - \mathcal{E}\hat{\sigma}_1 \tag{18}$$

where $\mathcal{D} = \frac{\mathbf{\Delta}}{\mathbf{\Delta}+\mathbf{1}^T\mathbf{V}^{-1}\mathbf{1}}$ and $\mathcal{E} = \frac{-\mathbf{1}^T\mathbf{V}^{-1}\mathbf{e}}{\mathbf{\Delta}+\mathbf{1}^T\mathbf{V}^{-1}\mathbf{1}}$. We may use equations (13), (14), (17) and (18) to write c_2 and B_2 in terms of c_1 and B_1, which is

$$c_2 = \exp\left(\frac{\ln c_1}{\mathcal{D}}\right)$$

$$B_2 = \left(\frac{\ln c_1}{\mathcal{D}}\right)\exp\left\{-\frac{1}{\mathcal{D}}\left[\ln c_1 \ln\left(\frac{\ln c_1}{B_1}\right) - \mathcal{E}\right]\right\}.$$

4. EXAMPLES

To demonstrate the results in this paper, we will use two data sets from a United States Air Force study that was conducted from 1964-1969 (see Yochmowitz, Wood and Salmon, 1985). In brief, the study dealt with effects of ionizing radiation on primates.

4.1. Complete Data. From the USAF study, of the 5 groups of rehsus monkeys that had expired as of 1989 was the group which received the highest level of radiation. This cohort size consisted of 9 rhesus monkeys and their survival times are $x_{1:9} = 1.9167$, $x_{2:9} = 2.4167$, $x_{3:9} = 2.9167$, $x_{4:9} = 2.9167$, $x_{5:9} = 3.4167$, $x_{6:9} = 4.1667$,

$x_{7:9} = 4.4167$, $x_{8:9} = 4.8333$, and $x_{9:9} = 6.25$. From (9) and (10), we see that the MLE's will be

$$\hat{c}_g = 2.113309 \text{ and } \hat{B}_g = .028611.$$

Using the tables provided by White(1964) for the expected values and variance/covariance matrix for the Extreme Value distribution, we find that the BLUE's are

$$\hat{c}_1 = 1.984435 \text{ and } \hat{B}_1 = .033093$$

and the BLIE's are

$$\hat{c}_2 = 2.097531 \text{ and } \hat{B}_2 = .027333.$$

4.2. Censored Data. The control group from the USAF study consisted of 33 lives, of which 14 were censored. The observed survival times, as of 1989, were $x_{1:33} = 5.0833$, $x_{2:33} = 6.6667$, $x_{3:33} = 6.8333$, $x_{4:33} = 7.0833$, $x_{5:33} = 13$, $x_{6:33} = 15$, $x_{7:33} = 15.5$, $x_{8:33} = 18.3333$, $x_{9:33} = 18.75$, $x_{10:33} = 19$, $x_{11:33} = 20.1667$, $x_{12:33} = 20.3333$, $x_{13:33} = 21$, $x_{14:33} = 21.4167$, $x_{15:33} = 21.5833$, $x_{16:33} = 21.9167$, $x_{17:33} = 22.8333$, $x_{18:33} = 22.9167$, $x_{19:33} = 22.9167$ and the remaining $x_{20:33}$ through $x_{33:33}$ are censored values and set to 25.

The MLE's for the generalized Gompertz were determined after taking the natural logarithm of the original observations, and from (11) and (12) were found to be

$$\hat{c}_g = 11.918494 \text{ and } \hat{B}_g = 7.086 \times 10^{-4}$$

White(1964) only provides tables for sample sizes of 20 or less, so no values for the BLUE's and BLIE's are presented.

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