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ACTUARIAL NOTE: A VALUATION METHOD FOR RETIREMENT INCOME ENDOWMENT POLICIES AFTER LIFE CONTINGENCIES HAVE CEASED

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THE valuation of Retirement Income Endowment policies, *i.e.*, policies under which the insurance benefit is the face amount or the cash value if greater, has become more of a problem in recent years for companies using the attained age valuation system. The number of policies reaching ages at which life contingencies cease has been increasing rapidly, so that mere volume presents a problem when these policies must be handled separately from the principal valuation. The valuation has been complicated further by a multiplicity of policy forms resulting from progressive increases in maturity values. These increases in maturity values have also resulted in longer durations between the time contingencies cease and maturity. When the valuation after contingencies cease is done by a seriatim method, calculation of the individual reserves becomes a serious problem, and when a group method is used, a large number of groups is required to produce the valuation reserve. The method presented in this Note was developed for a company using the attained age system and appears to embody most of its specific advantages. The resulting formulae permit a valuation procedure requiring separation only by duration since issue and rate of interest. For companies which have had numerous changes in maturity values, the advantage of combining all such policies for valuation is easily recognized.

While this method may be used for many types of level premium interest accumulations, this Note will be confined to its application to the Retirement Income Endowment policy.

Let us assume, for convenience, a policy under which modified reserves are built up to net level at the end of twenty years. Let:

x = Age at issue

$x+n$ = Age at maturity

$x+a$ = Age at which contingencies cease

β = Modified renewal net premium for age at issue x

P = Net level premium for age at issue x

$1+k$ = Maturity value per dollar face amount.

Then, for the most general case, $n > 20 > t > a$, the tth terminal reserve may be expressed as:

$$\begin{aligned}
{}_tV &= (1+k)\,v^{n-t} - \beta\ddot{a}_{\overline{20-t}|} - P(\ddot{a}_{\overline{n-t}|} - \ddot{a}_{\overline{20-t}|}) \\
&= (1+k)\,v^{n-t} - (\beta - P)\,\ddot{a}_{\overline{20-t}|} - P\ddot{a}_{\overline{n-t}|} \\
&= (1+k)\,v^{n-t} - (\beta - P)\left(\frac{1-v^{20-t}}{d}\right) - P\left(\frac{1-v^{n-t}}{d}\right) \\
&= v^{-t}\left[(1+k)\,v^n + (\beta - P)\frac{v^{20}}{d} + P\frac{v^n}{d}\right] - \frac{1}{d}\beta \\
&= (1+i)^t\left[(1+k)\,v^n + \beta\frac{v^{20}}{d} - P\left(\frac{v^{20}-v^n}{d}\right)\right] - \frac{1}{d}\beta \\
&= (1+i)^t F_x^{\text{MOD.}} - \frac{1}{d}\beta
\end{aligned}$$

where

$$F_x^{\text{MOD.}} = \left[(1+k)\,v^n + \beta\frac{v^{20}}{d} - P\left(\frac{v^{20}-v^n}{d}\right)\right].$$

For policies net level from issue, or net level after twenty years,

$$\begin{aligned}
{}_tV &= (1+k)\,v^{n-t} - P\ddot{a}_{\overline{n-t}|} \\
&= (1+k)\,v^{n-t} - P\left(\frac{1-v^{n-t}}{d}\right) \\
&= (1+i)^t\left[(1+k)\,v^n + P\frac{v^n}{d}\right] - \frac{1}{d}P \\
&= (1+i)^t F_x^{NL} - \frac{1}{d}P
\end{aligned}$$

where

$$F_x^{NL} = \left[(1+k)\,v^n + P\frac{v^n}{d}\right].$$

For policies modified on the full preliminary term or Commissioners reserve valuation method, the tth reserve is the same as for the net level policy, above, if P is replaced by the modified net premium.

For paid-up policies, either limited payment or fractional paid-up,

$$\begin{aligned}
{}_tV &= (1+k)\,v^{n-t} \\
&= (1+i)^t(1+k)\,v^n \\
&= (1+i)^t F_x^{PU}
\end{aligned}$$

where

$$F_x^{PU} = (1+k)\,v^n.$$

The F factors as produced by the above formulae are one and one-half to two times the face amount, and may be conveniently used without further adjustment for decimal point. The factors may readily be computed for an entire series of policies by computing tables of $(1+k)v^n$,

v^{20}/d, $(v^{20}-v^n)/d$, and v^n/d. The F factors may then be punched in the card field previously used for the Karup θ_x factor on the individual punched cards, and, since the F factor itself is independent of duration, it may be used as long as the premium is not changed.

A general formula for the tth mean reserve for a group of policies bearing the same year of issue and interest rate may be expressed as:

$$\Sigma_t M = \tfrac{1}{2}\left[(1+i)^{t-1}+(1+i)^{t}\right]\Sigma F-\left(\frac{2-d}{2d}\right)\Sigma P'$$

where ΣF is the total F factor for the group, and
$\Sigma P'$ is the total premium for the group.

This formula is independent of age at issue, plan, mortality table, and modification method. Therefore, the only divisions required for valuation are interest rate and year of issue. Reserves which are based on the same interest rate may be computed for all years of issue by accumulating the products of the respective interest multipliers, $\tfrac{1}{2}[(1+i)^{t-1}+(1+i)^{t}]$, times the year of issue F totals, and making a single subtraction of the total premium for all years of issue times the factor $(2-d)/2d$, since this $([2-d]/2d)\Sigma P'$ term is independent of duration. When the same number of decimals is used in computing and punching the F factors as is used for the factors for insurance plans, the reserves produced will, obviously, have the same degree of accuracy as any attained age valuation system. Computing the subtractive term, $([2-d]/2d)\Sigma P'$, for the total group, rather than for each year of issue, should tend to minimize any error introduced by the use of the larger multiplier.

It should be noted, further, that this formula may also be used to produce reserves beyond the normal maturity age for forms which contain optional maturity ages. The accumulation of the reserves after the normal maturity date has, in practice, been made on several different bases, three of which will be discussed here.

One common plan has been to base the reserves after the normal maturity age on the accumulation of the net premium. The tth terminal reserve on this basis, $t>n$, for a net level policy may be expressed as:

$$\begin{aligned}
{}_tV &= (1+k)(1+i)^{t-n}+P\ddot{s}_{\overline{t-n|}}\\
&= (1+k)\,v^{n-t}+P\left[\frac{(1+i)^{t-n}-1}{d}\right]\\
&= (1+k)\,v^{n-t}-P\left[\frac{1-v^{n-t}}{d}\right]\\
&= v^{-t}\left[(1+k)\,v^{n}+P\frac{v^{n}}{d}\right]-\frac{1}{d}P\\
&= (1+i)^{t}F_x^{NL}-\frac{1}{d}P\,.
\end{aligned}$$

Thus, when reserves beyond the normal maturity age are based on the accumulation of the net premium, the F factor and premium on the valuation card need not be changed. Application of the proper interest multiplier will automatically produce the correct reserve.

Another plan for reserves beyond the normal maturity date has been to base the reserves on an accumulation of the gross premium. In this case, the tth terminal reserve, $t > n$ is:

$$\begin{aligned} {}_tV &= (1+k)(1+i)^{t-n} + G\ddot{s}_{\overline{t-n}|} \\ &= (1+k)\, v^{n-t} - G\left(\frac{1-v^{n-t}}{d}\right) \\ &= v^{-t}\left[(1+k)\, v^n + G\frac{v^n}{d}\right] - \frac{1}{d}G \\ &= (1+i)^t\, F_x^G - \frac{1}{d}G \end{aligned}$$

where

$$G = \text{Gross premium}$$

$$F_x^G = \left[(1+k)\, v^n + G\frac{v^n}{d}\right].$$

When reserves beyond the normal maturity age are based on an accumulation of the gross premium, then a new F factor must be computed, and the gross premium punched in the net premium field. After this adjustment is made, however, these policies may be valued together with all other policies bearing the same interest rate.

A third plan for reserves beyond the normal maturity age is to accumulate the maturity value under interest without further premium payments. In this case the tth terminal reserve, $t > n$, is:

$$\begin{aligned} {}_tV &= (1+k)(1+i)^{t-n} \\ &= (1+i)^t(1+k)\, v^n \\ &= (1+i)^t F_x^{PU}. \end{aligned}$$

Thus when reserves beyond the normal maturity age are based on an accumulation of the maturity value without further premium payments, the F factor used is the same as for a policy paid-up prior to maturity, and the only adjustment needed at the normal maturity age is to change the F factor on the punched card and remove the net premium.

DISCUSSION OF PRECEDING PAPER

RICHARD A. GETMAN:

Our company has used the attained age method of valuation for both insurance and annuity contracts for a good many years.

As each Hollerith card is initially prepared there is punched on the card a "Transfer Year" representing the first calendar year in which a change in status occurs. A change in status consists of completion of premium payments, change in rate of premium payment, maturity or expiry, or some similar type of occurrence. For Retirement Income Endowment policies under which the insurance benefit is the face amount or the cash value if greater the transfer year is the last year in which the face amount is payable.

At the beginning of each calendar year all cards bearing that calendar year as a transfer year are removed from the file, and new cards are punched reflecting the new status of each policy. For Retirement Income Endowment policies the change consists of changing the θ-factor to the maturity value and the transfer year to the year of maturity. These cards are then valued thereafter separately for each interest rate according to duration to maturity by means of the formula

$$_tV = (1+k)\, v^{n-t} - P\ddot{a}_{\overline{n-t}|} \, ,$$

where $1+k$, the maturity value per dollar ace amount, and P, the net level premium, appear on the Hollerith card and v^{n-t} and $\ddot{a}_{\overline{n-t}|}$ are factors applied to the Hollerith tabulation separately for each duration to maturity. This is the formula which the authors use as their starting point for policies net level from issue or net level after twenty years (provided, of course, that $a \geqq 20$).

The method is very simple and no special calculations of any kind are required. Our present file, comprising over 5,000 cards, includes both premium-paying and paid-up without distinction, except for the fact that the paid-up cards do not carry any net premium. Accumulation after the maturity date could be performed, if desired, in any of the three ways suggested in the paper, by making the indicated changes either in the premium column of the card or in the factors applied to the tabulation.

The only feature of the authors' paper which our method cannot handle is modified preliminary term valuation involving both a modified renewal net premium β and a net level premium P concurrently. Even then, the use of only P throughout the remainder of the modified preliminary term valuation period would produce but a slight excess reserve of a rapidly decreasing nature.

Apparently the fundamental difference between the method described in the paper and the one described herein lies in the mechanics of the valuation system used by a particular company. The authors no doubt had good reason for preferring tabulations by year of issue. However, for a company for which there is no obstacle in using tabulations by year of maturity, the method outlined herein should prove more satisfactory.

JOHN M. BOERMEESTER:

I wish to compliment the authors on the paper. Their method requires a grouping of policies by year of issue; however, I want to point out that Kermit Lang's article in *TASA* XLVII contains a clue as to a method which will eliminate the requirement of groupings by year of issue. Mr. Lang's article, which discusses the valuation of optional settlements not involving life contingencies, shows that a valuation can be made under a system which, in effect, discounts values from a proper, so-called, fixed "base" year. The discount factor under this system then is simply a function of the year of valuation.

HARWOOD ROSSER:

The authors suppose that a, β, and P are already known for this plan. By way of review, and because some earlier formulas contained errors, a table is shown below which gives sources of formulas for most of the major reserve modification methods. Those for the Canadian method, believed to be hitherto unpublished, are developed.

EXPLANATION OF TABLE

1. Table 1 relies on the distinction between valuation "standard" and valuation "method" made in Menge's classic article in *RAIA* XXV. Thus the Illinois and Commissioners Standards both prescribe the full preliminary term method for certain age-groups.

2. No column for α is shown, for as soon as β is known, α can be calculated by Menge's relationships.

3. Some alternate equivalent formulas are listed.

4. When P appears under P', it means that the benefits are unaffected by the modification of reserves. This is amplified below.

5. *Illinois Method.*—While it is unnecessary to calculate P' for Case II, since it is not required in computing reserves, it may be noted that for all three cases

$$\beta' = P' + \frac{{}_{19}P_{x+1} - P^{1}_{x:\overline{1}|}}{\ddot{a}_{x:\overline{m}|}},$$

where m is the smaller of n and 20.

TABLE 1

SUMMARY OF RESERVE MODIFICATION FORMULAS FOR INCOME ENDOWMENT POLICIES

CASE	CRITERION FOR a	NET LEVEL PREMIUM FOR MODIFIED BENEFITS P'	MODIFIED RENEWAL NET PREMIUM β(or β')
	Full Preliminary Term Method		
All.	Same as NLP for age $x+1$, with period $n-1$	Not needed	P at age $x+1$, with period $n-1$
	Illinois Method		
I. $20 \leqq a < n$.	Same as NLP	P	$P + {}_{19}P_{x+1} - {}_{20}P_x$
II. $a < n \leqq 20$.	*RAIA* XX, 74, (2); also: *RAIA* XXXI, 401, (6)	Not needed	*RAIA* XX, 74, foll. (2); also: *RAIA* XXXI, 401, (5)
III. $a < 20 < n$.	*RAIA* XXXII, 169, (8a) or (8b)	*RAIA* XXXI 403, (9)	$P' + {}_{19}P_{x+1} - {}_{20}P_x$; also: *RAIA* XXXII 168, (7a)
	New Jersey Method		
I. $20 \leqq a < n$. . . . . (The other cases under New Jersey Standard call for the Illinois Method.)	Same as NLP	P	$P + \frac{P - P_{x:\overline{n}|}}{a_{x:\overline{19}|}}$

TABLE 1—*Continued*

CASE	CRITERION FOR α	NET LEVEL PREMIUM FOR MODIFIED BENEFITS P'	MODIFIED RENEWAL NET PREMIUM β (or β')
	Canadian Method		
All	(8) or (9) below	Not needed	(5) or (6) below
	Commissioners Method: *Maximum* Reserves		
I. $\beta'_{FPT} \leqq {}_{19}P_{x+1}$	Same as FPT	Not needed	Same as FPT
II. $\beta'_{FPT} > {}_{19}P_{x+1}$	Same as Illinois, Case II	Not needed	Same as Illinois, Case II
	Commissioners Method: Smaller Reserves		
II-A. ${}_{19}P_{x+1} \leqq \beta'_{FPT} \leqq (1+h)\,{}_{19}P_{x+1}$	Same as FPT	Not needed	Same as FPT
II-B. $\beta'_{FPT} > (1+h)\,{}_{19}P_{x+1}$	*TASA* XLVII, 48, par. 17*	Not needed	*TASA* XLVII, 47, par. 13*

* With $E = (1+h)\,{}_{19}P_{x+1} - P^1_{x:\overline{1}|}$.

6. *Commissioners Reserve Valuation Method.*—This does not produce a unique formula here (as brought out by Hahn and Menge in *RAIA* XXXV and by Espie in *TASA* XLVII), except at the younger ages (Case I). For the ambiguous area, two sets of formulas are given. Results of the second vary according to the choice of "equivalent level amount" $1 + h$. More as to this later.

PRELIMINARY TERM METHODS IN GENERAL

Fassel's formula (2) on page 234 of *RAIA* XIX for the net level premium P is equivalent to

$$P\ddot{a}_{x:\overline{n}|} = A^{1}_{x:\overline{a}|} + {}_aE_x\left[\,(1+k)\,v^{n-a} - P\left(\ddot{a}_{\overline{n-a}|} - \ddot{a}_{x+a:\overline{n-a}|}\right)\right], \quad (1)$$

using modern notation. A parallel form is

$$P'\ddot{a}_{x:\overline{n}|} = A^{1}_{x:\overline{a}|} + {}_aE_x\left[\,(1+k)\,v^{n-a} - \beta'\left(\ddot{a}_{\overline{n-a}|} - \ddot{a}_{x+a:\overline{n-a}|}\right)\right] \quad (2)$$

$$= \alpha' + \beta' \cdot a_{x:\overline{n-1}|}\,. \quad (3)$$

Apparently no explanation by general reasoning of Fassel's basic formula has hitherto been offered, although it has been attempted for a more complicated one (cf. *RAIA* XXXII, 163).

Equation (1) states that the present value of the premiums actually to be collected will purchase:

1. A unit of term insurance for a years, plus
2. Pure endowment at the end of that time for:
   A. The discounted maturity value (assuming that all remaining premiums for each survivor at time a will be paid when due), less
   B. The then present value of such premiums not collected by reason of deaths between a and n.

Equation (2) recognizes a reserve modification. Usually this alters the death benefits for the higher durations, and a revised net level premium, P', arises. This is probably always lower than P. The premiums not collected after a become β'. If more than one such renewal premium is involved, as after 20 years, obvious modifications of (2) and (3) can be made. When $P = P'$, the primes may be dropped throughout. This ordinarily occurs only if the modification period is shorter than the premium-paying period, and not always then.

Let us use the following designations, corresponding to (1) and (2), respectively:

Plan 1: Fassel's original plan, with a death benefit of face or net level premium reserve, if greater.

Plan 2: A plan providing the revised death benefits mentioned above; *i.e.*, face or modified reserve, if greater (denoted by primes herein).

While the differences are slight, these are two distinct plans if the death benefits are not identical throughout.

There is still a third possibility, if the cash values granted in the later policy years do not coincide with reserves by some method:

Plan 3: A plan with a death benefit of face or cash value, if greater. Espie and Hahn have both dealt with this. The authors of the current note mention a "cash value if greater" benefit, but apparently their formulas assume a "reserve if greater" benefit. This discussion does likewise. The distinction is of minor practical importance.

If desired, either net level or modified reserves could be calculated for each of the three plans. Obviously, only the former is required on Plan 1, and the latter on Plan 2.

CANADIAN METHOD

Since the premiums will always exceed whole-life premiums, the Canadian Standard prescribes the Canadian modification throughout for this plan. As the modification period is not limited, we will be dealing with Plan 2. By definition in the law (*Cf. RAIA* XXV, 197),

$$\beta' = P' + \frac{P_x - P^1_{x:\overline{1}|}}{a_{x:\overline{n-1}|}}. \tag{4}$$

Using (2) and (4) to eliminate P' gives

$$\beta' = \frac{M_x - M_{x+a} + D_{x+a}(1+k)v^{n-a} + D_x(P_x - P^1_{x:\overline{1}|})(1 + 1/a_{x:\overline{n-1}|})}{N_x - N_{x+a} + D_{x+a}\ddot{a}_{\overline{n-a}|}} \tag{5}$$

$$= \frac{A^1_{x:\overline{a}|} + {}_aE_x(1+k)v^{n-a} + (P_x - P^1_{x:\overline{1}|})(1 + 1/a_{x:\overline{n-1}|})}{\ddot{a}_{x:\overline{a}|} + {}_aE_x\ddot{a}_{\overline{n-a}|}} \tag{6}$$

Using (3) and (4) together, we obtain

$$\alpha = \beta' - (P_x - P^1_{x:\overline{1}|})\frac{a_{x:\overline{n}|}}{a_{x:\overline{n-1}|}}. \tag{7}$$

The result in (5) could also be reached by equating prospective and retrospective reserves at time a, and then utilizing (7).

Following the reasoning of Lang's paper in *RAIA* XXXI, we may state that a is the greatest integer for which

$$\frac{(1+k)v^{n-a} - 1}{\ddot{a}_{\overline{n-a}|}} \leqq \beta' \leqq P'_{x:\overline{a}|} = P_{x:\overline{a}|} + \frac{P_x - P^1_{x:\overline{1}|}}{a_{x:\overline{a-1}|}}, \tag{8}$$

where $P'_{x:\overline{a}|}$ is the Canadian Standard renewal net premium for an a-year endowment. If tables of such premiums are available, this is the preferable form for the criterion. The β' in (8) can, of course, be ignored in determining a.

If tables are not available, an alternate form, derivable from (8), is that a is the greatest integer for which

$$\frac{1}{\ddot{a}_{x:\overline{a}|}}+\frac{P_x-P^1_{x:\overline{1}|}}{\ddot{a}_{x:\overline{a}|}-1}\geq\frac{k}{(1+i)\ s_{\overline{n-a}|}}. \tag{9}$$

As has been noted before, it is sometimes easier to "guess and test," using either of the following:

$$P'_{x:\overline{a+1}|}<\beta'\leqq P'_{x:\overline{a}|} \tag{10}$$

$$_aV'\leqq 1<{}_{a+1}V'\ . \tag{11}$$

The a for Plan 1 is a good first guess.

COMMISSIONERS METHOD

Equivalent Level Amount.—Practical considerations affecting the choice in opposite directions are:

1. Maximum surrender charge of $25 still retained by some states.
2. Until recently, insistence upon Illinois Standard by Oklahoma.

Also, reserves should at least equal the cash values granted.

Among the theoretically possible selections for $1+h$ (Hahn's S) are:*

1. Unity (Cases I and II of Table 1).
2. That used for minimum cash values: *TASA* XLVII, 46, par. 9; also p. 372 or p. 375.
3. That used for other cash values: *ibid.*, p. 48, par. 16.
4. Menge's s, based on renewal years only: *RAIA* XXXV, 282, (61). See also pages 287–288.
5. The preceding, using the "endowment ratio" rather than the "term ratio" (*cf. RAIA* XXXVI, 95–96).

These are roughly in descending order as regards size of reserves produced. This is more readily seen for Case II-B if we employ an alternate form,

$$\beta'=P'+\frac{(1+h)\ {}_{19}P_{x+1}-P^1_{x:\overline{1}|}}{\ddot{a}_{x:\overline{n}|}}, \tag{12}$$

and keep the prospective reserve formula in mind. The interpretations in items 4 and 5 each apply to all preceding ones, except the first, so that the list is really longer. This may explain my reluctance to specify "minimum" reserves, however desirable.

* Under a "reserve if greater" benefit, item 2 would not appear, and item 3 would be confined to cash values equal to modified reserves.

Boundaries for Reserves.—There is, however, a theoretical lower boundary for Commissioners Method reserves; namely, those by the Full Preliminary Term Method. This boundary will not always be reached, and thus cannot serve as a legal minimum for all ages at issue. Its application follows automatically from Table I, as follows. For Case II-A, the equivalent level amount $1 + h$ enters only to the extent of determining which ages fall into the case; it does not affect the reserve calculations for those that do. As h is increased, fewer ages remain in Case II-B, where the results do depend on h.

At the other end there is a clear-cut maximum. If h is zero, Case II-A becomes vacant, and Case II-B reduces to Case II. The result is the same as the Illinois Standard with the 20-year limit removed (*cf.* *RAIA* XXXV, 267–269).

AN ACTUARIAL ODDITY

One is a little startled to find modified reserves exceeding net level ones, even though the former are on Plan 2 and the latter on Plan 1. This occurs after 20 years, under the Illinois Standard, for the case where $a < 20 < n$, and means that the modification results in slightly higher death benefits thereafter until maturity.

This will be apparent from comparison of the prospective reserve formulas after 20 policy years:

$$ {}_tV = (1 + k)\, v^{n-t} - P\ddot{a}_{\overline{n-t}|}\,, \tag{13} $$

$$ {}_tV_I' = (1 + k)\, v^{n-t} - P'\ddot{a}_{\overline{n-t}|} = {}_tV'\,. \tag{14} $$

If $P' < P$, then ${}_tV_I' > {}_tV$.

CONCLUSION

Walker and Lewis have put a new roof on the house that Fassel built. It needs no painting; I can only admire their clever workmanship. But Fassel's house now has many rooms, and I have tried to draw up a floor plan for those who may visit, and perhaps to light up some of the darker passageways.

(AUTHORS' REVIEW OF DISCUSSION)

CHARLES N. WALKER AND WILLIAM E. LEWIS:

The authors wish to express their appreciation for the criticism and elaboration of the valuation method presented in their paper. It is hoped that the comments of those who contributed, and our reply thereto, will present a more detailed analysis of why the method was developed than was considered desirable in the original paper.

Mr. Getman's method is designed for a company which maintains net level reserves, and, when such is the case, his procedure is probably more simple and direct. One defect appears to be in the treatment of extended maturity cases. There seem to us to be serious practical objections in having policies in force with transfer years prior to the date of valuation. It appears that these policies would be troublesome from a control point of view, for, in our practice at least, a new transfer year must be assigned to show the advanced year of maturity which is predetermined in the vast majority of cases. In addition, it would perhaps be necessary to consult original records in order to verify the correctness of the transfer year whenever sorting of the punched cards by transfer year was required.

A second thought concerning this net level method relates to the error involved in valuing policies calculated on a modified preliminary term basis. Our method was designed specifically to avoid this error, as the business it is applied to consists principally of policies which are valued other than net level. The amount of this error, as shown in the second step of our development of $F_x^{MOD.}$, is $(\beta - P)\ddot{a}_{\overline{20-t}|}$. While this is a decreasing error, it may not be a minor error. For example, for a Retirement Income Endowment at 55, age 34 at issue, with a maturity value of $2,365 per thousand face amount, valued on the CSO table at $2\frac{1}{4}\%$ interest, Illinois Standard, contingencies cease in 10 years. At this point the correct reserve is $959.54, while the reserve as computed by Mr. Getman's formula is $976.22, a difference of $16.68.

Kermit Lang's formulas for annuities certain, to which Mr. Boermeester referred, have been in use in our company for some time for the valuation of settlement options not involving life contingencies and have proved very satisfactory. This type of formula could, moreover, be applied to the valuation of Retirement Income Endowment policies. For the net level case the formula would be as follows:

$$ {}_tV = (1+k)\, v^{n-t} - P\ddot{a}_{\overline{n-t}|} \quad (1)$$

$$ = (1+k)\, v^{n-t} - P\left(\frac{1-v^{n-t}}{d}\right) \quad (2)$$

$$ = v^{-t}\left[(1+k)\, v^{n} + P\frac{v^{n}}{d}\right] - P\frac{1}{d} \quad (3)$$

$$ = v^{z-t}\left[(1+k)\, v^{-(z-n)} + P\frac{v^{-(z-n)}}{d}\right] - P\frac{1}{d}. \quad (4)$$

The z introduced in the last step of this development is a "base year"; consequently the $(z-t)$ term becomes the duration from the year of valuation to the base year, and is therefore constant for all years of issue. The $(z-n)$ term is the duration from the year of maturity to the base year. The bracketed term in step (4) would correspond to the F factors developed in our paper, *i.e.*, would be calculated for each policy and punched in the valuation card.

The difficulty in applying the above method to the Retirement Income Endowment policies is in the calculation of the bracketed term. Since $(z-n)$ is the duration from the year of maturity to the base year, this factor would be different for policies which mature in different calendar years, but which are identical in all other respects. This means that a different series of factors must be computed for each calendar year of issue of a particular policy series. In the method outlined in our paper, the F factor is independent of the calendar year of issue, this variable being taken into account by the year of issue separation and appropriate reserve multipliers. Since a particular policy series may cover several calendar years of issue, it is felt that the over-all amount of work required for the valuation procedure would be considerably less under our method than under a method following the principles outlined by Mr. Lang.