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A CLASS OF ASYMMETRIC DISTRIBUTIONS

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Abstract. We discuss a class of asymmetric distributions arising in a random summation scheme. We call members of the class asymmetric Laplace distributions as the standard Laplace distributions, which are symmetric, constitute a proper subclass. Among distributions which are limits in random summation schemes asymmetric Laplace distributions play an analogous role to that of normal distributions among distributional limits of non-random sums. Asymmetric Laplace laws are more "peaky" and have heavier tails than normal laws. They have stability properties and are convenient in applications, as their densities have explicit forms and estimation procedures are easily implemented. Anticipating increasing interest in this class of distributions we present statistical tools which can be utilized in practice, including algorithms for simulation and estimation. We also discuss more general classes of distributions, where asymmetric Laplace distributions appear as special cases.

Keywords: Aggregate claims; collective risk models; compound distribution; estimation; geometric stable law; heavy tailed modeling; interest rates; insurance mathematics; Laplace distribution; Linnik distribution; mathematical finance; maximum likelihood; Paretian-stable law; random summation; risk theory; ν-stable distribution.

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1 Introduction and notation

Probably the most widely known and used theorem of the probability theory is the Central Limit Theorem (CLT). This theorem in its most often used form gives necessary and sufficient conditions for the convergence of sums of independent and identically distributed (i.i.d.) random variables to the normal law. Consequently, many scientists and practitioners believe that, provided the number of summands is large, their sum can always be approximated by a normal distribution. This, however, may not be the case. If the summands have infinite variance, then the sum may converge to a *stable* law (see for example Samorodnitsky and Taggu [20]). Moreover, even if the variables are independent and normally distributed, the sum of their *random* number may not be distributed according to the normal law.

In Figure 1, we compare two histograms, each obtained for 5000 observations of the sums of i.i.d. random variables. For the one on the left, the observations were generated as non-random sums of 1000 independent normal random variables with a non-zero mean. On the right hand side we generated sums of random variables having the same normal distribution but this time with a random number of terms distributed according to the geometric distribution with parameter $p = 1/1000$ and independently of the terms themselves. The data were centered on their mean and scaled by their variation. We clearly see asymmetry and peakedness in the case of random summation scheme.

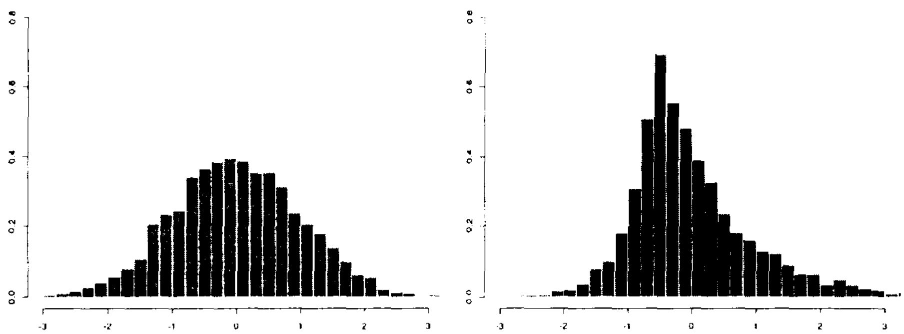

Figure 1: Histograms of non-random sums (left) and sums with geometrically distributed number of terms (right).

Apart from its interesting theoretical properties, the random summation scheme appears naturally in various fields, particularly in insurance mathematics. In risk theory, we are interested in the distribution of aggregate claims generated by the portfolio of insurance polices. If the individual claims are denoted by X_i's (usually assumed to be i.i.d.) and the random variable ν_p denotes the number of claims in a given time period, than the aggregate claim S_p is given by

$$S_p = X_1 + \cdots + X_{\nu_p}. \tag{1}$$

We consider a class of distributions that approximate *geometric compounds*, that is compound distributions (1) with the geometric number of terms:

$$P(\nu_p = k) = p(1-p)^{k-1},\ k = 1, 2, \ldots, \tag{2}$$

when the parameter p converges to zero (so that the average number of terms in (2) converges to infinity). Geometric compounds frequently appear in applied problems from various fields, including actuarial science, as discussed in Kalashnikov [5]. As shown in Mittnik and Rachev [16], the geometric compounds (1), appropriately normalized, converge (in distribution) to a *geometric stable* random variable, which is a location - scale mixture of stable random variables.

In this paper we focus on an important special case, where the random variables X_i's have a finite second moment (variance). Then, the limiting distribution of normalized S_p is a random variable with the following characteristic function:

$$\psi(t) = [1 + \sigma^2 t^2 - i\mu t]^{-1} \tag{3}$$

(see for example Mittnik and Rachev [16]). By specifying $\mu = 0$, we obtain Laplace distributions which are the only symmetric distributions within this class. Thus, it seems pertinent to name the distribution with ch.f. (3) an *asymmetric Laplace distribution* (AL).

We introduce AL laws in Section 2, where we present their basic properties and procedures for simulation and estimation. We show that the probability distribution of every AL random variable is the same as that of the difference of two independent exponentially distributed random variables. This crucial observation leads to explicit formulas for densities and distribution functions of AL distributions, facilitating their practical implementation. We also define a time dependent random process through an AL distribution, which plays an analogous role to Brownian motion. In the symmetric case, this process was applied to model financial data in Madan and Seneta [14], where it was termed the Variance Gamma process.

We think that AL laws should provide an alternative to normal distributions as distributional models in a variety of settings. This class is particularly well suited for modeling phenomena where the variable of interest results from of a large *random* number of independent innovations, while the empirical distribution appears to be asymmetric, "peaky", and has tails heavier than those allowed by normal distribution. One area of application where modeling with AL laws should be explored is mathematical finance, where the empirical data often have the above features. The idea that the price change during a period of time is produced by a random number of "individual effects" first appeared in Mandelbrot and Taylor [15] and Clark [3], and was further explored in Mittnik and Rachev [16, 17] and Kozubowski and Rachev [8]. In Section 3, we apply the AL model to the interest rates data studied by Klein [6], showing the consistency with our model. In the Appendix, we collect main properties of AL laws and comment on their various further extensions. The results are brief and presented without proofs, as the more detailed treatment of AL laws will appear elsewhere.

Notation.

- Z_θ – exponential random variable with the density

$$f_\theta(x) = \frac{1}{\theta}\exp(-x/\theta), \quad x > 0, \tag{4}$$

- $Z = Z_1$ – standard exponential random variable,
- For a vector (or matrix) $\mathbf{t}$, $\mathbf{t}'$ denotes the transpose of $\mathbf{t}$,
- $\mathbf{s}'\mathbf{t} = \sum_{i=1}^d t_i s_i$ – the inner product of $\mathbf{s} = (s_1, \ldots, s_d)'$ and $\mathbf{t} = (t_1, \ldots, t_d)'$,
- $||\mathbf{t}|| = (\mathbf{t}'\mathbf{t})^{1/2} = (\sum_{i=1}^d t_i^2)^{1/2}$ – the Euclidean norm in R^d,
- "$\stackrel{d}{\rightarrow}$" – convergence of distributions,
- "$\stackrel{d}{=}$" – equality of distributions,
- For $\nu > 0$, $\Gamma(\nu) = \int_0^\infty x^{\nu-1}e^{-x}dx$ (the gamma function),
- $\mathrm{sign}(x)$ equals 1 for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$,
- $\mu \in R$, $\sigma \geq 0$ – location and scale parameters of AL distribution,
- $\kappa = 2\sigma/(\mu + \sqrt{4\sigma^2 + \mu^2})$ – scale invariant parameter.

2 Asymmetric Laplace distributions

In this Section we define univariate asymmetric Laplace distributions and derive their basic properties. We omit most proofs and refer an interested reader to Kozubowski and Podgórski [11] for a more detailed treatment.

Definition 2.1 *A random variable is said to have an asymmetric Laplace (AL) distribution if there are parameters $\mu \in R$ and $\sigma \geq 0$ such that its characteristic function has the form (3). We denote such r.v. and its distribution as $Y_{\sigma,\mu}$ and $AL(\sigma, \mu)$, respectively, and write $Y_{\sigma,\mu} \sim AL(\sigma, \mu)$.*

Note the following relations among the parameters:

$$\frac{1}{\kappa} - \kappa = \frac{\mu}{\sigma}, \quad \frac{1}{\kappa} + \kappa = \sqrt{4 + \left(\frac{\mu}{\sigma}\right)^2}, \quad \frac{1}{\kappa^2} + \kappa^2 = 2 + \left(\frac{\mu}{\sigma}\right)^2.$$

2.1 Special cases

While the distribution makes sense for every $\mu \in R$ and $\sigma \geq 0$, we have several special cases.

1. If $\mu = \sigma = 0$, then $\psi(t) = 1$ for every $t \in R$, and the distribution is degenerate at 0.
2. For $\sigma = 0$ and $\mu > 0$, we have an exponential distribution with mean μ, denoted throughout as Z_μ. Similarly, if $\sigma = 0$ and $\mu < 0$, we have $-Z_{-\mu}$.
3. If $\mu = 0$ and $\sigma \neq 0$, we have the Laplace distribution with location zero and scale σ, whose density is
$$f(x) = \frac{1}{2\sigma} e^{-|x/\sigma|}, \quad x \in R. \tag{5}$$

2.2 Mixture representations

In this section we present various representations of AL distributions. The representations lead to explicit formulas for AL densities and distribution functions, and facilitate computer simulations of AL random variates.

Mixture of normal distributions. Let N and Z be independent and standard normal and exponential distributions, respectively. Then, the following relation takes place:

$$Y_{\sigma,\mu} \stackrel{d}{=} \mu Z + \sqrt{2\sigma^2 Z} \cdot N. \tag{6}$$

Thus, conditionally on $Z = z$, the r.v. $Y_{\sigma,\mu} \sim AL(\sigma, \mu)$ is normal with mean μz and variance $2\sigma^2 z$.

An exponential mixture. Let I_κ be a discrete r.v. taking values $-\kappa$ and $1/\kappa$ with probabilities $p = \kappa^2/(1+\kappa^2)$ and $q = 1/(1+\kappa^2)$, respectively. Let Z be standard exponential independent of I_κ. Then

$$Y_{\sigma,\mu} \stackrel{d}{=} \sigma \cdot I_\kappa \cdot Z. \tag{7}$$

In the symmetric case ($\mu = 0$ and $\kappa = 1$), the random variable I_κ takes values ± 1 with probabilities $1/2$ each, and we obtain the well-known representation of symmetric Laplace distribution.

Mixture of exponentials. The ch.f. of $Y_{\sigma,\mu}$ can be factored as

$$\psi(t) = \frac{1}{(1 + it\sigma\kappa)(1 - it\sigma/\kappa)}, \tag{8}$$

which shows that every AL random variable has the same distribution as the difference of two independent exponential random variables:

$$Y_{\sigma,\mu} \stackrel{d}{=} \sigma \cdot Z_{1/\kappa} - \sigma \cdot Z_\kappa. \tag{9}$$

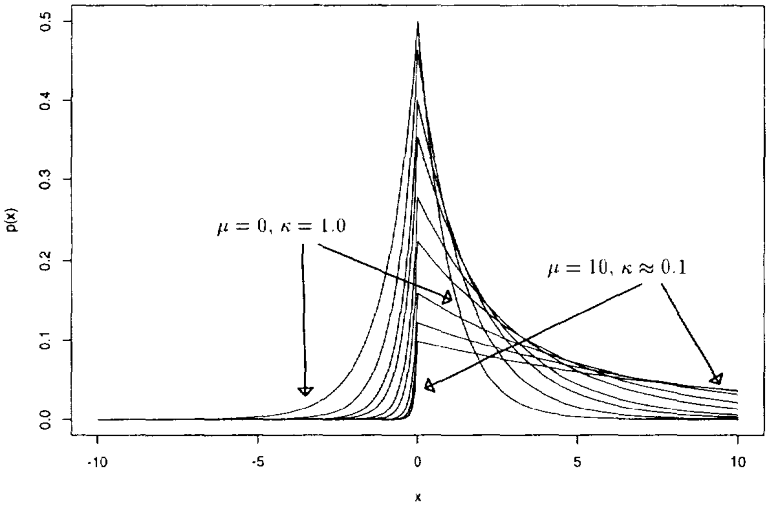

Figure 2: Asymmetric Laplace densities, $\sigma = 1$ and $\mu = 0$, 0.8, 1.5, 2, 3, 4, 6, 8, 10 which correspond to $\kappa \approx$ 1.0, 0.68, 0.50, 0.41, 0.30, 0.24, 0.16, 0.12, 0.1.

2.3 Densities

Let $p_{\sigma,\mu}$ and $F_{\sigma,\mu}$ denote probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of an $AL(\sigma, \mu)$ distribution, respectively. The representation (9) produces the following explicit formulas:

$$p_{\sigma,\mu}(x) = \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases} \exp\left(-\frac{\kappa}{\sigma}x\right), & \text{if } x \geq 0 \\ \exp\left(\frac{1}{\sigma\kappa}x\right), & \text{if } x < 0, \end{cases} \tag{10}$$

and

$$F_{\sigma,\mu}(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2}\exp\left(-\frac{\kappa}{\sigma}x\right), & \text{if } x > 0 \\ \frac{\kappa^2}{1+\kappa^2}\exp\left(\frac{1}{\sigma\kappa}x\right), & \text{if } x \leq 0. \end{cases}$$

Figure 2 shows AL densities for various values of the parameters. It is clear that the distribution is unimodal with the mode equal to zero. We see the characteristic peakedness of the density at zero. Some basic properties of AL densities are collected in the Appendix.

2.4 Moments and related parameters

Since the density of an AL law is a simple exponential function, the values of moments and other related parameters of AL laws follow. We summarize them in Table 1.

Parameter	Definition	Value
Absolute moment	$E\|Y\|^a$	$\left(\frac{\sigma}{\kappa}\right)^a \Gamma(a+1)\frac{1+\kappa^{2(a+1)}}{1+\kappa^2}$
nth moments	EY^n	$n!\left(\frac{\sigma}{\kappa}\right)^n \frac{1+(-1)^n\kappa^{2(n+1)}}{1+\kappa^2}$
Mean	EY	μ
Variance	$E(Y-EY)^2$	$\mu^2+2\sigma^2$
Mean deviation	$E\|Y-EY\|$	$\frac{2\sigma e^{(\kappa^2-1)}}{\kappa(1+\kappa^2)}$
Coefficient of Variation	$\frac{\sqrt{Var(X)}}{\|EX\|}$	$\sqrt{2\frac{\sigma^2}{\mu^2}+1}=\frac{\sqrt{1/\kappa^2+\kappa^2}}{1/\kappa-\kappa}$
Coefficient of Skewness	$\gamma_1=\frac{E(X-EX)^3}{(E(X-EX)^2)^{3/2}}$	$2\frac{1/\kappa^3-\kappa^3}{(1/\kappa^2+\kappa^2)^{3/2}}$
Kurtosis (adjusted)	$\gamma_2=\frac{E(X-EX)^4}{(Var(X))^2}-3$	$6-\frac{12}{(1/\kappa^2+\kappa^2)^2}$

Table 1: **Moments and related parameters of $Y \sim AL(\sigma,\mu)$.**

Remark 1. The mean deviation equals σ for $\mu=0$. Further, we have

$$\frac{\text{mean deviation}}{\text{standard deviation}}=\frac{2e^{\kappa^2-1}}{(1+\kappa^2)\sqrt{1+\kappa^4}}. \tag{11}$$

For the symmetric Laplace distribution ($\mu=0$, $\kappa=1$), the above ratio is equal to $1/\sqrt{2}$.

Remark 2. For a distribution with finite third moment and standard deviation greater than zero, the coefficient of skewness is a measure of symmetry (for symmetric distributions its value is zero), and is independent of scale.

Remark 3. For a distribution with a finite 4th moment, kurtosis (adjusted, so that $\gamma_2=0$ for normal distribution) measures peakedness, and is independent of scale. If $\gamma_2>0$, the distribution is said to be *leptokurtic*, and if $\gamma_2<0$, the distribution is said to *platykurtic.*

We see that an $AL(\sigma, \mu)$ distribution is leptokurtic and γ_2 varies from 3 (the least value for the symmetric Laplace distribution, where $\kappa = 1$) to 6 (the greatest value for exponential distribution, where $\kappa = 0$).

2.5 The median and skewness

The calculation of the median and other percentiles is straightforward. We have the following equation for the median m of an $AL(\sigma, \mu)$ distribution

$$m = \mu \frac{\log(2/(1+\kappa^2))}{1-\kappa^2}. \tag{12}$$

Note if $\mu = 0$, equation (12) yields $m = 0$, which is the median of symmetric Laplace distribution. Similarly, for $\sigma = 0$, we get $m = \mu \log 2$, which is the median of an exponential distribution with mean μ (to which AL law simplifies in this case).

Further, the following inequalities hold for the three common measures of the center:

$$\text{If } \mu > 0, \text{ then } Mode < Median < Mean.$$

$$\text{If } \mu < 0, \text{ then } Mode \geq Median \geq Mean.$$

All three measures are equal to zero if $\mu = 0$ (and the distribution is symmetric).

Finally, we note that another measure of skewness of a distribution with c.d.f. F, provided by the limit

$$\lim_{x \to \infty} \frac{1 - F(x) - F(-x)}{1 - F(x) + F(-x)},$$

in case of an $AL(\sigma, \mu)$ distribution is equal to $\mathrm{sign}(\mu)$.

2.6 Simulation

Since the distribution function of an AL distribution, as well as its inverse, can be written in closed form, the inversion method of simulation is straightforward to implement. Alternatively, mixture representations (6), (7), and (9) can be used for simulation. Below is a generator of a random variate from an AL distribution, based on the representation (7) for $\sigma > 0$ and on the representation (6) for $\sigma = 0$.

An $AL(\sigma, \mu)$ generator.

- Generate a standard exponential variate Z.
- IF $\sigma = 0$

  THEN $Y \leftarrow \mu \cdot Z$.

  ELSE {

Set $\kappa \leftarrow \frac{2\sigma}{\mu+\sqrt{\mu^2+4\sigma^2}}$.

Generate uniform $[0,1]$ variate U, independent of Z.

IF $U < \kappa^2/(1+\kappa^2)$

THEN Set $I \leftarrow -\kappa$.

ELSE Set $I \leftarrow 1/\kappa$.

Set $Y \leftarrow \sigma \cdot I \cdot Z$. }

- RETURN Y.

Numerical subroutines (written in SPlus©) for simulating AL distributions as well as for computing densities, quantiles, c.d.f.'s, and estimators are available from the authors upon request.

2.7 Estimation

Here we derive moment and maximum likelihood estimators for AL parameters σ and μ. We assume that $Y_1, \ldots, Y_n$ is an i.i.d. random sample from an $AL(\sigma, \mu)$ distribution given by ch.f. (3), and write our parameters in vector notation as $\theta = [\mu, \sigma]'$.

Method of moments. Let

$$m_1 = E(Y_{\sigma,\mu}) = \mu \text{ and } m_2 = E(Y_{\sigma,\mu})^2 = 2\mu^2 + 2\sigma^2 \tag{13}$$

be the first two moments of an $AL(\sigma, \mu)$ distribution (see Table 1). When we solve equations (13) for μ and σ and substitute the sample moments,

$$\widehat{m}_{1n} = \frac{1}{n}\sum_{i=1}^{n} Y_i \quad \text{and} \quad \widehat{m}_{2n} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2, \tag{14}$$

for m_1 and m_2, we obtain the method of moments estimators:

$$\widehat{\theta}_n = \begin{bmatrix} \widehat{\mu}_n \\ \widehat{\sigma}_n \end{bmatrix} = \begin{bmatrix} \widehat{m}_{1n} \\ \sqrt{\widehat{m}_{2n}/2 - \widehat{m}_{1n}^2} \end{bmatrix}. \tag{15}$$

Standard arguments of the large sample theory show that the method of moments estimators of μ and σ are consistent and asymptotically normal. Namely, if $\sigma > 0$, then $\widehat{\theta}_n$ given by (15) is strongly consistent estimator of θ and $\sqrt{n}(\widehat{\theta}_n - \theta)$ is asymptotically normal with (vector) mean zero and covariance matrix

$$\Sigma_{MME} = \sigma^2 \begin{bmatrix} 2 + \mu^2/\sigma^2 & \frac{1}{2}\mu/\sigma \\ \frac{1}{2}\mu/\sigma & \mu^2/\sigma^2 + \frac{1}{4}\mu^4/\sigma^4 + \frac{5}{4} \end{bmatrix}. \tag{16}$$

Maximum likelihood. Maximum likelihood estimators (MLE's) in this case are efficient (their asymptotic covariance matrix is the inverse of the Fisher information matrix). The following standard notation allows for a compact formulas for estimators. For real y, let $y^+ = \max(y, 0)$ and $y^- = \max(-y, 0)$ be the positive and negative parts of y, respectively. Applying the above notation to the random sample $Y_1, \ldots, Y_n$, we write $\overline{Y}^+ = \sum_{i=1}^n Y_i^+/n$, and $\overline{Y}^- = \sum_{i=1}^n Y_i^-/n$. Now, we can express the MLE's for κ, σ, and μ as follows:

$$\widehat{\kappa}_n = \sqrt[4]{\overline{Y}^-/\overline{Y}^+}, \quad \widehat{\mu}_n = \overline{Y}, \quad \widehat{\sigma}_n = \sqrt[4]{\overline{Y}^-}\sqrt[4]{\overline{Y}^+}\left(\sqrt{\overline{Y}^+} + \sqrt{\overline{Y}^-}\right). \tag{17}$$

The MLE $\widehat{\theta}_n = [\widehat{\mu}_n, \widehat{\sigma}_n]'$ is consistent and asymptotically normal. The asymptotic distribution of $\sqrt{n}(\widehat{\theta}_n - \theta)$ is bivariate normal with (vector) mean zero and covariance matrix

$$\Sigma_{MLE} = \frac{\sigma^2}{8}\begin{bmatrix} 8(1/\kappa^2 + \kappa^2) & 4(1/\kappa - \kappa) \\ 4(1/\kappa - \kappa) & 1/\kappa^2 + \kappa^2 + 6 \end{bmatrix}. \tag{18}$$

2.8 Generalized Laplace laws – Laplace motion

We can define a Lévy process on $[0, \infty)$ with independent increments, Laplace motion $\{Y(t),\ t \geq 0\}$, so that $Y(0) = 0$, $Y(1)$ is given by (3), and for $0 < \nu$ the distribution of $Y(\nu)$ is given by the ch.f.

$$\psi(t) = [1 + \sigma^2|t|^2 - i\mu t]^{-\nu}. \tag{19}$$

One may call it a *generalized asymmetric Laplace distribution.* Denote the corresponding r.v. by $Y_{\sigma,\mu,\nu}$. It is clear that $Y_{\sigma,\mu,\nu} \stackrel{d}{=} \sigma Y_{1,\delta,\nu}$, where $\delta = \mu/\sigma$, so we study the latter in the sequel. Note that factorization (8) shows that the ch.f. of $Y_{1,\delta,\nu}$ can be written as the product of two gamma ch.f.'s:

$$\psi_{1,\delta,\nu}(t) = \left(\frac{1}{1 + it\kappa}\right)^\nu \left(\frac{1}{1 - it/\kappa}\right)^\nu. \tag{20}$$

Recall that $\left(\frac{1}{1-it/\kappa}\right)^\nu$ is the ch.f. of the gamma r.v. $\Gamma_{\nu,1/\kappa}$, whose density is given by

$$g_{\nu,1/\kappa}(x) = \frac{\kappa^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\kappa x}, \quad x \geq 0. \tag{21}$$

(For $\nu = 1$ it reduces to the exponential distribution with mean $1/\kappa$). Thus, we have the relation

$$Y_{1,\delta,\nu} \stackrel{d}{=} \Gamma_{\nu,1/\kappa} - \Gamma_{\nu,\kappa}, \tag{22}$$

that generalizes (9). Since these distributions will be considered in another paper, we limit our discussion here to their densities. We can get the following expression for the density of $Y_{1,\delta,\nu}$ via the standard transformation theorem of random variables:

$$p_{1,\delta,\nu}(\pm x) = [\Gamma(\nu)]^{-2} e^{-\kappa^{\pm 1} x} \int_0^\infty y^{\nu-1}(x + y)^{\nu-1} e^{-\eta y} dy, \quad x > 0, \tag{23}$$

where $\eta = \sqrt{4+\delta^2}$. Note that in case $\nu = 1$ the above simplifies to (10), as it should. We can write the density (23) as

$$p_{1,\delta,\nu}(\pm x) = \frac{1}{\Gamma(\nu)\sqrt{\pi}}\left(\frac{x}{\eta}\right)^{\nu-1/2} e^{\pm\delta x/2} K_{\nu-1/2}\left(\frac{\eta x}{2}\right), \quad x > 0, \tag{24}$$

where K_u is the modified Bessel function of the third kind:

$$K_u(z) = \frac{(z/2)^u\Gamma(1/2)}{\Gamma(u+1/2)}\int_1^\infty e^{-zt}(t^2-1)^{u-1/2}dt. \tag{25}$$

If $\nu = k$ is an integer, then the density (23) is a mixture of k densities on $(-\infty,\infty)$ and has an explicit form. For $j = 0,\ldots,k-1$, the jth density has the form

$$f_{k,j}(x) = p_{k,j}g_{k-j,1/\kappa}(x)I_{[0,\infty)}(x) + q_{k,j}g_{k-j,\kappa}(-x)I_{(0,\infty)}(-x), \tag{26}$$

where $g_{\nu,\kappa}$ stands for a gamma density as before, and

$$p_{k,j} = \frac{p^kq^j}{p^kq^j+p^jq^k}, \quad q_{k,j} = 1 - p_{k,j} = \frac{p^jq^k}{p^kq^j+p^jq^k}, \tag{27}$$

with $p = 1/(1+\kappa^2)$ and $q = \kappa^2/(1+\kappa^2)$. Note that $p_{1,0} = p$ and $q_{1,0} = q$. For $k = 1$ and $j = 0$ the density (26) coincides with (10). Under the above notation, the density of $Y_{1,\delta,k}$ takes the form

$$p_{1,\delta,k}(x) = \sum_{j=0}^{k-1}\frac{(k+j-1)!}{j!(k-1)!}(p^kq^j+p^jq^k)f_{k,j}(x). \tag{28}$$

3 Applications

In this section we present two applications of AL distributions. The first one is in modeling interest rates on 30-year Treasury bonds. Klein [6] studied yield rates on average daily 30-year Treasury bonds from 1977 to 1990, finding that the empirical distribution is too "peaky" and "fat-tailed" to have been from a normal distribution. He rejected the traditional lognormal hypothesis and proposed the stable Paretian hypothesis, which would "account for the observed peaked middle and fat tails". The paper was followed by several discussions, where some researchers objected to the stable hypothesis and offered alternative models, including a first-order moving average model of Huber. In our approach, we assume that the successive logarithmic changes in interest rates are i.i.d. observations from an AL distribution. Our model is simple, allows for peakedness, fat-tails, skewness, and high kurtosis observed in the data. We were inspired by the ideas of Mittnik and Rachev [17], regarding the interest rate change as the *random* sum of a large number of small changes:

$$\text{interest rate change } = \sum_{i=1}^{\nu_p} \text{(small changes)},$$

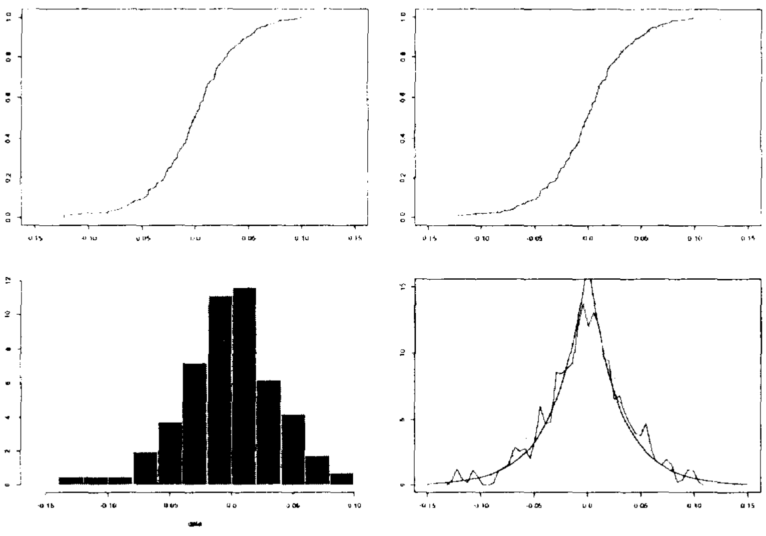

Figure 3: *Top-left:* Empirical c.d.f. vs. normal c.d.f. *Top-right:* Empirical c.d.f. vs. AL c.d.f. *Bottom-left:* Histogram of interest rates on 30-year Treasury bonds. *Bottom-right:* Non-parametric estimator of the density (thin solid line) vs. the theoretical ones (normal - dashed line, AL - thick solid line).

where the number of terms, ν_p, that has a geometric distribution. Thus, provided the small changes have finite variance, the AL law (3) can approximate the distribution of the interest rate change. We think of ν_p as the moment when the probabilistic structure governing the interest rates breaks down. Such event could be a new information, political, economical or other event that affect the fundamentals of the financial market.

Our goal is to present the idea of modeling interest rates using AL laws. The data set consists of interest rates on 30-year Treasury bonds on the last working day of the month and is published in Huber's discussion of Klein's paper [6], p. 156. The data covers the period of February 1997 through December 1993. We convert the data to the logarithmic changes according to the formula: $Y_t = \log(i_t/i_{t-1})$, where i_t is the is the interest rate on 30-year Treasury bonds on the last working day of the month t. There were the total of 202 values of the logarithmic changes Y_i.

First, we have plotted the histogram of the data set (Figure 3 (bottom-left). We can see the typical shape of a AL density: the distribution has high peak near zero, and appears to have tails thicker than that of the normal distribution. Comparisons of the c.d.f. of the normal distribution and the empirical c.d.f. seen on Figure 3 (top-left) and the density functions Figure 3 (bottom-right) confirm these findings. We see a disparity around the center

Parameter	Theoretical value	Empirical value
Mean	−0.001018163	−0.001018163
Variance	0.001733809	001372467
Mean deviation	0.02944785	0.02945773
Mean dev./ Std dev.	0.7072175	0.7582487
Coefficient of Skewness	−0.07334177	−0.2274964
Kurtosis (adjusted)	3.003586	3.599207

Table 2: **Theoretical versus empirical moments and related parameters of** $Y \sim AL(\widehat{\sigma}, \widehat{\mu})$

of the distribution due to a high peak in the observed data. In order to fit an AL model, we need to estimate the parameters μ and σ. We used the maximum likelihood estimators finding $\widehat{\mu} = -0.001018163$ and $\widehat{\sigma} = 0.029434439$. Further, we calculated the parameter κ as well as the theoretical values of various parameters presented in Table 2. We also calculated the empirical counterparts of the parameters, where we used the following statistics:

- Mean: $\frac{1}{n}\sum Y_i$.
- Variance: $\frac{1}{n}\sum(Y_i - \overline{Y})^2$.
- Mean deviation: $\frac{1}{n}\sum |Y_i - \overline{Y}|$.
- Coefficient of skewness: $\widehat{\gamma}_1 = \frac{1}{n}\sum(Y_i - \overline{Y})^3/(\frac{1}{n}\sum(Y_i - \overline{Y})^2)^{3/2}$.
- Kurtosis (adjusted): $\widehat{\gamma}_2 = \frac{1}{n}\sum(Y_i - \overline{Y})^4/(\frac{1}{n}\sum(Y_i - \overline{Y})^2)^2$.

We present the empirical and theoretical values in Table 2. Except for a slight discrepancy for the skewness, the match between empirical and theoretical values is striking. In addition, we show, in Figure 3 (top-right), the theoretical AL c.d.f. compared with the empirical c.d.f. and, in Figure 3 (bottom-right), the density kernel estimator based on the data against the theoretical densities of normal and AL distributions with the estimated parameters. We see that at the mode agreement is better for the AL distribution than for the normal one.

The second example illustrates how AL laws can account for asymmetry in the data. The data consists of currency exchange rates: the German Deutschemark versus the US Dollar (DMUS). The observations are daily exchange rates from 1/1/80 to 12/7/90 (2853 data points). As usual, we consider the change in the log(rate) from day t to day $t+1$. First, we plotted the histogram (Figure 4 (left)) We see the typical shape of a AL density. The distribution has a high peak near zero, and appears to have non-symmetric tails thicker than that of the normal distribution. A normal QQ plot (see Figure 4 (middle)) confirms these findings. We used maximum likelihood for estimating the AL parameters obtaining $\widehat{\mu} = 0.0007558$ and $\widehat{\sigma} = 0.00521968$. The quantile plot of the data set and the theoretical AL distribution is presented in Figure 4 (right). It shows only very slight departures from the straight line. We conclude that AL distributions model this data set more correctly than normal distributions.

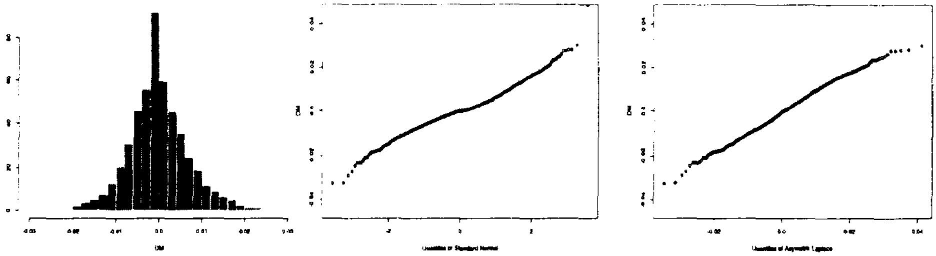

Figure 4: Analysis of the currency data.

4 Appendix

4.1 Properties

We collect here the main properties of AL laws. We omit most proofs and refer an interested reader to Kozubowski and Podgórski [11] for a more detailed treatment.

4.1.1 Stability

Paretian stable distributions, that include normal laws, have two fundamental properties. First, they are limiting laws for appropriately normalized sums of i.i.d. random variables. Thus, they work well as approximations to sums of i.i.d. random variables. Second, they are stable: the sum of i.i.d. normal (Paretian stable) r.v.'s has a normal (stable Paretian) distribution. These properties are shared by AL laws, if *deterministic summation* is replaced by *geometric summation*. By definition, AL laws are limits of geometric compounds of i.i.d. components. The stability properties of exponential and Laplace distributions are well known (see for example Arnold [2] for exponential and Lin [12] for Laplace):

$$Y \stackrel{d}{=} a_p \sum_{i=1}^{\nu_p} Y_i, \tag{29}$$

where ν_p is geometric (2), Y_i's are i.i.d. copies of Y, and ν_p and (Y_i) are independent (the constants $a(p)$ are equal to $\sqrt{p}$ for Laplace and p for exponential distributions). Although general AL r.v.'s do not satisfy (29), they have the following characterization (see Kozubowski [7]): A r.v. Y with finite variance is AL if and only if

$$a_p \sum_{i=1}^{\nu_p} (Y_i + b_p) \xrightarrow{d} Y, \tag{30}$$

where ν_p is geometric (2), Y_i's are i.i.d. copies of Y, and ν_p and (Y_i) are independent. Moreover, if $\sigma > 0$, the normalizing constants in (30) can be taken as

$$a_p = Cp^{1/2}, \quad b_p = (p^{1/2} - C)\mu/C, \quad C = \sqrt{2/(2 + (\mu/\sigma)^2)}.$$

In addition, all AL laws are *geometric infinitely divisible*, that is

$$Y \stackrel{d}{=} \sum_{i=1}^{\nu_p} Y_p^{(i)}, \tag{31}$$

where ν_p is geometric (2), $Y \sim AL(\sigma, \mu)$, $Y_p^{(i)}$'s are i.i.d. with $AL(\sigma\sqrt{p}, \mu p)$ distribution for each p, and ν_p, $(Y_p^{(i)})$ are independent.

Remark. AL laws are infinitely divisible in the classical sense as well. Please see Kozubowski and Podgórski [11] for the exact form of their Lévy measure.

4.1.2 Self-decomposability

Relations (7) and (9) are special cases with $c = 0$ of the more general representations:

$$Y_{\sigma,\mu} \stackrel{d}{=} c \cdot Y_{\sigma,\mu} + (\delta_1/\kappa - \delta_2\kappa)Z \stackrel{d}{=} c \cdot Y_{\sigma,\mu} + \delta_1\sigma Z_{1/\kappa} - \delta_2\sigma Z_\kappa, \quad 0 \le c \le 1, \tag{32}$$

where (δ_1, δ_2) has the following joint distribution:

$$\begin{aligned} P(\delta_1 = 0, \delta_2 = 0) &= c^2, \\ P(\delta_1 = 1, \delta_2 = 0) &= (1 - c)\left(c + \frac{1 - c}{1 + \kappa^2}\right), \\ P(\delta_1 = 0, \delta_2 = 1) &= (1 - c)\left(c + \frac{(1 - c)\kappa^2}{1 + \kappa^2}\right), \end{aligned}$$

and the r.v.'s $Y_{\sigma,\mu}$, (δ_1, δ_2), and Z (correspondingly, $Y_{\sigma,\mu}$, (δ_1, δ_2), $Z_{1/\kappa}$, and Z_κ) are mutually independent. Written in terms of ch.f.'s, the relation (32) takes the form

$$\psi(t) = \psi(ct)\psi_c(t), \quad 0 \le c \le 1, \tag{33}$$

where ψ and ψ_c are ch.f.'s of $Y_{\sigma,\mu}$ and $(\delta_1/\kappa - \delta_2\kappa)Z$, respectively. Recall that a ch.f. ψ that satisfy (33) is said to be self-decomposable, and the corresponding distribution (and r.v.) is said to be in class L. Thus, all AL laws are in class L, which implies that they are unimodal, as self-decomposability implies unimodality (Yamazato [21]). It is clear from the explicit formula for the density, that modes of all AL laws are actually equal to zero.

Remark. We note that self-decomposability and unimodality of AL laws was established in Ramachandran [19].

4.1.3 Properties of the densities

We discuss here basic properties of AL densities.

Values at zero. We have the following relations:

$$p_{\sigma,\mu}(0) = \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}, \quad F_{\sigma,\mu}(0) = P(Y_{\sigma,\mu} \le 0) = \frac{\kappa^2}{1+\kappa^2},$$

$$\lim_{\mu\to\pm\infty} p_{\sigma,\mu}(0) = \frac{1}{2\sigma}, \quad \lim_{\sigma\to\infty} p_{\sigma,\mu}(0) = \frac{1}{|\mu|}, \quad \lim_{\mu,\sigma\to 0} p_{\sigma,\mu}(0) = \infty.$$

Symmetry with respect to μ. The densities $p_{\sigma,\mu}$ are symmetric with respect to the parameter μ:

$$p_{\sigma,\mu}(x) = p_{\sigma,-\mu}(-x),$$

which follows from (10) and the relation

$$\kappa = \frac{2\sigma}{\sqrt{4\sigma^2+\mu^2}+\mu} = \frac{\sqrt{4\sigma^2+\mu^2}-\mu}{2\sigma}.$$

Asymmetry of the density. The density $p_{\sigma,\mu}$ of a AL distribution with $\mu > 0$ satisfies the relation

$$p_{\sigma,\mu}(x) > p_{\sigma,\mu}(-x), \quad x > 0.$$

By symmetry, an analogous results hold for $\mu < 0$. In fact, we have

$$\frac{p_{\sigma,\mu}(x)}{p_{\sigma,\mu}(-x)} = \exp(x(1/\kappa-\kappa)/\sigma) = \exp(x\mu/\sigma^2),$$

so that $p_{\sigma,\mu}(-x)/p_{\sigma,\mu}(x) \to 0$ as $x \to \infty$.

Derivatives. Except for $x = 0$, AL densities have derivatives of any order $n > 0$:

$$f^{(n)}_{\sigma,\mu}(x) = \begin{cases} (-1)^n \quad \left(\frac{\kappa}{\sigma}\right)^{n+1} \quad \frac{1}{1+\kappa^2} \quad e^{-\kappa x/\sigma}, & \text{if } x > 0 \\ \quad\quad \frac{1}{(\sigma\kappa)^{n+1}} \quad \frac{\kappa^2}{1+\kappa^2} \quad e^{x/(\kappa\sigma)}, & \text{if } x < 0 \end{cases}. \tag{34}$$

Further, we have

$$\lim_{x\to 0^+} (-1)^n p^{(n)}_{\sigma,\mu}(x) = \frac{1}{1+\kappa^2}\left(\frac{\kappa}{\sigma}\right)^{n+1}, \quad \lim_{x\to 0^-} p^{(n)}_{\sigma,\mu}(x) = \frac{\kappa^2}{1+\kappa^2}\left(\frac{1}{\kappa\sigma}\right)^{n+1}.$$

The two limits are equal if either $n = 0$ (showing the continuity of the density at zero) or $\kappa = 1$ (and thus $\mu = 0$), producing the symmetric Laplace distribution.

Complete monotonicity. A function f defined on $I \subset R$ is called completely monotonic (respectively, absolutely monotonic) if it is infinitely differentiable on I and $(-1)^k f^{(k)}(x) \ge 0$ (respectively, $f^{(k)}(x) \ge 0$) for any $x \in I$ and any $k = 0, 1, 2, \ldots$. The complete and absolute monotonicity of AL densities follow directly from (34). Namely, if $p_{\sigma,\mu}$ is the density of an $AL(\sigma,\mu)$ distribution, then the functions $p_{\sigma,\mu}(\pm x)$ are completely monotonic on $(0,\infty)$ and absolutely monotonic on $(-\infty, 0)$.

4.2 Further extensions

The class of AL laws can be extended in various ways. First, the distributions may be shifted, allowing for arbitrary modes. Next, one can consider a more general class of distributions given by an AL ch.f. (3) raised to a positive power. These are marginal distributions of the Lévy process $\{Y(t), t \geq 0\}$ with independent increments, for which $Y(1) \sim AL(\sigma, \mu)$ as it was described at the end of Section 2. Further, one obtains a richer class of limiting distributions, consisting of geometric stable laws, by allowing for infinite variance of the components in the geometric compounds (1). More generally, if the random number of components in the summation (1) is not geometrically distributed, a wider class of *ν-stable* laws is obtained as the limiting distributions. Finally, if the components in (1) are multi-dimensional, the multivariate AL distributions are obtained.

4.2.1 Translated AL laws

If $Y \sim AL(\sigma, \mu)$, then $Y + \xi$ is a r.v. with a three-parameter density

$$p_{\sigma,\mu,\xi}(x) = \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases} \exp\left(-\frac{\kappa}{\sigma}(x-\xi)\right), & \text{if } x \geq \xi \\ \exp\left(\frac{1}{\sigma\kappa}(x-\xi)\right), & \text{if } x < \xi, \end{cases} \tag{35}$$

and distribution function

$$F_{\sigma,\mu\xi}(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2}\exp\left(-\frac{\kappa}{\sigma}(x-\xi)\right), & \text{if } x > \xi \\ \frac{\kappa^2}{1+\kappa^2}\exp\left(\frac{1}{\sigma\kappa}(x-\xi)\right), & \text{if } x \leq \xi. \end{cases}$$

Although these three-parameter distributions are no longer limiting laws for geometric compounds (1), nor do they have stability property (29), they do provide more flexibility in data modeling by allowing for arbitrary modes.

4.2.2 Geometric stable laws

If the random variables in (1) have infinite variance, than the geometric compound no longer converge to an AL law. Instead, the limiting distributions form a broader class of *geometric stable* (GS) laws. It is a four-parameter family best described in terms of characteristic function:

$$\psi(t) = [1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t]^{-1}, \tag{36}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \operatorname{sign}(x)\tan(\pi\alpha/2), & \text{if } \alpha \neq 1, \\ 1 + i\beta\frac{2}{\pi}\operatorname{sign}(x)\log|x|, & \text{if } \alpha = 1. \end{cases}$$

The parameter $\alpha \in (0, 2]$ is the *index* that determines the tail of the distribution: $P(Y > y) \sim Cy^{-\alpha}$ (as $y \to \infty$) for $0 < \alpha < 2$. For $\alpha = 2$ the tail is exponential and the distribution reduces to AL law, as $\omega_{2,\beta} \equiv 1$. The parameter $\beta \in [-1, 1]$ is the skewness parameter, while

$\mu \in R$ and $\sigma \geq 0$ control the location and scale, respectively. We briefly comment on some of the features of GS laws, and refer an interested reader to Kozubowski and Rachev [10] for an up to date information and extensive references on GS laws and their special cases.

Remark 1. Special cases of GS laws include Linnik distribution, for which $\beta = 0$ and $\mu = 0$ (see Linnik [13]), and Mittag-Leffler laws, which are GS with $\beta = 1$ and either $\alpha = 1$ and $\sigma = 0$ (exponential distribution) or $0 < \alpha < 1$ and $\mu = 0$. The latter are the only non-negative GS r.v.'s (see Pillai [18]).

Remark 2. GS laws share many, but not all, properties of *Paretian stable* distributions, which were discussed in the actuarial context in Klein [6]. In fact, stable and GS laws are related through their characteristic functions, φ and ψ, as shown in Mittnik and Rachev [16]:

$$\psi(t) = \gamma(-\log \varphi(t)), \tag{37}$$

where $\gamma(x) = 1/(1+x)$ is the Laplace transform of the standard exponential distribution. Relation (37) produces the representation (36), as well as the mixture representation of a GS random variable Y in terms of independent standard stable and exponential r.v.'s , X and Z:

$$Y \stackrel{d}{=} \begin{cases} \mu Z + Z^{1/\alpha}\sigma X, & \alpha \neq 1, \\ \mu Z + Z\sigma X + \sigma Z\beta(2/\pi)\log(Z\sigma), & \alpha = 1. \end{cases} \tag{38}$$

Note that the above representation reduces to (6) in case $\alpha = 2$, as then X has the normal distribution with mean zero and variance 2.

Remark 3. The asymmetric Laplace distribution, which is GS with $\alpha = 2$, plays the same role among GS laws, as normal distribution does among stable laws. As normal distribution is convenient in application, so is AL law, as its p.d.f., c.d.f. have explicit expressions.

Remark 4. Like stable laws, GS laws lack explicit expressions for densities and distribution functions, which handicap their practical implementation. Also, they are "fat-tailed", have stability properties (with respect to random summation), and generalize the central limit theorem (as they are the only limiting laws for geometric compounds). *However, they are different from stable (and normal) laws in that their densities are more "peaked", while still being heavy-tailed.* Unlike stable densities, GS densities "blow-up" at zero if $\alpha < 1$. Since many financial data are "peaked" and "fat-tailed", they are often consistent with the GS model (see for example Kozubowski and Rachev [8]).

4.2.3 ν-stable laws

Suppose that the random number of terms in the summation (1) is any integer-valued random variable, and, as p converges to zero, ν_p approaches infinity (in probability) while $p\nu_p$ converges in distribution to a r.v. ν with Laplace transform γ. Then, the normalized compounds (1) converge in distribution to a *ν-stable distribution*, whose characteristic function is (37) (see for example Kozubowski and Panorska [9]). The class of ν-stable laws contains

GS and generalized AL laws as special cases: if ν_p is geometric (2), then $p\nu_p$ converges to the standard exponential and (37) produces (36); similarly, if ν_p is negative binomial with parameters ν and p, then $p\nu_p$ converges to a gamma distribution and (37) produces (19). The tail behavior of ν-stable laws is essentially the same as that of stable and GS laws (see Kozubowski and Panorska [9]).

4.2.4 Multivariate extension

The theory of AL laws can be extended to random vectors. Namely, a multivariate AL law can be defined as the limit (in distribution) as $p \to 0$ of appropriately normalized random sums

$$\mathbf{S}_p = \mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(\nu_p)}. \tag{39}$$

Here, $(\mathbf{X}^{(i)})$ is a sequence i.i.d. random vectors with *finite second moments* and ν_p has a geometric distribution (2), independent of $\mathbf{X}^{(i)}$'s. It follows that the limiting distributions for normalized geometric compounds (39) are laws with the following ch.f.:

$$\Psi(\mathbf{t}) = \left[1 + \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t} - i\mathbf{t}'\mathbf{m}\right]^{-1}, \tag{40}$$

where $\mathbf{m}$ is an arbitrary vector in R^d and Σ is a $d \times d$ non-negative symmetric matrix (see for example Mittnik and Rachev [16]). We shall call a distribution given by (40) a *multivariate asymmetric Laplace* law and denote it by $AL(\Sigma, \mathbf{m})$. The symmetric case with $\mathbf{m} = \mathbf{0}$ was discussed in the literature before (see Johnson and Kotz [4], Madan and Seneta [14]). If Σ is positive-definite, the distribution is truly d-dimensional and has a probability density function

$$g(\mathbf{y}) = \frac{2e^{\mathbf{y}'\Sigma^{-1}\mathbf{m}}}{(2\pi)^{d/2}|\Sigma|^{1/2}} \left(\frac{\mathbf{y}'\Sigma^{-1}\mathbf{y}}{2 + \mathbf{m}'\Sigma^{-1}\mathbf{m}}\right)^{v/2} K_v\left(\sqrt{(2 + \mathbf{m}'\Sigma^{-1}\mathbf{m})(\mathbf{y}'\Sigma^{-1}\mathbf{y})}\right), \tag{41}$$

where $v = (2-d)/2$ and K_v is the modified Bessel function (25). In the symmetric case ($\mathbf{m} = \mathbf{0}$) this density was derived in Anderson [1]. In the one-dimensional case, where $\Sigma = \sigma_{11}$, we have $v = (2-1)/2 = 1/2$ and the Bessel function simplifies to $K_{1/2}(u) = \sqrt{\pi/(2u)}e^{-u}$. Consequently, the density (41) simplifies to the density (10) of a univariate AL law with parameters $\sigma = \sqrt{\sigma_{11}/2}$ and $\mu = \mathbf{m}$. For $d > 1$ the density (41) blows up at zero. Further, if the dimensionality d is odd, $d = 2r + 3$, the density has a closed form:

$$g(\mathbf{y}) = \frac{C^r e^{\mathbf{y}'\Sigma^{-1}\mathbf{m} - C||\mathbf{y}||_{\Sigma^{-1}}}}{(2\pi||\mathbf{y}||_{\Sigma^{-1}})^{r+1}|\Sigma|^{1/2}} \sum_{k=0}^{r} \frac{(r+k)!}{(r-k)!k!}(2C||\mathbf{y}||_{\Sigma^{-1}})^{-k}, \quad \mathbf{y} \neq \mathbf{0}, \tag{42}$$

where $v = (2-d)/2$, $C = \sqrt{2 + \mathbf{m}'\Sigma^{-1}\mathbf{m}}$, and $||\mathbf{y}||_{\Sigma^{-1}} = \sqrt{\mathbf{y}'\Sigma^{-1}\mathbf{y}}$ is a norm in R^d. In the three dimensional space, we have $r = 0$ and the density is particularly simple:

$$g(\mathbf{y}) = \frac{e^{\mathbf{y}'\Sigma^{-1}\mathbf{m} - C||\mathbf{y}||_{\Sigma^{-1}}}}{2\pi||\mathbf{y}||_{\Sigma^{-1}}|\Sigma|^{1/2}}, \quad \mathbf{y} \neq \mathbf{0}. \tag{43}$$

Multivariate AL laws share many, but not all, properties of univariate AL laws. Since a more extensive study of the multivariate case will appear elsewhere, we only give few general remarks.

Stability. Multivariate AL laws are geometrically infinitely divisible as well as infinitely divisible in the classical sense, and satisfy the stability property (29) whenever either $\mathbf{m}$ or $\mathbf{\Sigma}$ equals zero. However, relation (30) does not generally hold for $d > 1$.

Mixture representation. Mixture representation (6) extends to the multivariate case as follows. Let $\mathbf{Y} \sim AL(\mathbf{\Sigma}, \mathbf{m})$ and let $\mathbf{X} \sim N(\mathbf{0}, \mathbf{\Sigma})$ (multivariate normal with mean zero and variance-covariance $\mathbf{\Sigma}$). Let Z be an exponentially distributed r.v. with mean 1, independent of $\mathbf{X}$. Then the following representation holds

$$\mathbf{Y} \stackrel{d}{=} \mathbf{m}Z + Z^{1/2}\mathbf{X}. \tag{44}$$

The mean and variance-covariance matrix. Representation (44) leads to the following formulas for the mean and variance-covariance of $\mathbf{Y} \sim AL(\mathbf{\Sigma}, \mathbf{m})$:

$$E\mathbf{Y} = \mathbf{m}, \quad E(\mathbf{Y} - E\mathbf{Y})(\mathbf{Y} - E\mathbf{Y})' = \mathbf{\Sigma} + \mathbf{m}\mathbf{m}'. \tag{45}$$

Linear transformations. Any linear transformation of an AL r.v. leads to another AL random variable. Let $\mathbf{Y} = (Y_1, \ldots, Y_d)' \sim AL(\mathbf{\Sigma}, \mathbf{m})$ and let $\mathbf{A}$ be an $l \times d$ real matrix. Then, the random vector $\mathbf{Y_A} = \mathbf{AY}$ is $AL(\mathbf{\Sigma_A}, \mathbf{m_A})$, where $\mathbf{m_A} = \mathbf{Am}$ and $\mathbf{\Sigma_A} = \mathbf{A\Sigma A}'$. In particular, multivariate and univariate marginals of an AL random vector are AL, as are all linear combinations of its components.

5 Summary

The class of AL laws plays an analogous role among geometric stable laws to that played by the class of normal distributions among stable laws. AL laws arise as limiting distributions for geometric compounds, as normal laws do for deterministic sums, of i.i.d. random variables with finite second moments. AL laws have a stability property with respect to geometric summation, as normal laws do with respect to classical summation. Both, AL and normal laws are convenient in applications, as their densities have explicit forms and estimation procedures are easily implemented. However, there are important differences between the two families of distributions: AL laws are more "peaky" and have tails heavier than normal laws and allow for asymmetry, whereas all normal distributions are symmetric. We hope that our survey of results and methods for AL laws will lead to more frequent applications of AL laws in actuarial science and other areas of applied research.

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