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Optimization of the Ultimate Ruin Probability in Risk Theory

by

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Abstract

In the classical risk models with constant and variable premium rate, the ultimate ruin probability depends on the choice of the initial surplus, of the premium rate

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and of the individual claimsize distribution. The optimization problem is to find the minimal and the maximal ruin probabilities given a fixed initial surplus, a premium rate and some moment constraints on the individual claimsize distribution. The individual claimsize distribution is concentrated on a closed interval and its first two moments have specific values. A numerical approach is used to solve the problem. In this approach, we apply a general optimization algorithm which requires a numerical method to approximate the ultimate ruin probability. One of the main practical interests is to derive the greatest and the lowest bounds of the ultimate ruin probability given some fixed constraints on the moments of the claimsize distribution. These bounds are obtained without the estimation of the claimsize distribution. For practical values of initial surplus, the difference between the bounds can be so small that they are good enough to approximate the ruin probability. The optimization problem can be extended to more general risk models. The numerical solution is derived with the same methodology which can also be applied to other optimization problems in actuarial science (stop-loss premiums, finite-time ruin probabilities).

1 Introduction

The objective of the paper is to present and to apply a numerical method in the calculation of the minimal and maximal ultimate ruin probabilities in two risk models given some moment constraints on the individual claimsize distribution. An application of this optimization problem is to find the extremal lower and upper bounds of the ultimate ruin probabilities given those constraints. We consider the classical risk models with constant and variable premium.

In both risk models, the ultimate ruin probability depends on the choice of the initial surplus, of the premium rate and of the individual claimsize distribution. The optimization problem is to find the minimal and the maximal ultimate ruin probabilities given a fixed initial surplus, a fixed definition of the premium rate and some moment constraints on the individual claimsize distribution. These constraints specify that the individual claimsize distribution is concentrated on a closed interval and its first two moments have specific values. A numerical approach is used to solve the optimization problem which is based on the application of a general optimization algorithm. The application of the optimization algorithm requires the numerical approximation of the ultimate ruin probability.

Often, in practice, we do not have a lot of information on the individual claimsize probability distribution. This knowledge may be the maximal amount, the mean and the variance of the individual claims. One of the main practical interest of finding the minimal and the maximal ultimate ruin probabilities is to derive the greatest and the lowest bounds of the ultimate ruin probability without the estimation of the claimsize distribution. For practical values of initial surplus, the difference between the bounds can be so small that they are good enough to approximate the ultimate ruin probability. The methodology used in this paper can also be applied to other optimization problems in actuarial science (stop-loss premiums, finite-time ruin probability) (see DeVylder (1996)).

The objective of this paper is to present the numerical approach to the optimization of ultimate ruin probability in two risk models. We do not explicit the proofs of the theorems and the propositions but give the references where they can be found in order to keep this article to a reasonable length. The paper is constructed as follows. We present the classical risk models with constant and variable premiums. We define the optimization problem and present the numerical methodology used to solve this problem, which involves a general optimization algorithm. Numerical examples are presented and discussed.

2 Classical risk model - with constant premium

In the classical risk model, the surplus process $\{U(t), t \geq 0\}$ is defined as follows

$$U(t) = u + ct - S(t), \; t \geq 0,$$

where

(1) u = U(0) = initial surplus

(2) c = premium rate

(3) S(t) = total claim amount over the time interval (0,t].

The process {S(t), t≥0} is a Compound Poisson process with

$$S(t) = \sum_{i=1}^{N(t)} X_i \tag{1}$$

where

(1) $\{X_1, X_2, \ldots\}$ is a sequence of *i.i.d.* random variables;

(2) {N(t),t≥0} is a Poisson process with parameter λ;

(3) $\{X_1, X_2, \ldots\}$ and {N(t),t≥0} are independent.

The common probability distribution of the X_i (i = 1, 2, ..) is F(x), with F(0) = 0. The nth moment of F is μ_n with $\mu_1 = \mu$. The probability distribution of S(t) is given by

$$F_{S(t)}(s) = P(S(t) \leq s) = \sum_{j=0}^{\infty} \exp(-\lambda t)\frac{(\lambda t)^j}{j!} F^{*j}(s),\ s > 0,$$

where F^{*j} = jth-convolution of F.

The premium rate c is

$$c = E(S(1))(1+\eta) = \mu\lambda\ (1+\eta) = \mu\lambda\ \gamma,$$

where η is the security loading which is assumed stricly positive.

We define by T the time of ruin

$$T = \begin{cases} \inf_{t>0} \{t, U(t) < 0\}, \text{ if U(t) falls below 0 at least once} \\ \infty, \text{ if U(t) never goes below 0} \end{cases}.$$

The ultimate ruin probability is denoted by $\psi(u, \eta, F)$, where

$$\psi(u, \eta, F) = P(T < \infty),$$

and its complement, denoted $\phi(u, \eta, F)$, is the ultimate non-ruin probability where

$$\begin{aligned} \phi(u, \eta, F) &= 1 - \psi(u, \eta, F) \\ &= P(U(t) > 0, \text{ for all } t > 0). \end{aligned}$$

The ultimate ruin probability $\psi(u, \eta, F)$ is function of the choice of the initial surplus u, the security loading η and the individual claimsize distribution F. The analytic expression of $\phi(u, \eta, F)$ is given in the following proposition.

Proposition 1 *We define G(x) by*

$$G(x) = \frac{1}{\mu} \int_0^x (1-F(y))\, dy , \; x > 0.$$

Then, we have

$$\phi(u, \eta, F) = p \sum_{j=0}^{\infty} q^j \; G^{*j}(u), \; u \geq 0,$$

where $p = \frac{\eta}{1+\eta}$ *and* $q = 1 - p$.

Proof: It is a known result. See Feller (1971), Gerber (1979), Grandell (1991) or Panjer and Wilmot (1992).

No explicit expression of $\psi(u, \eta, F)$ exists except for special cases of F such as the exponential probability distributions or mixtures of exponential probability distributions (see, for instance, Dufresne and Gerber (1989) or DeVylder and Marceau (1994)). Number of approximations have been proposed in the actuarial literature. A review and numerical comparisons of some of these methods are made in Marceau (1993).

3 Classical risk model with variable premium

A certain number of extensions to the classical risk model with constant premium rate were proposed in the actuarial litterature. We consider here the classical risk model with a variable premium rate. In this risk model, the surplus process {U(t), t≥0} defined as

$$U(t) = u + \int_0^t c(U(s))ds - S(t),\ t \geq 0,$$

where c(r) is the premium rate which depends on the current reserve with p(r) > 0 for r > 0.

The process {S(t),t≥0} is a Compound Poisson process as it is defined in (1). The surplus process can also be defined by a stochastic differential equation

$$dU(t)=c(U(t))dt-dS(t),t\geq 0.$$

We assume that the premium rate c(r) is function of the current surplus level U(t) = r. The classical risk model with variable premiums could be applied in two special cases. In the first case, we consider the situation when interests are earned on the surplus. The function c(r) has the following form

$$c(r) = c + \delta r, \tag{2}$$

where δ is the force of interest. If $c = (1+\eta)\lambda\mu$, (2) becomes

$$\begin{aligned} c(r) &= c + \delta r \\ &= ((1+\eta)+\frac{\delta}{\lambda\mu}r)\lambda\mu \\ &= ((1+\eta)+\rho r)\lambda\mu \\ &= ((1+\eta(r))\lambda\mu \\ &= \gamma(r)\ \lambda\mu. \end{aligned} \tag{3}$$

In the second case, the premium rate function c(r) is defined in such way that the premiums rates are charged by layers. In this case, the function c(r) has the following form

$$c(r) = \begin{cases} c_1, 0 = u_0 \leq r \leq u_1 \\ c_2, u_1 < r \leq u_2 \\ \dots \\ c_k, u_{k-1} < r < u_k = \infty \end{cases} \tag{4}$$

with $c_1 > c_2 > \dots > c_k > \lambda\mu$. The premium rate c(r) decreases as the surplus level increases. This can occur when the company decides to reduce the premium rate when the surplus becomes greater since the risk of ruin decreases with the surplus level. Another interpretation of (4) is to consider the reduction of the premium rate as a form of dividend payment which increases as the surplus level grows up.

Another special case is obviously the classical risk model with constant premium rate where $c(r)$ is equal to c for $r > 0$.

For studies on the classical risk model with variable premium rate, see Asmussen and Petersen (1988), Petersen (1990), Dickson (1991), Sundt and Teugel (1995), Michaud (1996), DeVylder (1996).

If T represents the first time that the surplus goes below zero i.e.

$$\mathrm{T} = \begin{cases} \inf\limits_{t>0} \{t, U(t) < 0\}, \text{ if U(t) falls below 0 at least once} \\ \infty, \text{ if U(t) never goes below 0} \end{cases},$$

then the ultimate ruin probability, denoted by $\psi(u, \eta(r), F)$, is

$$\psi(u, \eta(r), F) = \mathrm{P}(\mathrm{T} < \infty).$$

The ultimate non-ruin probability is denoted by $\phi(u, \eta(r), F)$ with $\phi(u, \eta(r), F) =$ 1 - $\psi(u, \eta(r), F)$. Again, $\psi(u, \eta(r), F)$ depends on the choice of the initial surplus u, the parameters of the function η(r) and the individual claimsize distribution F. In the following proposition, we give the integral equation for the function $\phi(u) = \phi(u, c(r), F)$ with known η(r) and F.

Proposition 2 *We define G(x) as*

$$G(x) = \frac{1}{\mu} \int_0^x (1-F(y))\, dy \ , \ x > 0. \tag{5}$$

Then, we have

$$G*\phi(u) = \int_0^u \gamma(x) d\phi(x), \; u \geq 0, \qquad (6)$$

where $\gamma(x) = 1 + \eta(x) = 1 + \eta + \frac{\delta}{\mu\lambda}$.

Proof: See DeVylder (1996).

The evaluation of $\phi(u)$ by numerical methods are proposed in Petersen (1990), Dickson (1991) and DeVylder (1996). A simulation method is also proposed in Michaud (1996).

4 The Optimization Problem

An excellent contribution to the study of optimization problems in actuarial science is given in DeVylder (1996). The optimization of the ultimate ruin probability in the classical risk model with constant premium rate corresponds to the Schmitter's problem (see Brockett, Goovaerts and Taylor (1991), Kaas (1991), DeVylder and Marceau (1996b), DeVylder, Goovaerts and Marceau (1997a), DeVylder, Goovaerts and Marceau (1997b)). In the present section, the optimization problem is formulated in the setting of the risk model with variable premium rate since the classical risk model with constant premium rate is one of its special cases. Another application is in the calculation of stop-loss premiums (see Goovaerts and al. (1986, 1990), DeVylder and Goovaerts (1982, 1983)). The reader is invited to consult DeVylder (1996) where he will find a fine contribution on the subject.

4.1 The problem

Consider that $\phi(u, \eta(r), F)$ represents the ultimate non-ruin probability within the classical risk model with variable premium rate. The conditions of the optimization problem are:

1. The initial risk reserve u is assumed fixed

2. The parameters η and δ of the function $\eta(r)$ are assumed fixed

3. The constraints on the individual claimsize distribution F are:

   - F is assumed to be concentrated on [a,b].
   - The mean μ_1 and the second moment μ_2 of F are assumed fixed. (7)

Additional contraints can be added on F (ex: unimodality, fixed third moment). The study of the optimization problem with these additional constraints and within the classical risk model with constant premium rate is made in Marceau (1996).

The **optimization problem** is, for fixed η, δ, u, a, b, $\mu_1$and μ_2, to **find** $F_{\min}$ **which minimize** $\phi(u, \eta, F)$ (or find $F_{\max}$ which maximize $\phi(u, \eta, F)$) with the constraints (7) on F .

It is important to mention that the functional $\phi(u, \eta, F) = \phi(F)$ is neither convex nor concave.

4.2 Application

For the application of the optimization problem, we define

$$\text{(i) } \phi(u,\eta,F_{\min}) = \inf_F \phi(u,\eta,F)$$

$$\text{(ii) } \phi(u,\eta,F_{\max}) = \sup_F \phi(u,\eta,F) \ .$$

Then, we have

$$\phi(u,\eta,F_{\min}) \leq \phi(u,\eta,F) \ \leq \phi(u,\eta,F_{\max}) \tag{8}$$

for all F with the constraints

- same mean μ_1;
- same second moment μ_2; (9)
- same support [a,b].

The extremal lower and upper bounds for $\phi(u,\eta,F)$ were found without estimating F. In the next section, we present the numerical approach that we use to find the solutions $F_{\min}$ and $F_{\max}$. We can express (10) in terms of ultimate ruin probabilities

$$\psi(u,\eta,F_{\max}) \leq \psi(u,\eta,F) \ \leq \psi(u,\eta,F_{\min}), \tag{10}$$

where

$$\psi(u,\eta,F_{min}) = 1-\phi(u,\eta,F_{min})$$

and

$$\psi(u,\eta,F_{\max}) = 1-\phi(u,\eta,F_{\max}).$$

5 Numerical approach to the problem

A presentation of the numerical approach to the problem of optimization of the ultimate non-ruin probability is made in Marceau (1996). In the present section, we give a summary of the basic elements of the numerical approach. An extensive presentation of this numerical approach and its application to a diversity of optimization problems is given in DeVylder (1996).

We define the following sets:

$$\mathrm{I} = [\mathrm{a},\mathrm{b}]$$

$$\mathrm{A}_n = \{i_0, i_1, ..., i_n\},$$

where A_n is a finite set of atoms such that $\mathrm{A}_n \subset \mathrm{I}$. For example,

$$\mathrm{i}_k = \mathrm{a} + (\mathrm{b}\text{-}\mathrm{a})\tfrac{k}{n},\ \mathrm{k} = 0,\ 1,\ ...,\ \mathrm{n}\ .$$

We also need the following definitions.

Definition 3 *Let Sp(I,μ,μ_2) be the set of all F with the same first two moments μ_1 and μ_2 and concentrated on I.*

Definition 4 *Let Sp(A_n,μ,μ_2) be the set of all F with the same first two moments μ_1 and μ_2 and concentrated on A_n.*

The set Sp(I,μ,μ_2) corresponds to the set of all F satisfying the constraints (9) of the optimization problem. All F in Sp(A_n,μ,μ_2) are finite-atomic. The probability masses of F belonging to $\mathrm{Sp}(\mathrm{A}_n,\mu,\mu_2)$ are denoted by $\mathrm{f}_{i_0}, \mathrm{f}_{i_1}, \ldots, \mathrm{f}_{i_n}$. It is clear that $\mathrm{Sp}(\mathrm{A}_n,\mu,\mu_2)$ is a subset of $\mathrm{Sp}(\mathrm{I},\mu,\mu_2)$. We use the following notations.

Definition 5 *Let $F_{\min}$ (or $F_{\max}$) be the solution to the optimization problem on the set $Sp(I,\mu,\mu_2)$.*

Definition 6 *Let $F_{\min,n}$ (or $F_{\max,n}$) be the solution to the optimization problem on the set $Sp(A_n,\mu,\mu_2)$.*

The basic idea of the numerical approach can be summarized in the following steps:

- Find $F_{\min,n}$ (or $F_{\max,n}$).
- By increasing n, the size of $\mathrm{Sp}(\mathrm{A}_n,\mu,\mu_2)$ increases and it follows that $F_{\min,n}$ converges to $F_{\min}$ ($F_{\max,n}$ converges to $F_{\max}$).

This approach is possible since $\mathrm{Sp}(\mathrm{I},\mu,\mu_2)$ is weakly compact. A space S is weakly compact if for each sequence $\mathrm{F}_n \in \mathrm{S}$, a subsequence F_{n_i} and a probability distribution F exists such that $\mathrm{F}_{n_i} \rightarrow \mathrm{F}$ weakly, for $i \uparrow \infty$. In the search of a solution, we apply a general optimization algorithm which is presented in the next section. The application of this algorithm requires the use of a numerical approximation method in order to calculate $\phi(u,\eta,F)$.

6 General Optimization algorithm

We denote by $\partial\phi(F_1, F_2)$ the directional derivative of $\phi(u, \eta(r), F)$ at F_1 in direction of F_2. Let F_{ext} represent an extremal point of either Sp(A_n,μ,μ_2) or Sp(I,μ,μ_2). A point Z of a given convex space S is said **extremal** if Z cannot be written as a convex combination of two points of S. It can be shown that F_{ext} is finite-atomic with at most three atoms (see Marceau (1996)). The number of extremal points in Sp(A_n,μ,μ_2) is **finite**.

Definition 7 *A point F_0 of Sp(A_n,μ,μ_2) is a local minimum if $\partial\phi(F_0, F) \geq 0$ for all $F \in$ Sp(A_n,μ,μ_2).*

In the following proposition, we give an important property of the set Sp(A_n,μ,μ_2).

Proposition 8 *Every point F of Sp(A_n,μ,μ_2) can be written as a convex combination of extremal points F_{ext} of Sp(A_n,μ,μ_2).*

Proof: See the DeVylder and Marceau (1996b).

Then we also need this result.

Proposition 9 *$\partial\phi(F_1, F_2)$ is linear in F_2.*

Proof: See in DeVylder and Marceau (1996b) and DeVylder (1996).

Given the two previous propositions, we obtain this propostion.

Proposition 10 *A point F_0 of $Sp(A_n,\mu,\mu_2)$ is a local minimum if $\partial\phi(F_0, F_{ext}) > 0$ for all F_{ext} of $Sp(A_n,\mu,\mu_2)$.*

Proof: See De Vylder and Marceau (1996b).

The application of the general optimization algorithm is based on the last proposition and the application is possible since the number of extremal points in Sp(A_n,μ,μ_2) is finite. The general optimization has three steps.

General Optimization Algorithm:

- Step1:
  - Let $F_0 \in$ Sp(A_n,μ,μ_2) be a starting point. Let k =0.
- Step2:
  - Calculate $\partial\phi(F_k, F_{ext})$ for all F_{ext} of Sp(A_n,μ,μ_2).
  - Let $F_{ext,k}$ producing the smallest $\partial\phi(F_k, F_{ext})$.
- Step3:
  - If $\partial\phi(F_k, F_{ext,k}) > 0$, then F_k is a local minimum
  - If $\partial\phi(F_k, F_{ext,k}) < 0$, then we find $\alpha = \alpha_k$ such that $\phi((1-\alpha)F_k + \alpha F_{ext,k})$ is minimal

- Let $F_{k+1} = (1 - \alpha_k)F_k + \alpha_k F_{ext,k}$ and k = k+1.

- Repeat steps 2 and 3.

This algorithm is of steepest descent type. The values of $\phi(u, \eta(r), F)$ and $\partial\phi(F_1, F_2)$ are obtained with numerical approximation methods.

7 Numerical approximation

For the calculation of the ultimate ruin probabilities, we use a different approximation method for each risk model. In the calculation of $\phi(u, \eta,\ F)$ within the classical risk model with constant premium rate, our numerical approximation method is based on the approximation of this risk model by the elementary risk model. The elementary model corresponds to the compound binomial model presented by Gerber (1988) and examined by Shiu (1989) and Wilmot(1993). The use of this risk model for numerical approximation of the (non-) ruin probabilities in the classical risk model with constant premium rate has been proposed and studied in DeVylder and Marceau (1996a) (see also Dickson (1994), DeVylder (1996), Marceau (1996), Dickson, Egidio Dos Reis and Waters (1995)). The ultimate non-ruin probability in the elementary risk model has an explicit expression and it is easy to evaluate. It is used as an approximation of $\phi(u, \eta, F)$. The quality of the approximation is very good. The numerical approximation methods of $\phi(u, \eta, F)$ proposed in Dufresne and Gerber

(1989), Panjer (1986) or Panjer and Wilmott (1992) can also be applied.

For the calculation of $\phi(u, \eta, F)$ within the classical risk model with variable premium rate, we use the numerical method proposed by DeVylder (1996). The method of DeVylder is based on the discretization of the probability distribution function G defined in (5). The methods proposed by Petersen (1990) are also appropriate. They are based on the utilization of numerical methods for the solution of integral equations. These methods are explained in Baker (1977).

According to DeVylder (1996), the directional derivative $\delta\phi(F_0, F_1)$ is estimated by

$$\delta\phi(F_0, F_1) \cong \frac{\phi(u,\eta,F_\varepsilon) - \phi(u,\eta,F_0)}{\varepsilon},$$

where $F_\varepsilon = (1-\varepsilon)\, F_0 + \varepsilon\, F_1$ and ε is a small positive real number (ex: 0.00001).

8 Numerical examples

In the numerical examples, we assume for both risk models that the probability distributions F are concentrated on the interval I = [0,1]. The first two moments are $\mu = 0.400$ and $\mu_2 = 0.225$. The parameter λ of the Poisson Process $\{N(t), t \geq 0\}$ is equal to 1. In the classical risk model with constant premium, the security loading η is 25%.

We also consider the classical risk model with interest on the surplus, which is a special case of the classical risk model with variable premium. The function $\eta(\mathrm{r})$ is given by

$$\eta(\mathrm{r}) = \eta + \frac{\delta}{\lambda\mu}\,\mathrm{r} = 0.25 + \frac{0.02}{(1)(0.400)}\,\mathrm{r}\ .$$

For the application of the general optimization algorithm, the finite set of atoms A_n is $\{i_0, i_1, ..., i_n\}$ with

$$\mathrm{i}_k = \frac{k}{n},\ \mathrm{k} = 0,\ 1,\ ...,\ \mathrm{n}$$

and n = 50.

In order to accelerate the performance of the general optimization algorithm, we choose as starting point F_0 the extremal point F^{ext} of the space $\mathrm{Sp}(\mathrm{A}_n,\mu,\mu_2)$ which minimizes (maximizes) the functional $\phi(u, \eta, F)$. The procedure needed to determine in a systematic way the extremal points F^{ext} is given in De Vylder, Goovaerts and Marceau (1997a) or Marceau (1996).

For the classical risk model with constant premium rate, the values of $\phi(u, \eta, F_{\min})$ and $\phi(u, \eta, F_{\max})$ for different initial surplus levels u are given in the tables 1 and 2 with the corresponding atoms and masses of $\mathrm{F}_{\min}$ and $\mathrm{F}_{\max}$. For the classical risk model with variable premium rate, the values of $\phi(u, \eta(r), F_{\min})$ and $\phi(u, \eta(r), F_{\max})$ for different initial surplus levels u are given in the tables 3 and 4. The solutions $\mathrm{F}_{\min}$ and $\mathrm{F}_{\max}$ in those tables are "amalgamated" . The solution obtained from the optimization algorithm is $F_{\min,n}$ (or $F_{\max,n}$). It is the solution to the optimization problem on $\mathrm{Sp}(\mathrm{A}_n,\mu,\mu_2)$. The solution $F_{\min,n}$ (or $F_{\max,n}$) may have successive atoms and isolated atoms. Successive atoms j_1, ..., j_k (k >1) with masses f_{j_1}, ..., f_{j_k} are amalgamated in the unique atom

$$\frac{1}{n}\frac{f_{j_1}j_1+\ldots+f_{j_k}j_k}{f_{j_1}+\ldots+f_{j_k}}.$$

The masses of the amalgamated solution $F^a_{\min,n}$ (or $F^a_{\max,n}$) are recalculated in order to achieve the constraints of the optimization problem. The solution $F^a_{\min,n}$ (or $F^a_{\max,n}$) is an approximation of the solution $F_{\min}$ (or $F_{\max}$).

In regards to the numerical results, we observe that $F_{\min}$ and $F_{\max}$ are always extremal points of Sp(I,μ,μ_2). The solutions $F_{\min}$ and $F_{\max}$ have at most three atoms. These solutions are not uniform in function of the initial surplus u. Also, in other numerical tests, we observe that the solutions $F_{\min}$ and $F_{\max}$ are not uniform in η.

For each risk model, it seems that there exists a u_0 for which the $F_{\min}$ and $F_{\max}$ are the same for all u above u_0. The existence of such u_0 is proven in DeVylder, Goovaerts and Marceau (1997b) within the classical risk model with constant premium rate. We can also obeserve that for a given small initial surplus u, the solution $F_{\min}$ (or $F_{\max}$) is not the same from one risk model to the other.

For practical values of ultimate non-ruin probability $\phi(u,\eta,F)$, the difference between $\phi(u,\eta,F_{\min})$ and $\phi(u,\eta,F_{\max})$ is small. This gives a good approximation of $\phi(u,\eta,F)$. Similarly, since the difference between $\phi(u,\eta(r),F_{\min})$ and $\phi(u,\eta(r),F_{\max})$ is small for practical values of $\phi(u,\eta(r),F)$, we obtain a good approximation of $\phi(u,\eta(r),F)$ without having to estimate the probability distribution F.

9 Conclusion

The optimization of the ultimate ruin probability in a more general risk model is examined. We use a numerical approach in order to find the solution of the optimization problem. The solutions $F_{\min}$ and $F_{\max}$ have at most three atoms when the constraints of the problem are a closed interval and fixed two first moments. We obtain extremal lower and upper bounds to the ultimate non-ruin probability without having to estimate the probability distributions of individual claimsize. The difference between these bounds is so small for practical values of ultimate ruin probabilities (i.e. less that 10%) that they represent good approximations to the ultimate non-ruin probability.

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Table 1 – Minimal non-ruin probabilities

Variation of the solution with u

u	a_{min}	b_{min}	c_{min}	$\phi(u,\eta,F_{min})$
1.0	0.0000	0.1400	0.6400	0.5975
1.5	0.2742		0.9167	0.7237
2.0	0.2917		1.0000	0.8081
2.5	0.2917		1.0000	0.8666
3.0	0.2917		1.0000	0.9073
3.5	0.2917		1.0000	0.9355
4.0	0.2917		1.0000	0.9552
4.5	0.2917		1.0000	0.9688
5.0	0.2917		1.0000	0.9783

I=[0,1] $\mu_1 = 0.4$ $\mu_2 = 0.225$ $\eta = 0.25$

Note: a_{min}, b_{min} and c_{min} are the atoms of F_{min}

Table 2 – Maximal non-ruin probabilities

Variation of the solution with u

u	a_{max}	b_{max}	c_{max}	$\phi(u,\eta,F_{max})$
1.0	0.2700	0.7900	1.0000	0.6030
1.5	0.0000		0.5625	0.7258
2.0	0.0000		0.5625	0.8130
2.5	0.0000		0.5625	0.8726
3.0	0.0000		0.5625	0.9131
3.5	0.0000		0.5625	0.9408
4.0	0.0000		0.5625	0.9596
4.5	0.0000		0.5625	0.9725
5.0	0.0000	--	0.5625	0.9812

I=[0,1] $\mu_1 = 0.4$ $\mu_2 = 0.225$ $\eta = 0.25$

Note: a_{max}, b_{max} and c_{max} are the atoms of F_{max}

Table 3 – Minimal non-ruin probabilities

Variation of the solution with u

u	a_{min}	b_{min}	c_{min}	$\phi(u,\eta,F_{min})$
1.0	0.1200		0.6321	0.7280
1.5	0.2750		0.9200	0.8525
2.0	0.2917		1.0000	0.9218
2.5	0.2917		1.0000	0.9601
3.0	0.2917		1.0000	0.9804
3.5	0.2917		1.0000	0.9907
4.0	0.2917		1.0000	0.9957
4.5	0.2917		1.0000	0.9981
5.0	0.2917		1.0000	0.9992

I=[0,1] $\mu_1 = 0.4$ $\mu_2 = 0.225$ $\eta = 0.25$ $\delta = 0.04$

Note: a_{min}, b_{min} and c_{min} are the atoms of F_{min}

Table 4 – Maximal non-ruin probabilities

Variation of the solution with u

u	a_{max}	b_{max}	c_{max}	$\phi(u,\eta,F_{max})$
1.0	0.2614	0.7800	1.0000	0.7333
1.5	0.0000		0.5625	0.8558
2.0	0.0000	-	0.5625	0.9269
2.5	0.0000	—	0.5625	0.9646
3.0	0.0000	—	0.5625	0.9837
3.5	0.0000	—	0.5625	0.9928
4.0	0.0000	—	0.5625	0.9970
4.5	0.0000	-	0.5625	0.9988
5.0	0.0000	-	0.5625	0.9995

I=[0,1] $\mu_1 = 0.4$ $\mu_2 = 0.225$ $\eta = 0.25$ $\delta = 0.04$

Note: a_{max}, b_{max} and c_{max} are the atoms of F_{max}