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Efficient Estimation of Ultimate Ruin Probability

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ABSTRACT An unbiased, consistent and asymptotically efficient estimator of $H^{*t}(x)$ based on importance sampling variance reduction technique, in the framework of Monte Carlo simulation methods, will be presented, where $H^{*t}(x)$ is the t-fold convolution of a c.d.f. $H(x)$. We will also show that the estimator is also highly efficient in terms of number of computations.
Using the Pollaczeck-Khinchine formula, we will extent the use of the above mentioned estimator to the calculation of ultimate ruin probabilities in the context of the Classical case of Risk Theory and compare our results with recent actuarial literature using the same methodology, Asmussen and Binswanger(1997).

1. INTRODUCTION

Defining a Classical Risk Process in continuous time $\{Z_t\}_{t\geq 0}$ with X_k claim sizes and premium c per time unit,

$$Z_t = u + ct - \sum_{k=1}^{N_t} X_k$$

where u are the initial reserves and N_t the total number of claims up to time t (distributed Poisson with parameter λt) where λ is the average number of claims in one year (or another time units considered). Let F denote the distribution function of claim sizes X_k with mean p_1 and $c = \lambda p_1(1+\theta)$, where θ is the premium loading factor.

Let us now define $\tau = \inf\{w > 0 : Z_w < 0\}$ as the ruin time and the ultimate ruin probability

$$\Psi(x) = P\{\tau < \infty\}$$

We will use the text-book Pollaczeck-Khinchine formula extensively cited in actuarial literature (see for example Panjer and Willmot(1992).Theorem 11.4.5.) for ruin probability in the Classical case of Risk Theory (exponential waiting times between claims or Poisson number of claims)

$$\begin{aligned}\Psi(x) &= 1-\Phi(x) = 1-\frac{\theta}{1+\theta}\sum_{t=0}^{\infty}\left(\frac{1}{1+\theta}\right)^{t} H^{*t}(x)\\ &= \frac{\theta}{1+\theta}\sum_{t=0}^{\infty}\left(\frac{1}{1+\theta}\right)^{t}\left(1-H^{*t}(x)\right) \qquad (1.1)\end{aligned}$$

where $H^{*t}(x)$ is the t-fold convolution function of the distribution function

$$H(x) = \int_0^x \frac{1-F(x)}{p_1}, \quad x > 0 \qquad (1.2)$$

The former t-fold convolution of $H(x)$ can be expressed using the following multiple integral

$$\begin{aligned}H^{*t}(x) &= \int_0^x \int_{s_1}^x \cdots \int_{s_{t-2}}^x H(x-s_{t-1})\, h(s_{t-1}-s_{t-2}) \cdots \\ &\cdots h(s_2-s_1)\, h(s_1)\, ds_1 \cdots ds_{t-1}\end{aligned}$$

In sections 2 and 3, following Usábel(1998), we will introduce a simple estimator(based on a importance sampling Monte Carlo method) of $H^{*t}(x)$, $\mathcal{H}^{*t}(x)$, and subsequently of $1-H^{*t}(x)$, and prove that it is unbiased and consistent, giving an upper bound of its variance.

In section 4, a sample mean estimator of simple estimators introduced in the former sections is presented. The new estimator $(\aleph^{*t}(x))$ inherits the properties of unbiasedness and consistency and we will prove, in section 5, that it is also asymptotically efficient. Following Heidelberger(1995) or Asmussen and Rubinstein(1995) or Asmussen and Binswanger(1997), we will use here the same standard current criterion for calling a rare events simulation estimator asymptotically(or logarithmically) efficient,

$$\lim_{x\to\infty} \inf \frac{\log\left(sd\left\{\Psi^{e}(x)\right\}\right)}{\log\left(\Psi(x)\right)} \geq 1$$

where $sd\left\{\Psi^{e}(x)\right\}$ is the standard deviation of the estimator.

Nevertheless, our final aim is approximating ultimate ruin probabilities. Outstanding results of efficient simulation of ruin probabilities using importance sampling are Siegmund(1976) and Asmussen (1984), in the case of light-tailed distribution for the claim sizes. Regarding heavy-tailed claim sizes distributions (most interesting in actuarial practice), recentlyAsmussen and Binswanger (1997) presented a very interesting efficient conditional Monte Carlo algorithm (algorithm III of the original paper) based on the idea that, for subexponential claim sizes, only the largest claim and not the sum of all claims causes ruin, following the formal definition of subexponential claim size distribution (7.2).

In the present work, we will first develop in section 6 an unbiased, consistent and efficient estimator, based on the results of former sections, for the ultimate ruin probability suitable to any tail behaviour of the distribution function of the claim sizes. Later, in section 7, a second method based on conditional Monte Carlo and the results of sections 2-5 will be presented. The estimator of the ultimate ruin probability obtained in section 7 is only valid for heavy-tailed claim sizes distributions.

Numerical illustrations are presented in section 8 and section 9 is devoted to concluding comments.

2. Simple Unbiased estimator

Let us now introduce an estimator of the t-fold convolution of $H(x)$,

$$H^{*t}(x) \simeq \mathcal{H}^{*t}(x) = H(x - S_{t-1}) H(x - S_{t-2}) \dots H(x - S_1) H(x) \qquad (2.1)$$

where S_i $i = 1, \dots, t-1$ are random numbers generated using the following density functions:

$$S_1 \longrightarrow d_1(s_1) = \frac{h(s_1)}{H(x)} \qquad s_1 \in [0, x]$$

$$S_j \longrightarrow d_j(s_j) = \frac{h(s_j - S_{j-1})}{H(x - S_{j-1})} \qquad s_j \in [S_{j-1}, x] \qquad j > 1 \qquad (2.2)$$

We will prove now that the above mentioned estimator is unbiased.

Theorem 1. *The function $\mathcal{H}^{*t}(x)$ is an unbiased estimator of the t-fold convolution of $H(x)$.*

Proof. The expected value of the estimator $\mathcal{H}^{*t}(x)$ can be expressed:

$$\begin{aligned}
& E\left\{\mathcal{H}^{*t}(x)\right\} \\
= & \int_0^x \int_{s_1}^x \cdots \int_{s_{t-2}}^x H(x - s_{t-1}) H(x - s_{t-2}) \\
& \dots H(x - s_1) H(x) \frac{h(s_1)}{H(x)} \frac{h(s_2 - s_1)}{H(x - s_1)} \\
& \dots \frac{h(s_{t-1} - s_{t-2})}{H(x - s_{t-2})} ds_1 \cdots ds_{t-1} \\
= & \int_0^x \int_{s_1}^x \cdots \int_{s_{t-2}}^x H(x - s_{t-1}) h(s_{t-1} - s_{t-2}) \\
& \dots h(s_2 - s_1) h_1(s_1) ds_1 \cdots ds_{t-1} \\
= & H^{*t}(x)
\end{aligned}$$

■

It is clear, using Theorem 1, that

$$E\left\{1-\mathcal{H}^{*t}(x)\right\}=1-H^{*t}(x)$$

3. Variance of the simple estimator

The variance of the unbiased estimator $1-\mathcal{H}^{*t}(x)$ can be bounded using the following theorem.

Theorem 2. *The variance of the estimator $1-\mathcal{H}^{*t}(x)$ has an upper bound:*

$$Var\left\{1-\mathcal{H}^{*t}(x)\right\}\leq\left(H(x)\right)^{t}\ H^{*t}(x)-\left[H^{*t}(x)\right]^{2} \qquad (3.1)$$

Proof. Using the properties of the variance

$$Var\left\{1-\mathcal{H}^{*t}(x)\right\}=Var\left\{\mathcal{H}^{*t}(x)\right\}$$

Due to the fact that $\mathcal{H}^{*t}(x)$ is an unbiased estimator

$$Var\left\{\mathcal{H}^{*t}(x)\right\}=E\left\{\left(\mathcal{H}^{*t}(x)\right)^{2}\right\}-\left[H^{*t}(x)\right]^{2} \qquad (3.2)$$

Let us study the former expected value:

$$\begin{aligned}
&E\left\{\left(\mathcal{H}^{*t}(x)\right)^{2}\right\}\\
=\ &\int_{0}^{x}\int_{s_1}^{x}\cdots\int_{s_{t-2}}^{x}H^{2}(x-s_{t-1})\,H^{2}(x-s_{t-2})\\
&\ldots H^{2}(x-s_1)\,H^{2}(x)\,\frac{h(s_1)}{H(x)}\frac{h(s_2-s_1)}{H(x-s_1)}\\
&\cdots\frac{h(s_{t-1}-s_{t-2})}{H(x-s_{t-2})}ds_1\cdots ds_{t-1}\\
\leq\ &H(x)\,H(x)\ldots H(x)\,H(x)\int_{0}^{x}\int_{s_1}^{x}\cdots\\
&\cdots\int_{s_{t-2}}^{x}H(x-s_{t-1})\,H(x-s_{t-2})\cdots H(x-s_1)\,H(x)\\
&\frac{h(s_1)}{H(x)}\frac{h(s_2-s_1)}{H(x-s_1)}\cdots\frac{h(s_{t-1}-s_{t-2})}{H(x-s_{t-2})}ds_1\cdots ds_{t-1}
\end{aligned} \qquad (3.3)$$

because

$$Max\left[H(x-s_{t-1})\,H(x-s_{t-2})\ldots H(x-s_1)\,H(x)\right]=\left(H(x)\right)^{t}$$

since $H(y)$ is non-decreasing.

Substituting (3.3) into (3.2) we get the statement of the theorem. ■

4. Sample mean estimator

Let us define now this new estimator as a sample mean of $1-\mathcal{H}^{*t}(x)$

$$\aleph^{*t}(x,n) = \frac{\sum_{i=1}^{n}\left(1-\mathcal{H}_i^{*t}(x)\right)}{n} = \frac{\sum_{i=1}^{n}\left(1-H\left(x-S_{t-1}^i\right)H\left(x-S_{t-2}^i\right)\dots H\left(x-S_1^i\right)H(x)\right)}{n} \quad (4.1)$$

using (2.1), where S_j^i are random numbers generated from the p.d.fs. (2.2) for $i=1,\dots,n$ and $j=1,\dots,t-1$

$$S_1^i \longrightarrow d_1^i\left(s_1^i\right) = \frac{h\left(s_1^i\right)}{H(x)} \qquad s_1^i \in [0,x]$$

$$S_j^i \longrightarrow d_j^i\left(s_j\right) = \frac{h\left(s_j^i - S_{j-1}^i\right)}{H\left(x_j - S_{j-1}^i\right)} \qquad s_j^i \in \left[S_{j-1}^i, x\right] \qquad j>1 \quad (4.2)$$

and $\{1-\mathcal{H}_i^{*t}(x)\}_{i=1}^{n}$ is a sample of independent estimators $1-\mathcal{H}^{*t}(x)$.

As a sample mean of an unbiased estimator, $\aleph^{*t}(x,n)$ is also unbiased and consistent with variance bounds (in the non trivial case $1-H^{*t}(x)>0$)

$$Var\left\{\aleph^{*t}(x,n)\right\} = \frac{Var\{1-\mathcal{H}^{*t}(x)\}}{n} < \frac{(H(x))^t H^{*t}(x) - \left[H^{*t}(x)\right]^2}{n} \quad (4.3)$$

under fairly general conditions (see for example [9])

$$\lim_{n\to\infty} \quad \aleph^{*t}(x,n) \longrightarrow N\left[1-H^{*t}(x), \sqrt{\frac{Var\{1-\mathcal{H}^{*t}(x)\}}{n}}\right] \quad (4.4)$$

and the interval estimation with a confidence level $1-\alpha$ is

$$\left[\aleph^{*t}(x,n) \mp \phi(1-\alpha)\sqrt{\frac{Var\{1-\mathcal{H}^{*t}(x)\}}{n}}\right] \quad (4.5)$$

we can use an estimator of the variance of the $\mathcal{H}^{*t}(x)$:

$$Var\left\{1-\mathcal{H}^{*t}(x)\right\} \simeq k = \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(1-\mathcal{H}_i^{*t}(x)\right)^2 - n\sum_{i=1}^{n}\left(1-\mathcal{H}_i^{*t}(x)\right)\right) \quad (4.6)$$

as recommended in [9, pg. 68], k is a strongly consistent estimator of

$$Var\left\{\mathcal{H}^{*t}(x)\right\} - Var\left\{1-\mathcal{H}^{*t}(x)\right\}$$

Then an asymptotically valid confidence interval can be

$$\left[\aleph^{*t}(x,n) \mp \phi(1-\alpha)\sqrt{\frac{k}{n}}\right] \tag{4.7}$$

We can also avoid the use of an estimator for the variance substituting the result of Theorem 2 into (4.5) and get a broader confidence interval

$$\left[\aleph^{*t}(x,n) + \phi(1-\alpha)\sqrt{\frac{T}{n}}\right]$$

where

$$T = (H(x))^t \aleph^{*t}(x) - \left[\aleph^{*t}(x)\right]^2$$

5. Effiency of the Estimator $\aleph^{*t}(x)$

We will follow here a standard current criterion (e.g. Heidelberger(1995) or Asmussen and Rubinstein(1995) or Asmussen and Binswanger(1997)) for calling a rare events simulation asymptotically or logarithmically efficient. Theorem 4 proves that the unbiased and consistent estimator $\aleph^{*t}(x,n)$ is also asymptotically efficient

$$\lim_{x\to\infty} \inf \frac{\log\left(sd\left\{\aleph^{*t}(x,n)\right\}\right)}{\log\left(H^{*t}(x)\right)} \geq 1$$

In order to prove Theorem 4 the following lemma is introduced.

Lemma 3. *Defining the functions $f(x), g(x) \geq 0$where*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$$

if

$$f(x) \leq (g(x))^2 \qquad as \quad x \to \infty \tag{5.1}$$

then

$$\lim_{x\to\infty} \frac{\log(\sqrt{f(x)})}{\log(g(x))} \geq 1$$

Proof. When 5.1 holds, with some simple arithmetical operations we can get for x large enough

$$\begin{aligned} -\log(f(x)) &\geq -\log((g(x))^2) \Rightarrow \\ -\log(f(x)) &\geq -2\log(g(x)) \Rightarrow \\ -\log(\sqrt{f(x)}) &\geq -\log(g(x)) \end{aligned}$$

it is easy to prove then that

$$\lim_{x\to\infty} \frac{\log(\sqrt{f(x)})}{\log(g(x))} \geq 1$$

■

And finally we will prove that the estimator $\aleph^{*t}(x,n)$ is unbiased, consistent and efficient.

Theorem 4. *The estimator $\aleph^{*t}(x,n)$ of the function, $1 - H^{*t}(x)$, is asymptotically efficient.*

$$\lim_{x\to\infty} \inf \frac{\log\left(sd\left\{\aleph^{*t}(x,n)\right\}\right)}{\log\left(1 - H^{*t}(x)\right)} \geq 1$$

Proof. Let us remember that

$$Var\left\{1 - \mathcal{H}^{*t}(x)\right\} = Var\left\{\mathcal{H}^{*t}(x)\right\}$$

It is clear that,

$$\inf \frac{\log\left(sd\left\{\aleph^{*t}(x,n)\right\}\right)}{\log\left(1 - H^{*t}(x)\right)} = \frac{\log\left(sd\left\{\mathcal{H}^{*t}(x)\right\}\right)}{\log\left(1 - H^{*t}(x)\right)} \tag{5.2}$$

because

$$\begin{aligned}\log\left(sd\left\{\aleph^{*t}(x,n)\right\}\right) &= \log\left(\frac{sd\left\{\mathcal{H}^{*t}(x)\right\}}{\sqrt{n}}\right)\\ &= \log\left(sd\left\{\mathcal{H}^{*t}(x)\right\}\right) - \frac{1}{2}\log(n)\end{aligned}$$

expression decreasing with increasing n.

Using Theorem 2 (3.1)

$$\frac{\log\left(sd\left\{\mathcal{H}^{*t}(x)\right\}\right)}{\log\left(1 - H^{*t}(x)\right)} \geq \frac{\log\left(\sqrt{(H(x))^t\ \ H^{*t}(x) - \left[H^{*t}(x)\right]^2}\right)}{\log\left(1 - H^{*t}(x)\right)} \tag{5.3}$$

It is clear that

$$\lim_{x\to\infty}\left((H(x))^t\ \ H^{*t}(x) - \left[H^{*t}(x)\right]^2\right) = \lim_{x\to\infty}\left(1 - H^{*t}(x)\right)^2 = 0$$

and as $x \to \infty$

$$\begin{aligned}&\left(1 - H^{*t}(x)\right)^2 - \left((H(x))^t H^{*t}(x) - \left[H^{*t}(x)\right]^2\right)\\ =&\ 1 - 2H^{*t}(x) + \left(2\left[H^{*t}(x)\right]^2 - (H(x))^t H^{*t}(x)\right)\\ \approx&\ 1 - 2H^{*t}(x) + \left[H^{*t}(x)\right]^2 = \left(1 - H^{*t}(x)\right)^2 \geq 0\end{aligned}$$

because

$$\lim_{x\to\infty} (H(x))^t = \lim_{x\to\infty} H^{*t}(x) = 1 \tag{5.4}$$

and finally

$$(1 - H^{*t}(x))^2 \geq (H(x))^t \; H^{*t}(x) - (H^{*t}(x))^2$$

following the result of lemma 3 we can conclude the statement of the theorem. ■

6. An Efficient Algorithm for approximating ultimate ruin probability

Using expression (1.1) and the family of estimators $\aleph^{*t}(x,n)$ $\quad t = 0, 1, \ldots$ we obtain

$$\begin{aligned}
\Psi(x) &= \frac{\theta}{1+\theta}\sum_{t=0}^{\infty}\left(\frac{1}{1+\theta}\right)^t (1 - H^{*t}(x)) \\
&\simeq \Psi_1^e(u,n) = \frac{\theta}{1+\theta}\sum_{t=0}^{L}\left(\frac{1}{1+\theta}\right)^t \aleph^{*t}(x,n)
\end{aligned} \tag{6.1}$$

where

$$\left(\frac{1}{1+\theta}\right)^{L+i} \aleph^{*L+i}(x,n) \approx 0 \qquad i = 1, 2, \ldots$$

as long as we have to estimate the functions $(1 - H^{*t}(x))$ one by one up to $t = L$, estimator $\Psi_1^e(u,n)$ of the ultimate ruin probability inherits the properties of estimator $\aleph^{*t}(x,n)$ and is, subsequently, unbiased, consistent and asymptotically efficient.

Let us now highlight an important property of this method. When we get the estimator from (4.1) and store these pairs of values

$$\left(1 - \mathcal{H}_i^{*t-1}(x), S_{t-2}^i\right) \qquad i = 1, \ldots, n$$

then

$$\begin{aligned}
&\aleph^{*t}(x,n) \\
&= \frac{\sum_{i=1}^{n}(1 - \mathcal{H}_i^{*t}(x))}{n} \\
&= \frac{\sum_{i=1}^{n}\left(1 - H\left(x - S_{t-1}^i\right) H\left(x - S_{t-2}^i\right) \ldots H\left(x - S_1^i\right) H(x)\right)}{n} \\
&= \frac{\sum_{i=1}^{n}\left(1 - H\left(x - S_{t-1}^i\right) \mathcal{H}_i^{*t}(x)\right)}{n}
\end{aligned}$$

where using (2.2)

$$S_{t-1}^{i} \longrightarrow d_{t-2}^{i}\left(s_{t-2}\right)=\frac{h\left(s_{t-1}^{i}-S_{t-2}^{i}\right)}{H\left(x-S_{t-2}^{i}\right)} \qquad s_{t-1}^{i}\in\left[S_{t-2}^{i},x\right] \qquad i=1,\ldots,n$$

This last result means that increasing one unit the order of the convolution only implies generating n random numbers more and the total amount of random numbers required is $n(t-1)$, where t is the dimension considered. The save of number of steps – random numbers in our case – become even more obvious when we need to evaluate $1-H^{*t}(x)$ for $t=1,2,\ldots$, one by one up to a certain integer L. **The total amount of calculations will still be $n(L-1)$, increasing linearly with L or the number of simulations per step, n.**

7. A Second Efficient Algorithm for Subexponential Claim Size Distributions

Let us introduce now a second estimator for ultimate ruin probability based on the compound method of generating random numbers.

Using expression (1.1)

$$\Psi(x)=\frac{\theta}{1+\theta}\sum_{t=0}^{\infty}\left(\frac{1}{1+\theta}\right)^{t}\left(1-H^{*t}(x)\right)$$

it is clear that the ultimate ruin probability is a geometric compound process and

$$\begin{aligned}\Psi(x) &= \lim_{n\to\infty}\frac{\sum_{i=1}^{n}(1-H^{*P_i}(x))}{n}\\ &\simeq \Psi_2^{c}(x,m,n)=\frac{\sum_{i=1}^{n}\aleph^{*P_i}(x,m)}{n} \qquad (7.1)\\ m &= 1,2,\ldots\end{aligned}$$

where P_i is a random number generated from a geometric distribution

$$P_i\rightarrow\frac{\theta}{1+\theta}\left(\frac{1}{1+\theta}\right)^{p}$$

It is easy to prove that estimator $\Psi_2^{c}(x,m,n)$ is unbiased,

$$\begin{aligned}E\left\{\Psi_2^{c}(u,m,n)\right\} &= E_p\left[E\left\{1-\mathcal{H}^{*p}(x)\,|p\right\}\right]=\\ &= \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^{p}E\left\{1-\mathcal{H}^{*p}(x)\right\}\\ &= \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^{p}\left(1-H^{*p}(x)\right)\\ &= \Psi(u)\end{aligned}$$

and consistent because

$$Var\{\Psi_2^c(x,m,n)\} = \frac{Var\{\aleph^{*P}(x,m)\}}{n} = \frac{Var\{\mathcal{H}^{*P}(x)\}}{mn}$$

We will now prove that this estimator is also asymptotically efficient for subexponential claim size distributions.

Theorem 5. *The unbiased and consistent estimator $\Psi_2^c(u,m,n)$ of the ultimate ruin probability is also asymptotically efficient*

$$\lim_{x\to\infty} \inf \frac{\log(sd\{\aleph^{*t}(x,n)\})}{\log(\Psi(x))} \geq 1$$

for subexponential claim size distributions.

Proof. A non-negative random variable X with distribution function H is called subexponential if, for $t > 2$

$$\lim_{x\to\infty} \frac{P(X_1 + \cdots + X_t > x)}{P(\max(X_1,...,X_t) > x)} = 1$$

where $X_1,...,X_t$ are i.i.d. copies of X(see for instance Asmussen and Binswanger(1997) Definition 1.1.).

It is obvious then that

$$\lim_{x\to\infty} \frac{P(X_1 + \cdots + X_t > x)}{P(\max(X_1,...,X_t) > x)} = \lim_{x\to\infty} \frac{1 - H^{*t}(x)}{1 - (H(x))^t} = 1 \tag{7.2}$$

concluding that $1 - H^{*t}(x)$ and $1 - (H(x))^t$ decrease asymptotically at the same order.

When claim sizes follow a subexponential distribution(F(x)), it is easy to prove that

$$H(x) = \int_0^x \frac{1 - F(x)}{p_1}, \quad x > 0$$

is also subexponential.

Let us study now the following variance

$$Var\{\Psi_2^c(x,m,n)\} = \frac{Var\{\aleph^{*P}(x,m)\}}{n} = \frac{Var\{\mathcal{H}^{*P}(x)\}}{mn}$$

using expressions 1.1 and 4.3.

Again, it is easy to prove that

$$\inf \frac{\log(sd\{\Psi_2^c(x,m,n)\})}{\log(\Psi(x))} = \frac{\log\left(sd\left\{\mathcal{H}^{*P}(x)\right\}\right)}{\log(\Psi(x))}$$

because

$$\begin{aligned}
\log\left(\frac{sd\left\{\aleph^{*P}(x,m)\right\}}{n}\right) &= \log\left(\frac{sd\left\{\mathcal{H}^{*P}(x)\right\}}{\sqrt{mn}}\right) \\
&= \log\left(sd\left\{\mathcal{H}^{*P}(x)\right\}\right) - \frac{1}{2}\log(mn)
\end{aligned}$$

expression decreasing with increasing n or m.

If we apply the variance decomposition theorem (see for example Bratley, Fox and Schrage(1987). Lemma 2.1.1.) and theorem 2 and expression (5.4) and (7.2), for x large enough

$$\begin{aligned}
& Var\left\{\left(\mathcal{H}^{*P}(x)\right)\right\} \\
= \; & E_p\left[Var\left\{\mathcal{H}^{*p}(x)\,|p\right\}\right] + Var_p\left\{E\left[\mathcal{H}^{*p}(x)\,|p\right]\right\} \\
= \; & \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^p Var\left\{\mathcal{H}^{*p}(x)\right\} \\
& + \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^p \left(H^{*p}(x) - \Phi(x)\right)^2 \\
\leq \; & \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{H(x)}{1+\theta}\right)^p H^{*p}(x) - (\Phi(x))^2 \\
\approx \; & \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^p (H^{*p}(x))^2 - (\Phi(x))^2 \\
= \; & \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^p \left(H^{*p}(x) - \Phi(x)\right)^2 \\
= \; & \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^p \left((1-\Phi(x)) - (1-H^{*p}(x))\right)^2 \\
= \; & \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^p \left(\Psi(x) - (1-H^{*p}(x))\right)^2 \\
= \; & \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^p (1-H^{*p}(x))^2 - (\Psi(x))^2
\end{aligned}$$

$$\approx \frac{\theta}{1+\theta}\sum_{p=0}^{\infty}\left(\frac{1}{1+\theta}\right)^{p}(1-(H(x))^{p})^{2}-(\Psi(x))^{2}$$
$$= 1-\frac{2\theta}{1+\theta-H(x)}+\frac{\theta}{1+\theta-(H(x))^{2}}-(\Psi(x))^{2}$$
$$= \frac{(1-H(x))-\theta}{(1-H(x))+\theta}+\frac{\theta}{1+\theta-(H(x))^{2}}-(\Psi(x))^{2} \qquad (7.3)$$

and finally, because (7.3) should be non-negative, for x large enough

$$\left[\frac{(1-H(x))-\theta}{(1-H(x))+\theta}+\frac{\theta}{1+\theta-(H(x))^{2}}-(\Psi(x))^{2}\right]-(\Psi(x))^{2}$$
$$\approx -1+\frac{\theta}{1+\theta-(H(x))^{2}}\leq 0$$
$$\Rightarrow \left[\frac{(1-H(x))-\theta}{(1-H(x))+\theta}+\frac{\theta}{1+\theta-(H(x))^{2}}-(\Psi(x))^{2}\right]\leq(\Psi(x))^{2}$$
$$\Rightarrow Var\left\{\left(\mathcal{H}^{*P}(x)\right)\right\}<(\Psi(x))^{2}$$

The former result and lemma 3 lead us to the statement of the theorem.
■

8. Numerical results

We will test the two estimators $\Psi_1^c(u,n)$(6.1) and $\Psi_2^c(u,m,n)$(7.1) in practical calculation of ruin probabilities and compare with similar results of actuarial literature.

Both estimators of the ultimate ruin probability are built upon the simple estimator $\aleph^{*t}(x,n)$ (4.1) of $1-H^{*t}(x)$. It is obvious that estimator $\Psi_1^c(u,n)$ should be better than $\Psi_2^c(u,m,n)$ because the second one, besides using $\aleph^{*t}(x,n)$, is based on conditional Monte Carlo.

Confidence intervals with significance level α of $\aleph^{*t}(x,n)$ were obtained using formulas (4.1), (4.2), (4.6) and (4.7) of section 4.

Figures for estimator $\Psi_1^c(u,n)$ were generated using the results of section 6

$$\Psi_1^c(u,n)=\frac{\theta}{1+\theta}\sum_{t=0}^{L}\left(\frac{1}{1+\theta}\right)^{t}\aleph^{*t}(x,n)$$

and the sum was truncated when the terms became neglectible

$$\left(\frac{1}{1+\theta}\right)^{L+1}\aleph^{*L+1}(x,n)<10^{-8}$$

For the second estimator of the ultimate ruin probability, $\Psi_2^c(u, m, n)$, results of section 7 are used,

$$\Psi_2^c(x, m, n) = \frac{\sum_{i=1}^n \aleph^{*P_i}(x, m)}{n}$$

where P_i is a random number generated from a geometric distribution

$$P_i \rightarrow \frac{\theta}{1+\theta}\left(\frac{1}{1+\theta}\right)^p$$

and subsequently following Fishman(1996)

$$\begin{aligned} P_i &\leftarrow \frac{\ln(u)}{\ln\left(\frac{1}{1+\theta}\right)} \qquad u \rightarrow \mathcal{U}(0,1) \text{ uniform distribution} \\ i &= 1, 2, ..n \end{aligned}$$

We will consider as an illustration, a subexponential claims size distribution, Pareto, and an exponentially tailed one, exponential.

8.1. Pareto distribution PAR(a,b).

The distribution function is

$$F(x) = \left(1 - \left(\frac{a}{x}\right)^b\right) I(x > a) \quad a > 0, b > 1 \text{ and } x > 0$$

the mean is $p_1 = ab/(b-1)$, and the density $f(x)$ and the c.d.f. of the integrated tail distribution $H(x)$ are respectively

$$f(x) = \frac{b-1}{ab}\left(I(x < a) + \left(\frac{a}{x}\right) I(x \geq a)\right)$$

$$H(x) = \frac{b-1}{ab} x I(x < a) + \left(1 - \frac{1}{b}\left(\frac{a}{x}\right)^{b-1}\right) I(x \geq a)$$

For the simulation, we can obtain the inverse function

$$H^{-1}(x) = \frac{ab}{b-1} x \; I\left(x < \frac{b-1}{b}\right) + \frac{a}{(b(1-x))^{\frac{1}{b-1}}} \; I\left(x \geq \frac{b-1}{b}\right)$$

Figures are compared in Table 1 with those obtained by Asmussen and Binswanger(1997), where only subexponential claim sizes was considered, using the efficient estimator of algorithm III (Table I), considered the best of the original paper. Their results were based on the idea that only the largest claim and not the sum of all claims causes ruin, following the formal definition of subexponential claim size distribution (7.2).

Table 1
Ultimate ruin probability confidence intervals for Pareto claims size distribution

PAR(1,2), $\theta = 0.1, n = 1,000, m = 1, \alpha = 0.05$			
U	$\Psi_1^c(u,n)$	$\Psi_2^c(x,m,n)$	Algorithm III(A&B)
10	$(5.57 \mp 0.05)10^{-1}$	$(5.69 \mp 0.25)10^{-1}$	$(5.5 \mp 0.3)10^{-1}$
50	$(1.92 \mp 0.06)10^{-1}$	$(1.93 \mp 0.18)10^{-1}$	$(1.9 \mp 0.2)10^{-1}$
100	$(8.86 \mp 0.75)10^{-2}$	$(7.92 \mp 0.91)10^{-2}$	$(8.6 \mp 1.2)10^{-2}$
500	$(1.14 \mp 0.07)10^{-2}$	$(1.15 \mp 0.09)10^{-2}$	$(1.0 \mp 0.2)10^{-2}$
1000	$(5.36 \mp 0.15)10^{-3}$	$(5.27 \mp 0.37)10^{-3}$	$(5.3 \mp 0.6)10^{-3}$

8.2. Exponential distribution

The distribution function is

$$F(x) = 1 - e^{-\frac{1}{m}x} \qquad x > 0$$

the inverse of the c.d.f. of the integrated tail distribution

$$H^{-1}(x) = -p_1 \ln(x)$$

Table 2
Ultimate ruin probability confidence intervalsfor exponential claims size distribution

$p_1 = 1, \theta = 0.1, n = 1,000, \alpha = 0.05$	
U	$\Psi_1^c(u,n)$
1	$(8.30 \mp 0.009)10^{-1}$
5	$(5.75 \mp 0.063)10^{-1}$
10	$(3.69 \mp 0.074)10^{-1}$
50	$(9.75 \pm 0.638)10^{-3}$
80	$(6.30 \mp 0.491)10^{-4}$

From Table 1, we can conclude that estimator $\Psi_1^c(u,n)$ is better than the methods based on conditional Monte Carlo (as it was expected) because generates smaller confidence intervals and can be used for any tail behaviour of the claims size distribution (see also table 2).

Regarding the results of the estimators based on conditional Monte Carlo; $\Psi_2^c(u,m,n)$ also showed smaller intervals than algorithm III of Asmussen and Binswanger (1997), in the example considered.

9. Concluding Comments

An unbiased, consistent and asymptotically efficient estimator based on importance sampling variance reduction technique, in the framework of Monte Carlo simulation methods, was obtained, $\aleph^{*t}(x,n)$ (4.1) for the reciprocal of the t-fold convolution function of a distribution function $1 - H^{*t}(x)$, $x > 0$.

Based on the former estimator, two new estimators, using the Pollaczeck-Khinchine formula, of the ultimate ruin probabilities in the context of the Classical case of Risk Theory were presented.

The first one, $\Psi_1^c(u,n)$ (6.1), estimates the set of convolutions $\{1 - H^{*i}(u)\}_{i=2}^{L}$ used in the Pollaczeck-Khinchine formula up to a certain integer L for which the rest of the terms of the sum are neglectible. In the second estimator of the ultimate ruin probability obtained, $\Psi_2^c(u,m,n)$ (7.1),instead of truncation, considering that the Pollaczeck-Khinchine formula represents a compound geometric process, conditional Monte Carlo method was used. Both estimators are unbiased and consistent.

However, $\Psi_1^c(u,n)$ can be considered better than the estimators based on Conditional Monte Carlo because

a) It is asymptotically efficient for any distribution of the claims size while $\Psi_2^c(u,m,n)$ and Algoritm III of Asmussen and Binswanger(1997) just for subexponential distributions.

b) Generates smaller confidence intervals.

Finally, it is very important to highlight that in estimator $\Psi_1^c(u,n)$, although we need to obtain approximations for the set $\{1 - H^{*i}(u)\}_{i=2}^{L}$, the number of computations required, random numbers in our case case, is just $n(L-1)$, increasing linearly with the precision parameter n. For the reason just cited above we can consider $\Psi_1^c(u,n)$ a very efficient estimator in terms of computational time despite the complexity involved when approximating sucesive convolutions, in other words, multiple integrals of increasing dimension.

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