"ANNUITIES AND LIFE INSURANCE UNDER RANDOM INTEREST RATES"

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Abstract

In this paper we study the effect of random interest rates on life insurance programs and on annuities both certain and non-certain. This is done under several assumptions on the stochastic structure of the interest rates. We find the moments and the distributions of several random actuarial functions of interest, such as $a_{x:\overline{n}|}$ $A_{x:\overline{n}|}$. Comparisons with the case of fixed interest rates are given. This is done using numerical results and graphical representations of our model.

1. Introduction and literature review

An important problem facing the insurance industry is estimating future interest rates. In particular the future interest rates are important in long term insurance contracts such a life insurance. The liquidity in financial markets make this problem even more significant.

Risks in life insurance are due to two factors (i) Randomness in the remaining lifetime of the insured (ii) Uncertainty in interest rates. The law of large numbers guarantees that the risk due to deaths can be reduced by selling many contracts. Risks due to fluctuating interest rates are difficult to reduce.

This observation motivated many researches to study the effect of volatility on pricing and reserving life insurance in random environment. In actuarial literature there is a distinction between analyzing the effect of randomness in the two cases above. As early as 1969 A.H. Pollard. and J.H. Pollard published a paper in which they treated actuarial functions as random variables. The randomness being caused only by variations in the age at death. Specifically they analyze A_x , a_x and ${}_t v_x$ as random variables giving their first two moments and the correlation between pairs of these random variables.

later on De Peril (1989) gave a survey of the distribution functions (d.f.) and the probability density functions (p.d.f.) of the benefit function of most common life insurance's and annuities.

Boyle (1976) studied the effect of the stochastic nature of interest rates on actuarial functions, assuming that the force of interest is generated by a white noise, that is forces of interest in successive years are assumed to be uncorrelated and normally distributed random variables.

Panjer and Bellhouse (1980,1981) developed a general theory for annuities and insurance functions assuming that the force of interest follows autoregressive process. The theory is further worked out for unconditional and conditional autoregressive processes of orders one and two. Beekman and Fuelling, (1990,1991) presented a model evaluating annuities when interest rates and future life times are random. Expressions for mean values and standard deviations of present value of future payments are obtained. This is done assuming the force of interest rate behaves either like Orenstein-Uhlenbeck process or a Wiener process.

2. The model

We study important random actuarial functions in discrete time, when the interest rates form a sequence of random variables. The actuarial functions include:

$\tilde{\ddot{s}}_{\overline{n}|}$ - Accumulated annuity - certain due under stochastic interest rates.

$\tilde{A}^{1}_{x:\overline{n}|}$ - Temporary life assurance (with term - n) under stochastic interest rates .

$\tilde{A}_{x:\overline{n}|}$ - n-year endowment assurance under stochastic interest rates .

$I\tilde{A}^{1}_{x:\overline{n}|}$ - Increasing whole- life assurance under stochastic interest rates.

$\tilde{\ddot{a}}_{x:\overline{n}|}$ - Temporary life annuity under stochastic interest rates.

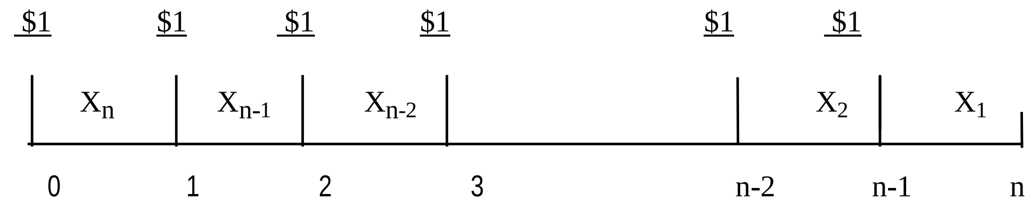
$I\tilde{\ddot{a}}_{x:\overline{n}|}$ - Increasing temporary life annuity under stochastic interest rates.

Hereafter ~ above an actuarial function denotes the value of an actuarial functions under random interest rates.

In this work the interest rates are assumed to be either i.i.d r.v's or markovian stream. We get the cumulative distribution functions of these actuarial functions as well as their moments. Graphs and numerical solutions are given for the distributions and moments of the above actuarial functions.

2.1 Distributions and moments of annuity certain - $\tilde{\ddot{s}}_{\overline{n}|}$

Let R_t be the annual interest rate during [n-t, n-t+1) for t=1,2,...,n. And let $X_t = 1 + R_t$ (i.e X_t is the value of $1 at n-t+1 if deposited at n-t.



Hence the random value of an annuity certain for n years is:

$$\tilde{\ddot{s}}_n = \sum_{i=1}^{n} \prod_{t=1}^{i} X_t = X_1(1+X_2+X_2X_3+\ldots\ldots+X_2X_3\cdots X_n) \tag{1}$$
$$= X_1(1+\tilde{\ddot{s}}_n^*)$$

Where $\tilde{\ddot{s}}_n^* = 1+\sum_{k=2}^{n} \prod_{j=2}^{k} X_j$

Note that i) $\tilde{\ddot{s}}_n^*$ is independent of X_1.

ii) $\tilde{\ddot{s}}_n^*$ is distributed as $\tilde{\ddot{s}}_{n-1}$.

Let $F_{\tilde{\ddot{s}}_n}(y) = p(\tilde{\ddot{s}}_n \le y)$, Conditioning on X_1 we get:

$$F_{\tilde{\ddot{s}}_n}(y) = \int F_{\tilde{\ddot{s}}_{n-1}}\left(\frac{y}{x_1}-1\right)dG_1(x) \tag{2}$$

Where G_1 is the distribution of X_1.

Observed that (2) is a recursive equation for $F_{\tilde{\ddot{s}}_n}(y)$ with initial condition

$$F_{\tilde{\ddot{s}}_n}(y) = p(X_1 \le y) = G_1(y).$$

$$E[(\tilde{\ddot{s}}_n)^k] = \int y^k dF_{\tilde{\ddot{s}}_n}(y) \tag{3}$$

Let $\mu_n = E(\tilde{\ddot{s}}_n)$, Then

$$\mu_n = E(\tilde{\ddot{s}}_n) = E_{x_1} E(\tilde{\ddot{s}}_n | X_1) = E_{x_1} E\{((1+\tilde{\ddot{s}}_{n-1}^*)X_1) | X_1\} \tag{4}$$

This yields the recursive relation:

$$\mu_n = (1+\mu_{n-1})\mu_1 \qquad \forall \;\; n \geq 1 \quad , \qquad \mu_0 = 0 \tag{5}$$

So

$$\mu_n = \mu_1 + \mu_1^2 + \cdots + \mu_1^n = \frac{\mu_1 - \mu_1^{n+1}}{1-\mu_1} \tag{6}$$

Similarly if we denote by σ_n^2 the variance of $\tilde{\ddot{s}}_n$, $Var(\tilde{\ddot{s}}_n)$. we get

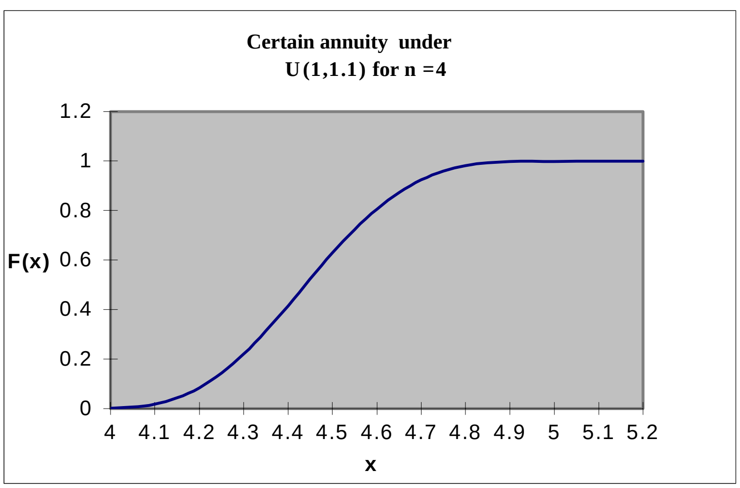
$$\sigma_n^2 = (1+\mu_{n-1})^2 \sigma_1^2 + \sigma_{n-1}^2 (\mu_1)^2 \qquad \forall \;\; n \geq 2 \tag{7}$$

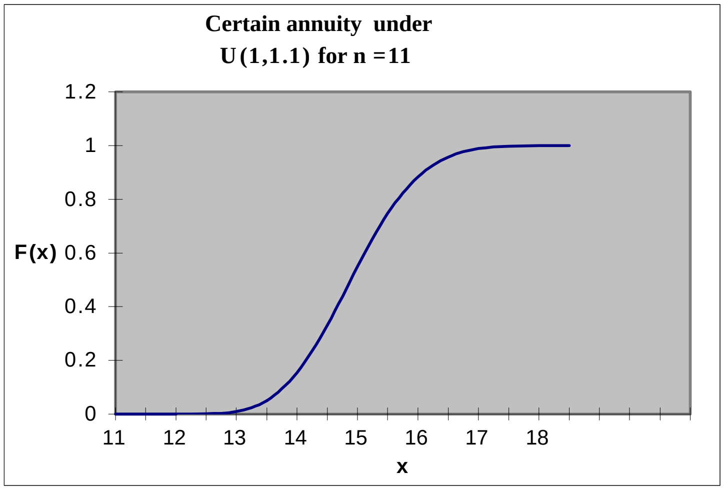
Note that the above recursive relations ((2), (6), (7)) applies for any distributions of i.i.d interest rates.

Numerics

First case

$X_t = (1+R_t) \sim U(1,1.1) \qquad \forall \quad t=1,...n.$





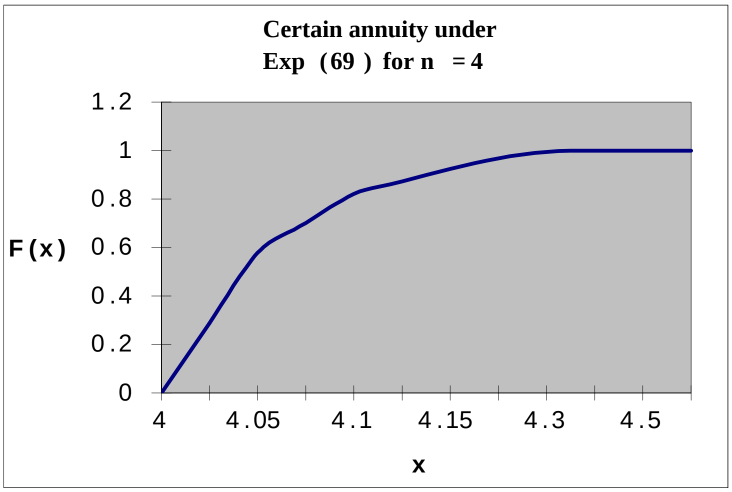
Note the larger the n the steeper the graph of the distribution.

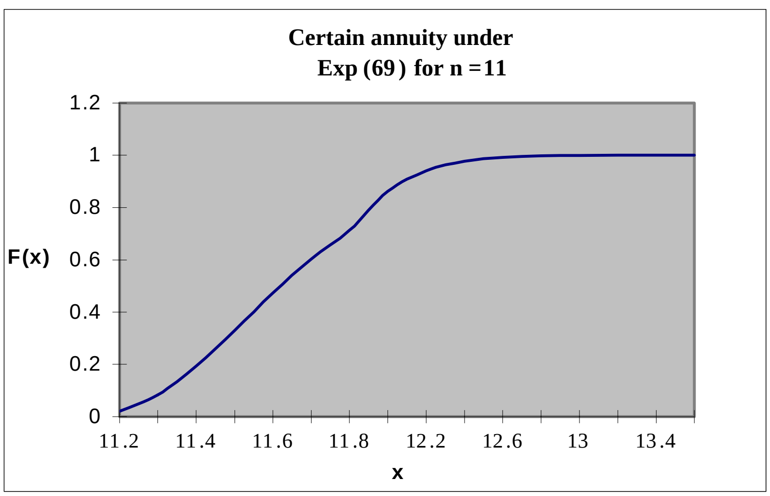
For the moments we get first two moments of $\tilde{\ddot{s}}_n$ assuming $R_t \sim U(1,1.1)$ $\forall$ t=1,...n.

n	**First moments**	**Second moments**
1	$\mu_1 = 1.05$	$\sigma_1^2 = 0.000833$
2	$\mu_2 = 2.157$	$\sigma_2^2 = 0.004421$
4	$\mu_4 = 4.565$	$\sigma_4^2 = 0.03001$
11	$\mu_{11} = 14.917$	$\sigma_{11}^2 = 0.78804$

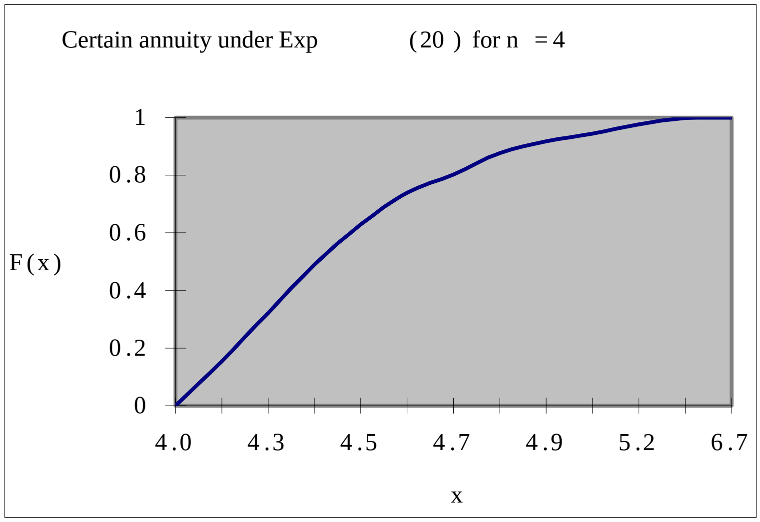
Second and third case

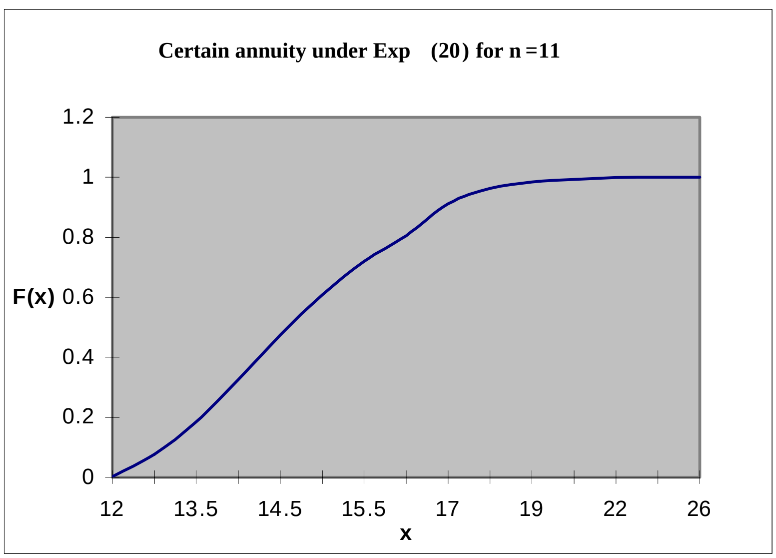
2nd case $X_t = 1 + \exp(\lambda = 69)$ $\forall$ t=1,...n. So with probability 0.999, X falls in (1,1.1).





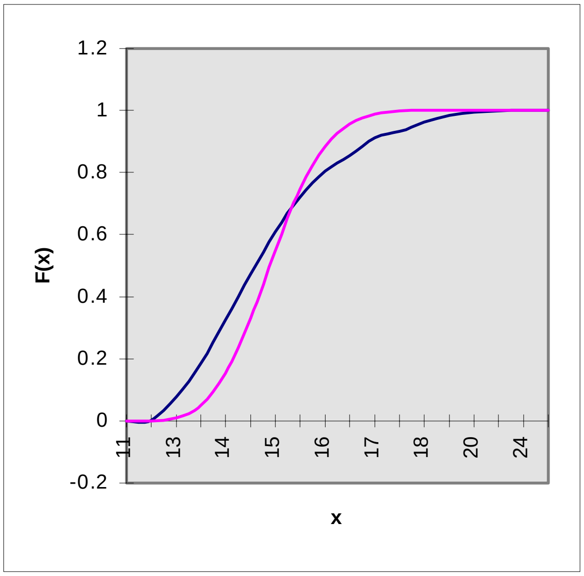
3rd case $X_t = 1 + \exp(\lambda = 20)\ \forall\ t=1,\ldots n$. So that mean interest rate is 5%.





We can see the slopes of the $\tilde{\ddot{s}}_n$ when X_t is exponential, steeper in the case when X_t has uniform distribution.

For example if we compare the uniform case with exp(λ=20) for n=11 unit time:



uniform F(x)	exp F(x)	n=11 x
0	0	11
0	0.0017	12
0.0098	0.0773	13
0.0498	0.185	13.5
0.1536	0.3257	14
0.3314	0.4737	14.5
0.5484	0.6086	15
0.7461	0.7198	15.5
0.8835	0.8053	16
0.9571	0.855	16.5
0.9881	0.9119	17
0.9975	0.9333	17.5
0.9997	0.9622	18
1	0.9843	19
1	0.9935	20
1	0.9989	22
1	0.9998	24
1	1	26

We see that the exponential distribution is more **risky** then the uniform distribution (i.e. one cross over), see Kaas(1994) ch. III.

Waters (1978) gave recursive formulas for the moments of the annuity certain under i.i.d random interest rates, calculated at the beginning of the annum and, solved it numerically, when the X_t 's are i.i.d r.v's having lognormal distribution. We compare the expected value of $\tilde{\ddot{s}}_n$ when (i) Interest rates have lognormal distribution and (ii) When interest rates have a uniform distribution, assuming the two distributions have the same first two moments. We find for $\tilde{\ddot{s}}_n$ in the lognormal case that $\mu_2 = 2.1524$, $\mu_4 = 4.5255$. while in the uniform case , we get $\mu_2 = 2.157$, $\mu_4 = 4.565$. Clearly in both cases the first moments of $\tilde{\ddot{s}}_n$ is 1.05 for both distributions.

2.2 Temporary assurance under random interest rates - $\tilde{A}^1_{x:\overline{n}}$

In this case the randomness is due to :

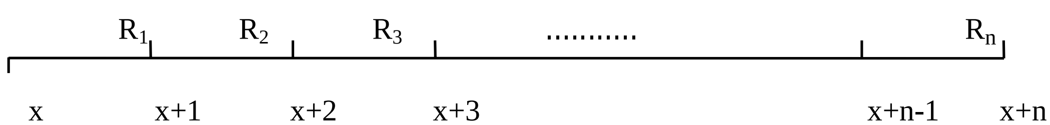
(i) The lifetimes of the insureds.

(ii) The interest rates.

More specifically the remaining life, T_x ,is clearly random. Moreover T_x, has known distribution as given by an appropriate life-table (such as A(1967-70)).

The sequence $\{R_t\}_{t\geq 0}$ assumed to be random, having properties as listed in subsection 2.1.

Specifically let R_t be i.i.d random variables which represent the interest rates at the year [t-1, t) for t=1,2,....,n.



Recall that

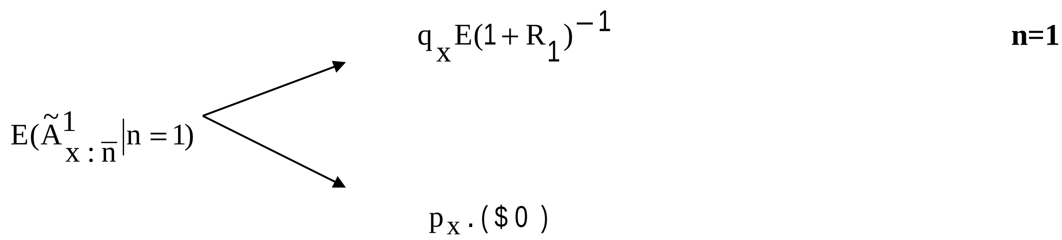
$$A^1_{x:\overline{n}} = \sum_{k=0}^{n-1} {}_{k|}q_x v^{k+1} \tag{8}$$

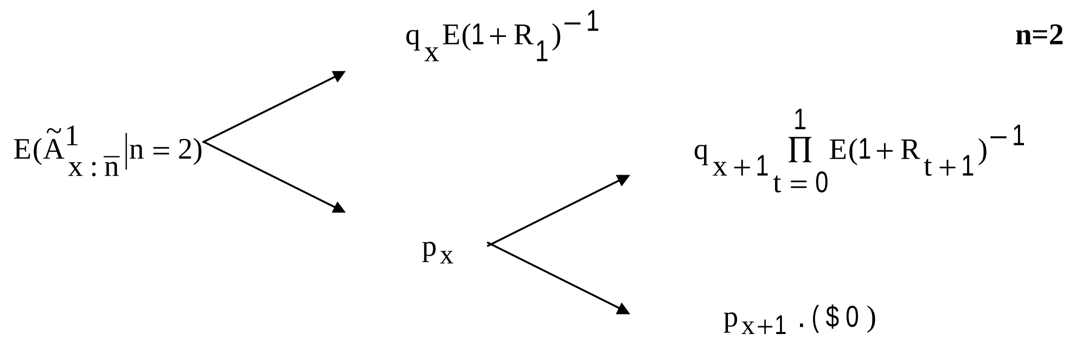
under random interest rates one has to replace v^k in (8) by $\prod_{t=0}^{k-1} E(1+R_{t+1})^{-1}$

thus we get :

$$E(\tilde{A}^1_{x:\overline{n}}) = \sum_{k=0}^{n-1} {}_{k|}q_x \prod_{t=0}^{k} E(1+R_{t+1})^{-1} \tag{9}$$

The tree diagram for $\mathbf{E(\tilde{A}^1_{x:\overline{n}})}$ is:





<u>The backward and forward recursive equation for</u> $E(\tilde{A}^1_{x:\overline{n}})$ are given respectively by (10) and (11) below

$$E(\tilde{A}^1_{x:n}) = E(\tilde{A}^1_{x:n-1}) + q_{x+n-1}\Big(\prod_{k=x}^{x+n-2} p_k\Big)\Big(\prod_{t=0}^{n-1} E(1+R_{t+1})^{-1}\Big) \qquad (10)$$

$\forall \quad n \geq 2$

$$E(\tilde{A}^1_{x:\overline{n}}) = p_x E(\tilde{A}^1_{x+1:\overline{n-1}}) E(1+R_1)^{-1} + q_x E(1+R_1)^{-1} \qquad \forall \ n \geq 2 \qquad (11)$$

With boundary conditions.

$$E(\tilde{A}^1_{x:\overline{0}}) = 0 \qquad \forall \quad x \qquad , n=0 \qquad (12)$$

with boundary conditions given by (12).

Conditioning on the status of the individual at the end of the first year (i.e - active or dead) we get:

$$P(\tilde{A}^1_{x:\overline{n}} \le z) = P(\tilde{A}^1_{x:\overline{n}} \le z | I=0)P(I=0) + P(\tilde{A}^1_{x:\overline{n}} \le z | I=1)P(I=1)$$

$$= P((1+R_1)^{-1} \le z)q_x + P((1+R_1)^{-1}\tilde{A}^1_{x+1:\overline{n-1}} \le z)p_x \qquad (13)$$

Where

$$I = \begin{cases} 0 - & \text{if the insured, aged x, dies during the first year i.e during } (x,x+1). \\ 1 - & \text{otherwise} \end{cases}$$

Let $\alpha^{(z)}_{x:\overline{n}}$ be the distribution function of $\tilde{A}^1_{x:\overline{n}}$, so by conditioning on R_1 we get from (13) the following equation:

$$\alpha^{(z)}_{x:\overline{n}} = q_x P(R_1 \ge \frac{1}{z} - 1) + p_x \int_r \alpha^{(z(1+r))}_{x+1:\overline{n-1}} f^{(r)}_{R_1} dr \qquad \forall \quad n \ge 2 \qquad (14)$$

with boundary conditions:

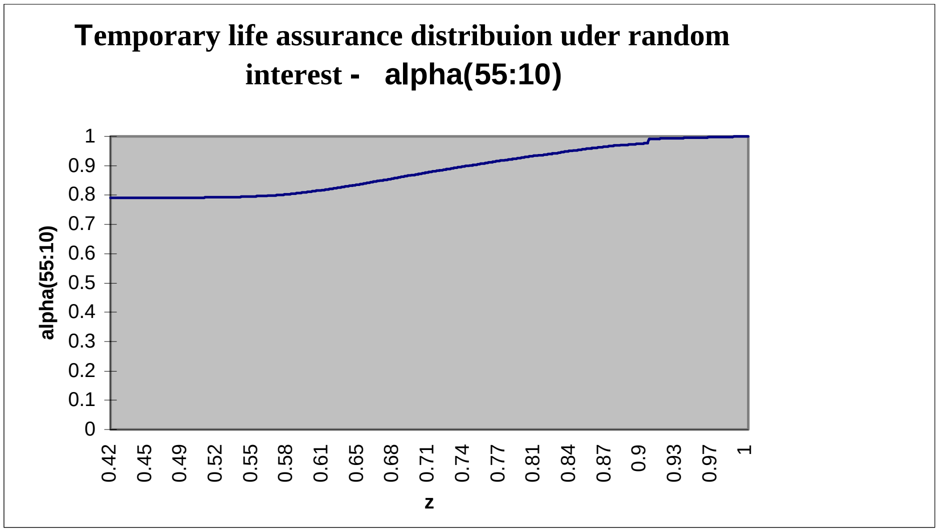
$$\alpha^{(z)}_{x:\overline{0}} = \begin{cases} 0 & z < 0 \\ 1 & z \ge 0 \end{cases} \qquad \forall \qquad x = 0,1,2,\ldots \qquad (15)$$

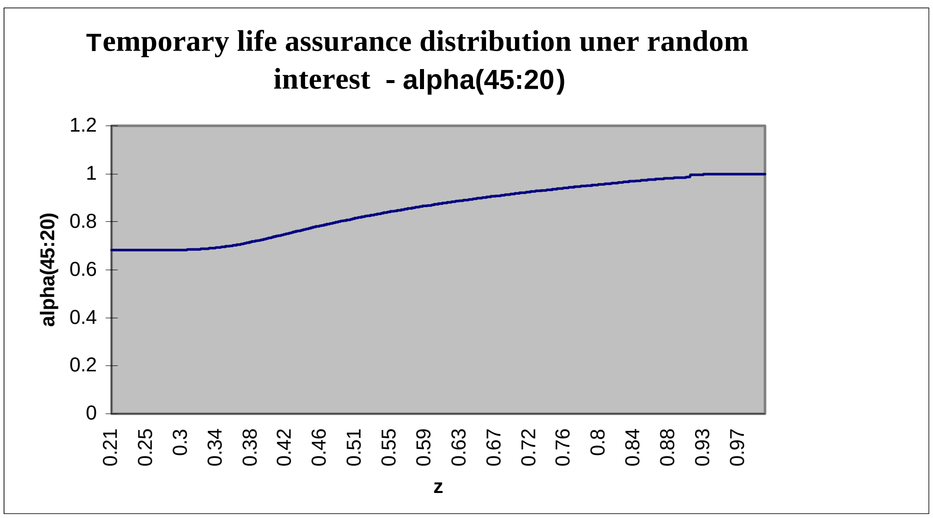
Numerics

Let $\mathbf{R_t} \sim \mathbf{U(0,0.1)}$ **i.i.d** r.v's $\forall$ $\mathbf{t = 1,2,\ldots n}$

Using life table A(1967-70) and equation (14) we get the following graphs for

$\alpha^{(z)}_{55:\overline{10}}$ and $\alpha^{(z)}_{45:\overline{20}}$





We can see that alpha is monotone increasing function of x .

Note that $\alpha^{(0)}_{x:\overline{n}}$ is the probability that $T_x \geq n$.

Remark

We would like to compare temporary insurance with discount factor v where $v = E(1+R_t)^{-1}$, with temporary insurance under random interest rates. Then by Jensen inequality $E(\tilde{A}^1_{x:\overline{n}}) \geq A^1_{x:\overline{n}}$. In particular for x=55 and n=10 , we get:

$$E(\tilde{A}^1_{55:\overline{10}}) = 0.1001328 \geq A^1_{55:\overline{10}} = 0.0966857$$

2.3 whole life assurance under random interest rates - $\tilde{\mathbf{A}}_{\mathbf{x}}$

For whole life assurance with random interest rates, one can derive formulas for the moments of $\tilde{\mathbf{A}}_{\mathbf{x}}$ and for its distribution by letting $n = \infty$ in the formulas obtained for $\tilde{A}^1_{x:\overline{n}}$. In particular we have:

$$E(\tilde{A}^1_x) = \sum_{k=0}^{\infty} {}_{k|}q_x \prod_{t=0}^{k} E(1+R_{t+1})^{-1} \tag{17}$$

Let $\zeta_x^{(z)}$ be the cumulative distribution function of $\tilde{A}^1_x$ **.** By conditioning on R_1 we get from the following recursive equation:

$$\zeta_x^{(z)} = q_x P(R_1 \geq \frac{1}{z} - 1) + p_x \int_r \zeta_{x+1}^{(z(1+r))} f_{R_1}^{(r)} dr \tag{18}$$

2.4 Term endowment assurance under random interest rates - $\tilde{\mathbf{A}}_{\mathbf{x}:\overline{\mathbf{n}}}$

For this random endowment assurance, we have:

$$E(\tilde{A}_{x:\overline{n}}) = p_x E(\tilde{A}_{x+1:\overline{n-1}}) E(1+R_1)^{-1} + q_x E(1+R_1)^{-1} \quad \forall \quad n \geq 2 \tag{19}$$

with initial condition

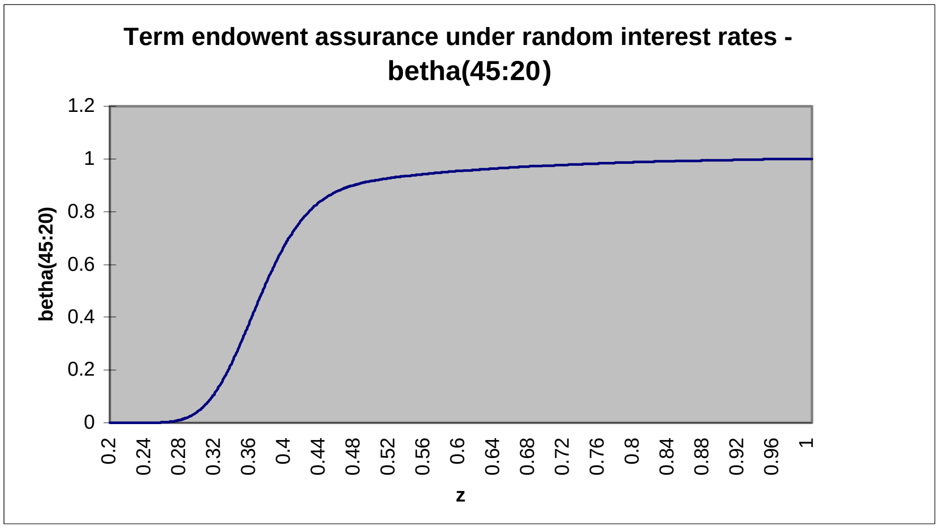
$$E(\tilde{A}_{x:\overline{1}}) = q_x E(1+R_1)^{-1} + p_x E(1+R_1)^{-1} = E(1+R_1)^{-1} \qquad n=1 \tag{20}$$

i.e $\tilde{A}_{x:\overline{n}}$ be the cumulative distribution function of $\beta^{(z)}_{x:\overline{n}}$ Let

$\beta^{(z)}_{x:\overline{n}} = P(\tilde{A}_{x:\overline{n}} \le z)$ then we get:

$$\beta^{(z)}_{x:\overline{n}} = q_x P\left(R_1 \ge \frac{1}{z} - 1\right) + p_x \int_r \beta^{(z(1+r))}_{x+1:\overline{n-1}} f^{(r)}_{R_1} dr \qquad \forall \quad n \ge 2 \qquad (21)$$

under the boundary b'1D()Tj47(e)7t629.5216(Tj4T9 1 T0 TD8)Tj-5w( )TjF2 1 Tf2BT6 TD05



Here we can see that as the insured is younger, z - the l.h.s of the support of betha is closer to 0, and the graph, the neighborhood of 0 is steeper.

2.5 . n-year life annuity under random interest rates - $\tilde{a}_{x:\overline{n}}$

Consider the case where the insured aged x pays 1$ per annum in the end of each year until the minimum between his remains life and the next n years. We would like to study such a program under stochastic interest rates.

We have the following basic relation

$$\tilde{a}_{x:\overline{n}} = \begin{cases} 0 & , \quad I = 0 \\ (\tilde{a}_{x+1:\overline{n-1}} + 1)(1+R_1)^{-1} & , \quad I = 1 \end{cases} \tag{23}$$

where I is given in (13).

Conditioning on I we have the following recursive relation for $\psi^{(z)}_{x:\overline{n}}$

$$\psi^{(z)}_{x:\overline{n}} = P(\tilde{a}_{x:\overline{n}} \le z) = P(\tilde{a}_{x:\overline{n}} \le z \mid I=0)P(I=0) + P(\tilde{a}_{x:\overline{n}} \le z \mid I=1)P(I=1)$$

$$= q_x + P\left[(1+R_1)^{-1}\left\{\tilde{a}_{x+1:\overline{n-1}} + 1\right\} \le z \right] \cdot p_x$$

$$= q_x + p_x \int_r \psi^{(z(1+r)-1)}_{x+1:\overline{n-1}} f_{R_1}^{(r)} dr \qquad (24)$$

$$\forall \quad n \ge 2$$

under the boundary condition:

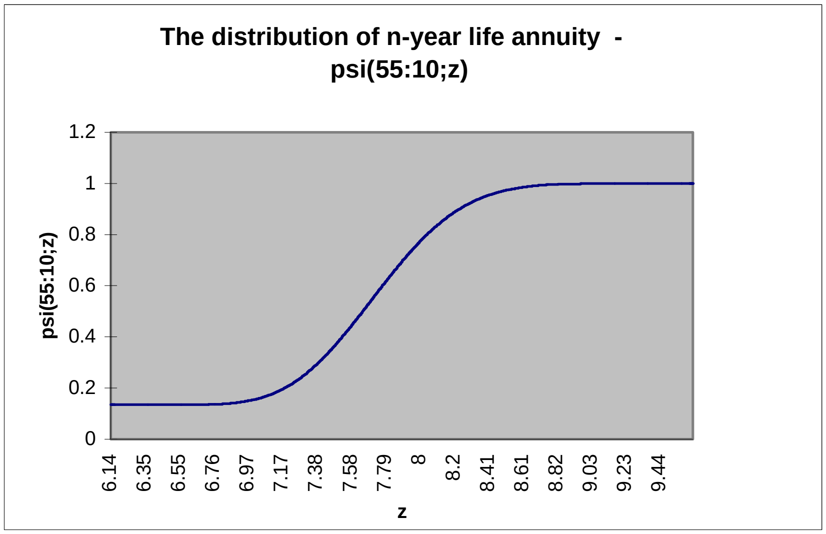
$$\psi^{(z)}_{x:\overline{1}} = q_x + p_x P(R_1 \ge \frac{1}{z} - 1) \qquad \forall \quad z \qquad (25)$$

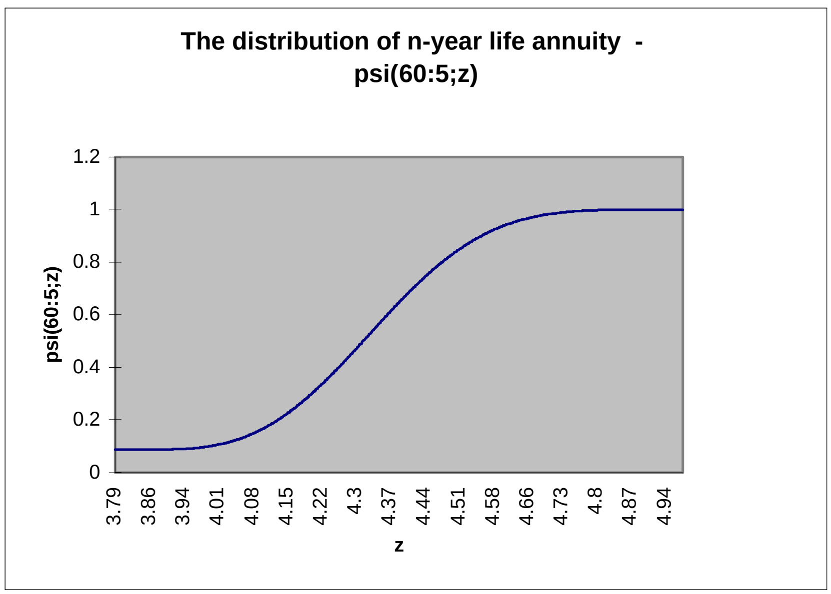
<u>Numerics</u>

Let $\mathbf{R_t} \sim \mathbf{U(0,0.1)}$ **i.i.d** r.v's $\forall$ $\mathbf{t = 1,2,... n}$

Using life table A(1967-70) and equation (24) we get the following graphs for

$\psi^{(z)}_{55:\overline{10}}$ and $\psi^{(z)}_{60:\overline{5}}$:





We can see here that as the insured is younger, z - the l.h.s of the support of psi is closer to 0.

We have derived also formulas for the distributions and the moments of :

$A^{1}_{x:\overline{n}}$

Pollard, A.H. & Pollard, J.H.A (1969). **A stochastic approach to actuarial functions.** *Journal of the Institute of Actuaries*.95, 79-85.

Waters, J.A. (1978). **The moments and distributions of actuarial functions**. *Journal of the Institute of Actuaries* 105, 61-75.