Pricing Dynamic Insurance Risks Using the Principle of Equivalent Utility

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Outline of Talk

1. Principle of equivalent utility—static model
2. Principle of equivalent utility—dynamic model
3. Examples with exponential utility

Equivalent Utility—Static Case

- Find the largest price that a (potential) buyer of insurance is willing to pay for insurance against a random loss—this is the so-called *reservation* price.
- This reservation price is determined within the context of expected utility theory.
- u = concave utility function of wealth of the buyer.
- w = initial wealth of the buyer.
- Y = random loss.
- P = maximum premium that the buyer is willing to pay for complete coverage against the loss Y.

$$u(w - P) = \mathrm{E}[u(w - Y)].$$

- See Bowers et al. (1997, equation 1.3.1).

Equivalent Utility—Dynamic Case

- Now, suppose our buyer of insurance has other decisions to make besides the maximum amount to pay for insurance.
- For example, the buyer chooses how much of her wealth to invest in riskless bond versus a risky stock.
- Riskless bond price X_t follows process $dX_t = rX_t\, dt$.
- Risky stock price S_t follows geometric Brownian motion

$$dS_t = S_t\,(\mu\, dt + \sigma\, dB_t),$$

  in which $\mu > r$ and $\sigma > 0$ are constants, and B_t is a standard Brownian motion with respect to a filtration $\{\mathsf{F}_t\}$ of the probability space $(\Omega, \mathsf{F}, \Pr)$.
- W_t is the wealth of the decision maker at time t.
- π_t is the amount that the decision maker invests in the risky asset at time t.

Dynamic Case (continued)

- On the left-hand side of $u(w - P) = \mathrm{E}[u(w - Y)]$, we have the utility of the buyer without the risk.
- In the dynamic case, we represent this as

$$V(w,\ t) = \sup_{\{\pi_t\} \in \mathsf{A}} E[u(W_T) \mid W_t = w],$$

in which u is the utility of terminal wealth at some specified time T, and A is the set of *allowable* investment policies.

- We will also allow the terminal time T to be a random time, such as the time of death τ.
- V solves the Hamilton-Jacobi-Bellman equation

$$\begin{cases} V_t + \max_{\pi} \left[(\mu - r)\pi V_w + \dfrac{1}{2}\sigma^2 \pi^2 V_{ww} \right. + rwV_w = 0, \\ V(w,\ T) = u(w). \end{cases}$$

Exponential Example

- $u(w) = -\frac{1}{\alpha} e^{-\alpha w}$, for some $\alpha > 0$.
- The value function V is given by

$$V(w,\ t) = -\frac{1}{\alpha} \exp\left(-\alpha w e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T-t) \right).$$

- The optimal amount of money invested in the risky stock at time t is given by

$$\pi *_t = \frac{(\mu - r)}{\sigma^2} \frac{e^{-r(T-t)}}{\alpha}.$$

- Note that this amount is independent of wealth, a common phenomenon with exponential utility.

Dynamic Case (continued)

- On the right-hand side of $u(w - P) = \mathrm{E}[u(w - Y)]$, we have the utility of the buyer with the risk.
- In the dynamic case, we represent this as

$$U(w,\ y,\ t) = \sup_{\{\pi_t\} \in \mathcal{A}} E\left[u(W_T) \,\middle|\, W_t = w,\ Y_t = y \right],$$

in which W_t follows the wealth process given by

$$dW_t = \left[rW_t + (\mu - r)\pi_t \right] dt + \sigma \pi_t dB_t - dY_t,$$

and Y_t is the loss process.

- For concreteness, suppose that Y_t follows a diffusion process

$$dY_t = \theta(W_t,\ Y_t,\ t)dt + \zeta(W_t,\ Y_t,\ t)d\tilde{B}_t,$$

in which $\tilde{B}_t$ is a standard Brownian motion, independent of the Brownian motion B_t for the stock process.

Dynamic Case (continued)

- U solves the Hamilton-Jacobi-Bellman equation

$$\begin{cases} U_t + \max\limits_{\pi} \left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2 \pi^2 U_{ww} \right. + (rw - \theta(w, y, t))U_w + \theta(w, y, t)U_y \\ + \frac{1}{2}\zeta^2(w,\ y,\ t)U_{ww} - \zeta^2(w,\ y,\ t)U_{wy} + \frac{1}{2}\zeta^2(w,\ y,\ t)U_{yy} = 0, \\ \\ U(w,\ y,\ T) = u(w). \end{cases}$$

- This equation simplifies considerably if θ and ζ are independent of y. We will see this in the next example.
- Once we have the value functions V and U, then we solve for the lump sum premium $P(w, y, t)$:

$$V(w - P(w, y, t)) = U(w, y, t).$$

Exponential Utility (continued)

- $u(w) = -\dfrac{1}{\alpha} e^{-\alpha w}$, for some $\alpha > 0$.
- θ and ζ are independent of w and y.
- Then, U is independent of y, and can be written as

$$U(w, t) = V(w, t)\, e^{\psi(t)},$$

  in which V is as in the previous example and ψ solves the ordinary differential equation

$$\begin{cases} \psi'(t) + \alpha e^{r(T-t)}\theta(t) + \dfrac{1}{2}\alpha^2 e^{2r(T-t)}\zeta^2(t) = 0, \\ \psi(T) = 0. \end{cases}$$

- It follows that the reservation price of the buyer is

$$P(w,\ t) = e^{-r(T-t)} \int_t^T \left(e^{r(T-s)}\theta(s) + \frac{1}{2}\alpha e^{2r(T-s)}\zeta^2(s) \right) ds,$$

  the discounted expected loss plus a loading proportional to the variance of the loss during the period $[t, T]$.

Exponential Utility (continued)

- In the fixed horizon case, the reservation price is independent of the stock process. In fact, we would get the same price if we allowed the buyer to invest only in the riskless bond.
- Now, suppose we randomize the time until "expiration" of the contract. Specifically, we replace T with the time of death of the buyer of insurance.
- For simplicity, suppose that r = 0. We omit the details, but we can show that the reservation price of the buyer is now

$$P(w,\ t) = \frac{1}{\alpha} \ln \left[\frac{\int_t^\infty e^{-\int_t^s (\delta - \alpha\theta(u) - \alpha^2 \zeta^2(u)/2) du} \lambda_x(s)\, {}_{s-t}p_{x+t} ds}{\bar{A}^{\delta}_{x+t}} \right],$$

in which $\delta = \dfrac{\mu^2}{2\sigma^2}$.

Exponential Utility (continued)

- If we set T equal to $t + \overset{\circ}{e}_{x+t}$, then we can show that the reservation price with the fixed horizon is less than the one when the horizon is random.
- Also, if we model the losses as a Poisson process, we can find the reservation prices in the cases of fixed and random horizons. If the expected loss and variance of the loss for the Poisson process equal those of the diffusion process, then the reservation price for the diffusion process is less than the price for the Poisson process.
- I find both results to be intuitively pleasing.

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